CHAPTER 5
SENSITIVITY ANALYSIS FOR SOME SYSTEMS OF
GENERALIZED NONLINEAR VARIATIONAL
INCLUSIONS

5.1 INTRODUCTION

In recent years, much attention has been devoted to developing general methods for the sensitivity analysis of solution set of variational inequality problems and variational inclusion problems. From the, mathematical and engineering points of view, sensitivity properties of various variational inequality problems can provide a new insight concerning the problems and can stimulate the ideas for solving the problems. The resolvent operator technique play a crucial role in lots of problems arising from several fields of research, especially from economics, optimization and control theory, operational research, transportation network modelling and mathematical programming. By using the projection technique, Dafermos [27], Mukherjee and Verma [86], Noor [94], Yen [118] dealt with the sensitivity analysis for variational inequalities with single-valued mappings. By using the implicit function approach that makes use of so-called normal mappings, Robinson [107] dealt with the sensitivity analysis for variational inequalities with single-valued mappings in finite-dimensional spaces. By using resolvent operator technique, Adly [1], Noor and Noor [96], and Agarwal et al. [2] studied the sensitivity analysis for quasi-variational inclusions with single-valued mappings. By using projection technique and the property of fixed point set of multi-valued contractive mappings, Ding and Luo [35], and Kazmi et al. [66] studied the behavior and sensitivity analysis of solution set for generalized quasi-variational inequalities. Recently, Liu et al. [82], Kazmi et al. [70], Park and Jeong [100] and Ding [31-33] studied the behavior and sensitivity analysis of solution set of generalized nonlinear implicit quasi-variational inclusions of several type with set-valued mappings. Agrawal et al. [2] and Jeong and Kim [61] studied the sensitivity analysis for systems of nonlinear variational inclusions with single-valued mappings and maximal monotone mappings. Ding and Yao [38] and Kazmi et al. [71] studied the sensitivity analysis for systems of nonlinear variational inclusions with multi-valued mappings and maximal monotone mappings.
In this chapter, we use the resolvent operator techniques and the property of fixed point set of multi-valued contractive mappings, we study the behavior and sensitivity analysis of the solution sets for some classes of systems of generalized nonlinear variational inclusions. In particular, we discuss that the solution sets of these systems of variational inclusions are non-empty, closed and Lipschitz continuous with respect to the parameters under some suitable conditions.

The remaining part of this chapter is organized as follows:

In Section 5.2, we consider a system of generalized nonlinear mixed quasi-variational inclusions considered by Agarwal et al. [2]. By using the resolvent operator technique, we shall discuss the behavior and sensitivity analysis of the solution set for this system of variational inclusions.

In Section 5.3, we consider a system of parametric generalized mixed quasi-variational inclusions in Hilbert spaces considered by Ding et al. [33]. By using the resolvent operator technique and the property of fixed point set of multi-valued contractive mapping, we study the behavior and sensitivity analysis of the solution set for this system of variational inclusions.

In the last Section, we consider a system of generalized nonlinear mixed quasi-variational inclusions in \( q \)-uniformly smooth Banach spaces considered by Jeong et al. [61]. By using the resolvent operator technique, we discuss the behavior and sensitivity analysis of the solution set for this system of variational inclusions.

This chapter is based on work of Agarwal et al. [2], Ding et al. [33] and Jeong et al. [61].

Throughout the Section 5.2 and 5.3, unless or otherwise stated, \( \mathcal{H} \) is a real Hilbert space endowed with norm \( \| \cdot \| \) and the inner product \( \langle \cdot , \cdot \rangle \) while in Section 5.4, \( E \) is a real Banach space endowed with dual space \( E^* \) and the dual pair \( \langle \cdot , \cdot \rangle \) between \( E \) and \( E^* \).

5.2 SYSTEM OF PARAMETRIC QUASI-VARIATIONAL INCLUSIONS

Let \( \Omega \) and \( \Lambda \) be two nonempty open subsets of \( \mathcal{H} \) in which the parameters \( \omega \) and \( \lambda \) take values. Let \( M : \mathcal{H} \times \Omega \to 2^\mathcal{H} \) and \( N : \mathcal{H} \times \Lambda \to 2^\mathcal{H} \) be two maximal
monotone mappings with respect to the first argument, \( A, S : \mathcal{H} \times \Omega \to \mathcal{H} \) and \( B, T : \mathcal{H} \times \Lambda \to \mathcal{H} \) be nonlinear single-valued mappings. The system of parametric quasi-variational inclusions (in short, SPQVI) is: Find \((x, y) \in \mathcal{H} \times \mathcal{H}\) such that

\[
\begin{align*}
0 & \in x - y + \rho(A(y, \omega) + S(y, \omega)) + \rho M(x, \omega), \\
0 & \in y - x + \gamma(B(x, \lambda) + T(x, \lambda)) + \gamma N(y, \lambda).
\end{align*}
\]

where \(\rho > 0\) and \(\gamma > 0\) are constants.

Now, we give the following definitions.

**Definition 5.2.1.** A mapping \( T : \mathcal{H} \times \Lambda \to \mathcal{H} \) is said to be \( k\)-strongly monotone with respect to the first argument if there exists a constant \( k > 0 \) such that

\[
(T(x, \lambda) - T(y, \lambda), x - y) \geq k\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.
\]

**Definition 5.2.2.** A mapping \( T : \mathcal{H} \times \Lambda \to \mathcal{H} \) is said to be \( s\)-Lipschitz continuous with respect to the first argument if there exists a constant \( s > 0 \) such that

\[
\|T(x, \lambda) - T(y, \lambda)\| \leq s\|x - y\|, \quad \forall x, y \in \mathcal{H}.
\]

**Remark 5.2.1.** Since \( M \) is a maximal monotone mapping with respect to the first argument, for any fixed \( \omega \in \Omega \), we can define

\[
J^p_{M(. , \omega)}(\mu) = (I + \rho M(. , \omega))^{-1}(\mu),
\]

which is called the parametric resolvent operator associated with \( M(. , \omega) \) [16].

Next, we give the following lemma.

**Lemma 5.2.1[2].** For any fixed \( \omega \in \Omega \) and \( \lambda \in \Lambda \), \((x(\omega, \lambda), y(\omega, \lambda))\) is a solution of SPQVI(5.2.1) if and only if

\[
x(\omega, \lambda) = J^p_{M(. , \omega)}[y(\omega, \lambda) - \rho(A + S)(y(\omega, \lambda), \omega)],
\]

\[
y(\omega, \lambda) = J^\gamma_{N(. , \lambda)}[x(\omega, \lambda) - \gamma(B + T)(x(\omega, \lambda), \lambda)],
\]

where

\[
J^p_{M(. , \omega)}(\mu) = (I + \rho M(. , \omega))^{-1}(\mu), \quad J^\gamma_{N(. , \lambda)}(\mu) = (I + \gamma N(. , \lambda))^{-1}(\mu).
\]
**Assumption 5.2.1**[2]. There are two constants $\xi > 0$ and $\sigma > 0$ such that
\[
\|J^s_{M(\omega)}(x) - J^s_{M(\omega)}(x)\| \leq \xi \|\omega - \tilde{\omega}\|, \quad \|J^s_{N(\lambda)}(y) - J^s_{N(\lambda)}(y)\| \leq \sigma \|\lambda - \tilde{\lambda}\|, \quad \forall x, y \in H.
\]

The following theorem gives the existence of a unique solution of SPQVI(5.2.1).

**Theorem 5.2.1**[2]. Let $S : H \times \Omega \rightarrow H$ be $k_1$-strongly monotone and $s_1$-Lipschitz continuous with respect to the first argument, $T : H \times \Lambda \rightarrow H$ be $k_2$-strongly monotone and $s_2$-Lipschitz continuous with respect to the first argument; $A : H \times \Omega \rightarrow H$ be $l_1$-Lipschitz continuous with respect to the first argument and $B : H \times \Lambda \rightarrow H$ be $l_2$-Lipschitz continuous with respect to the first argument. Suppose that $M : H \times \Omega \rightarrow 2^H$ and $N : H \times \Omega \rightarrow 2^H$ are two maximal monotone mappings with respect to the first argument. If
\[
0 < \rho < \min\left\{\frac{2(k_2 - l_1)}{s_2 - s_1}, \frac{1}{l_1}\right\}, \quad l_1 < k_1,
\]
\[
0 < \gamma < \min\left\{\frac{2(k_2 - l_2)}{s_2 - s_1}, \frac{1}{l_2}\right\}, \quad l_2 < k_2,
\]
(5.2.2)
then, for any fixed $\omega \in \Omega$ and $\lambda \in \Lambda$, SPQVI(5.2.1) has a unique solution $(x(\omega, \lambda), y(\omega, \lambda))$.

The following theorem shows that the solution of SPQVI(5.2.1) is continuous (or Lipschitz continuous).

**Theorem 5.2.2**[2]. Suppose that the mappings $M, N, A, S, B$ and $T$ are the same as in Theorem 5.2.1, and for any fixed $x, y \in H$, the mappings $\omega \mapsto A(x, \omega)$, $\omega \mapsto S(x, \omega)$, $\lambda \mapsto B(y, \lambda)$, and $\lambda \mapsto T(y, \lambda)$ are all continuous (or Lipschitz continuous). If Assumption 5.2.1 and condition (5.2.2) hold, then the solution $(x(\omega, \lambda), y(\omega, \lambda))$ for SPQVI(5.2.1) is continuous (or Lipschitz continuous).

**5.3 SYSTEM OF PARAMETRIC GENERALIZED MIXED QUASIVARIATIONAL INCLUSIONS**

Let $H_1$ and $H_2$ be two real Hilbert spaces, $\Omega \subseteq H_1$ and $\Lambda \subseteq H_2$ be two non empty open subsets, the parameters $\omega$ and $\lambda$ take values in $\Omega$ and $\Lambda$, respectively. Let $A, B, F, G, Q, S : H_1 \times \Omega \rightarrow C(H_1)$ and $C, D, E : H_2 \times \Lambda \rightarrow C(H_2)$ be multi-valued mappings and $n : H_1 \times \Omega \rightarrow H_1$, $N_1 : H_1 \times H_1 \times H_2 \times H_2 \times \Omega \rightarrow H_1$, $N_2 : H_1 \times H_1 \times \Lambda \rightarrow H_2$ and $T : H_1 \times H_2 \times \Lambda \rightarrow H_2$ be single-valued mappings.
Let $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega \to \mathcal{H}_1$, $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda \to \mathcal{H}_2$ be Lipschitz continuous mappings with Lipschitz constants $\tau_1 > 0$ and $\tau_2 > 0$ respectively, $K_1 : \mathcal{H}_1 \to \mathcal{H}_1$ be $(r_1, \eta_1)$-strongly monotone with $(G - \eta)(\mathcal{H}_1 \times \Omega) \cap \text{dom}K_1 \neq \emptyset$, and $K_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be $(r_2, \eta_2)$-strongly monotone. Let $M_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \to 2^{\mathcal{H}_1}$ be a multi-valued mapping such that for each given $(z_1, \omega) \in \mathcal{H}_2 \times \Omega$, $M_1(\cdot, z_1, \omega)$ is $(K_1, \eta_1)$-monotone mappings with $(G - \eta)(\mathcal{H}_1 \times \Omega) \cap \text{dom}M_1(\cdot, z_1, \omega) \neq \emptyset$, and $M_2 : \mathcal{H}_2 \times \mathcal{H}_1 \times \Lambda \to 2^{\mathcal{H}_2}$ be a multi-valued mapping such that for each given $(z_2, \lambda) \in \mathcal{H}_2 \times \Lambda$, $M_2(\cdot, z_2, \lambda)$ is $(K_2, \eta_2)$-monotone mappings.

We consider the following system of parametric generalized mixed quasi-variational inclusions (in short, SPGMQVI): For each given $(w_1, \omega) \in \mathcal{H}_1 \times \Omega$ and $(w_2, \omega) \in \mathcal{H}_2 \times \Lambda$, find $(x, y) = (x(\omega), y(\lambda)) \in \mathcal{H}_1 \times \mathcal{H}_2$, such that $a = a(x, \omega) \in A(x, \omega)$, $b = b(x, \omega) \in B(x, \omega)$, $c = c(y, \lambda) \in C(y, \lambda)$, $d = d(y, \lambda) \in D(y, \lambda)$, $e = e(y, \lambda) \in E(y, \lambda)$, $f = f(x, \omega) \in F(x, \omega)$, $g = g(x, \omega) \in G(x, \omega)$, $q = q(x, \omega) \in Q(x, \omega)$, and $s = s(x, \omega) \in S(x, \omega)$ and

$$
\begin{align*}
& w_1 \in N_1(a, b, c, d, \omega) + M_1((g - \eta)(x, \omega), e, \omega), \\
& w_2 \in -T(x, y, \lambda) + \mu M_2(y, s, \lambda),
\end{align*}
$$

(5.3.1)

where $\mu > 0$ is a constant.

Now, we give the following definitions.

**Definition 5.3.1[38].** Let $N_1 : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \Omega \to \mathcal{H}$ and $N_2 : \mathcal{H} \times \mathcal{H} \times \Omega \to \mathcal{H}$ be single-valued mappings and $A, B, F, G : \mathcal{H} \times \Omega \to C(\mathcal{H})$ be multi-valued mappings. Then

(i) $G$ is said to be $\alpha_G$-strongly monotone in the first argument if there exists $\alpha_G > 0$ such that

$$
\langle g - \bar{g}, x - \bar{x} \rangle \geq \alpha_G \|x - \bar{x}\|^2, \quad \forall (x, \bar{x}, \lambda) \in \mathcal{H} \times \mathcal{H} \times \Omega, \ g \in G(x, \lambda), \ \bar{g} \in G(\bar{x}, \lambda);
$$

(ii) $G$ is said to be $l_G$-Lipschitz continuous in the first argument if there exists $l_G > 0$ such that

$$
\hat{H}(G(x, \lambda), G(\bar{x}, \lambda)) \leq l_G \|x - \bar{x}\|, \quad \forall (x, \bar{x}, \lambda) \in \mathcal{H} \times \mathcal{H} \times \Omega;
$$

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(iii) \( N_1 \) is said to be \((l_1, l_2)\)-mixed Lipschitz continuous if there exist \( l_1, l_2 > 0 \) such that for all \( a, \tilde{a}, b, \tilde{b}, c, \tilde{c}, d, \tilde{d} \in \mathcal{H} \) and \( \lambda \in \Omega \),

\[
\|N_1(a, b, c, d, \lambda) - N_1(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \lambda)\| \leq l_1\|a - \tilde{a}\| + l_2\|b - \tilde{b}\|.
\]

**Definition 5.3.2**[122]. Let \( \eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) and \( K : \mathcal{H} \to \mathcal{H} \) be two single-valued mappings. Then \( K \) is said to be

(i) \( \eta \)-monotone if

\[
\langle K(x) - K(y), \eta(x, y) \rangle \geq 0, \quad \forall x, y \in \mathcal{H};
\]

(ii) \( \eta \)-strictly monotone if

\[
\langle K(x) - K(y), \eta(x, y) \rangle > 0, \quad \forall x, y \in \mathcal{H},
\]

and equality holds if and only if \( x = y \);

(iii) \((r, \eta)\)-strongly monotone if there exists a constant \( r > 0 \) such that

\[
\langle K(x) - K(y), \eta(x, y) \rangle \geq r\|x - y\|^2, \quad \forall x, y \in \mathcal{H};
\]

(iv) \( l_K \)-Lipschitz continuous if there exists a constant \( l_K > 0 \) such that

\[
\|K(x) - K(y)\| \leq l_K\|x - y\|, \quad \forall x, y \in \mathcal{H}.
\]

**Definition 5.3.3**[122]. Let \( \eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) be a single-valued mapping and let \( M : \mathcal{K} \to 2^\mathcal{H} \) be a multi-valued mapping. Then \( M \) is said to be

(i) \( \eta \)-monotone if

\[
\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in \mathcal{H}, u \in Mx, v \in My;
\]

(ii) \((r, \eta)\)-strongly monotone if there exists a constant \( r > 0 \) such that

\[
\langle u - v, \eta(x, y) \rangle \geq r\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in Mx, v \in My;
\]
(iii) \((m, \eta)\)-relaxed monotone if there exists a constant \(m > 0\) such that
\[
\langle u - v, \eta(x, y) \rangle \geq -m\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in Mx, v \in My;
\]
(iv) maximal monotone if \(\langle u - v, x - y \rangle \geq 0\) for any \((v, y) \in G_r(M)\) implies \((u, x) \in G_r(M)\), where \(G_r(M) = \{(u, x) \in \mathcal{H} \times \mathcal{H} : u \in Mx\};
(v) maximal \(\eta\)-monotone if \(\langle u - v, \eta(x, y) \rangle \geq 0\) for any \((v, y) \in G_r(M)\) implies \((u, x) \in G_r(M)\).

**Definition 5.3.4[122]**. Let \(K : \mathcal{H} \rightarrow \mathcal{H}\) and \(\eta : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}\) be two single-valued mappings and \(M : \mathcal{H} \rightarrow 2^{\mathcal{H}}\) be a multi-valued mapping. Then \(M : \mathcal{H} \rightarrow 2^{\mathcal{H}}\) is said to be \((K, \eta)\)-monotone if

(i) \(M\) is \((m, \eta)\)-relaxed monotone;
(ii) \((K + \rho M)(\mathcal{H}) = \mathcal{H}, \quad \forall \rho > 0.\)

Next, we give the following lemmas.

**Lemma 5.3.1[111]**. Let \(\eta : \mathcal{H} \times 2^{\mathcal{H}} \rightarrow \mathcal{H}\) be a \(\tau\)-Lipschitz continuous mapping, \(K : \mathcal{H} \rightarrow \mathcal{H}\) be a \((r, \eta)\)-strongly monotone mapping and \(M : \mathcal{H} \rightarrow 2^{\mathcal{H}}\) be a \((K, \eta)\)-monotone mapping. Then the operator \((K + \rho M)^{-1}\) is single-valued for \(0 < \rho < \frac{r}{m}\).

Based on Lemma 5.3.1, we can define the generalized resolvent operator \(J_{\rho, K}^M\) associated with \(K\) and \(M\) as follows.

Let \(\eta : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}\) be a \(\tau\)-Lipschitz continuous mapping, \(K : \mathcal{H} \rightarrow \mathcal{H}\) be a \((r, \eta)\)-strongly monotone mapping and \(M : \mathcal{H} \rightarrow 2^{\mathcal{H}}\) be a \((K, \eta)\)-monotone mapping. Then the **generalized resolvent operator** \(J_{\rho, K}^M(x)\) is defined by
\[
J_{\rho, K}^M(x) = (K + \rho M)^{-1}(x), \quad \forall x \in \mathcal{H}.
\]

**Lemma 5.3.2[111]**. Let \(\eta : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}\) be a \(\tau\)-Lipschitz continuous mapping, \(K : \mathcal{H} \rightarrow \mathcal{H}\) be a \((r, \eta)\)-strongly monotone mapping and \(M : \mathcal{H} \times \mathcal{H} \times \Omega \rightarrow 2^{\mathcal{H}}\) be a \((K, \eta)\)-monotone in the first argument. Then for each \((z, \eta) \in \mathcal{H} \times \Omega\), the generalized resolvent operator associated with \(M(., z, \lambda)\) defined by
\[
J_{\rho, K}^{M(., z, \lambda)}(x) = (K + \rho M(., z, \lambda))^{-1}(x), \quad \forall x \in \mathcal{H},
\]

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is Lipschitz continuous with constant \( \frac{\tau}{r-\rho m} \) for \( 0 < \rho < \frac{\tau}{m} \), i.e.,
\[
\|J^{M_{\mu,K_2}(x)}_{\rho,K}(x) - J^{M_{\mu,K_2}(x)}_{\rho,K}(y)\| \leq \frac{\tau}{r-\rho m}\|x-y\|, \quad \forall x,y \in \mathcal{H}.
\] (5.3.4)

The following lemma show that SPGMQVI(5.3.1) is equivalent to a system of relations.

**Lemma 5.3.3**[33]. For each given \((w_1, \omega) \in \mathcal{H}_1 \times \Omega\) and \((w_2, \lambda) \in \mathcal{H}_2 \times \Lambda\), \(a = a(x, \omega) \in A(x, \omega), \ b = b(x, \omega) \in B(x, \omega),\ c = c(y, \lambda) \in C(y, \lambda), \ d = d(y, \lambda) \in D(y, \lambda),\ e = e(y, \lambda) \in E(y, \lambda),\ f = f(x, \omega) \in F(x, \omega),\ g = g(x, \omega) \in G(x, \omega),\ q = q(x, \omega) \in Q(x, \omega),\) and \(s = s(x, \omega) \in S(x, \omega)\) is a solution of SPGMQVI(5.3.1) if and only if
\[
\begin{aligned}
(g - \bar{n})(x) &= J^{M_{\mu,K_1}(x,\omega)}_{\rho,K_1}(K_1(g - \bar{n})(x) + \rho(\omega_1 - N_1(a, b, c, d, \omega))) \\
y &= J^{M_{\mu,K_2}(x,\lambda)}_{\rho,K_2}(\omega_2 + T(x, y, \lambda) - \mu N_2(f, q, \lambda) + K_2(y)),
\end{aligned}
\] (5.3.5)

where \(\rho, \mu > 0\) are constants.

Now, for each given \((w_1, \omega) \in \mathcal{H}_1 \times \Omega\) and \((w_2, \lambda) \in \mathcal{H}_2 \times \Lambda\), define multivalued mappings \(\phi_{p} : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda \to 2^{\mathcal{H}_1}\), \(\psi_{\mu} : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda \to 2^{\mathcal{H}_2}\) and \(\Gamma_{p,\mu} : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \times \Lambda \to 2^{\mathcal{H}_1 \times \mathcal{H}_2}\) by

\[
\begin{aligned}
\phi_{p}(x, y, \omega, \lambda) &= \{u \in \mathcal{H}_1 : \exists a \in A(x, \lambda), b \in B(x, \lambda), c \in C(y, \lambda), d \in D(y, \lambda),
\quad e \in E(y, \lambda), g \in G(x, \lambda) \text{ such that } \} \\
\psi_{\mu}(x, y, \omega, \lambda) &= \{v \in \mathcal{H}_2 : \exists s \in S(x, \lambda), f \in F(x, \lambda), q \in Q(x, \lambda) \text{ such that } \} \\
\Gamma_{p,\mu}(x, y, \omega, \lambda) &= \{(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2 : u \in \phi_{p}(x, y, \omega, \lambda) \text{ and } v \in \psi_{\mu}(x, y, \omega, \lambda)\},
\end{aligned}
\]

where \(\rho, \mu > 0\) are constants.

By Lemma 5.3.3 and the definition of multi-valued mappings \(\phi_{p}, \psi_{\mu}\) and \(\Gamma_{p,\mu}(x, y, \omega, \lambda)\), we have the following result.

**Lemma 5.3.4**[33]. For each given \((w_1, \omega) \in \mathcal{H}_1 \times \Omega\) and \((w_2, \omega) \in \mathcal{H}_2 \times \Lambda\), \((x, y) = (x(\omega), y(\lambda)) \in \mathcal{H}_1 \times \mathcal{H}_2, a = a(x, \omega) \in A(x, \omega), b = b(x, \omega) \in B(x, \omega), c = c(y, \lambda) \in C(y, \lambda),\) and \(d = d(y, \lambda) \in D(y, \lambda)\), we have the following result.
C(y, \lambda), d = d(y, \lambda) \in D(y, \lambda), e = e(y, \lambda) \in E(y, \lambda), f = f(x, \omega) \in F(x, \omega),
g = g(x, \omega) \in G(x, \omega), q = q(x, \omega) \in Q(x, \omega), \text{ and } s = s(x, \omega) \in S(x, \omega) \text{ is a solution of SPGMQVI(5.3.1) if and only if } (x, y) = (x(\omega, \lambda), y(\omega, \lambda)) \text{ is a fixed point of the mapping } G_{\rho,\mu}(x, y, \omega, \lambda).

Remark 5.3.1. By Lemma 5.3.4, we observe that the sensitivity analysis of the solution set of SPGMQVI(5.3.1) with respect to the parametric \((\omega, \lambda) \in \Omega \times \Lambda\) is essentially the sensitivity analysis of the fixed point set of the multi-valued mapping \(G_{\rho,\mu}(x, y, \omega, \lambda)\) with respect to the parametric \((\omega, \lambda) \in \Omega \times \Lambda\).

Now the following theorem shows that the solutions set of SPGMQVI(5.3.1) is non empty and closed.

Theorem 5.3.1[33]. Let \(A, B, F, G, Q, S : \mathcal{H}_1 \times \Omega \to C(\mathcal{H}_1)\) and \(C, D, E : \mathcal{H}_2 \times \Lambda \to C(\mathcal{H}_2)\) be multi-valued mappings. Let \(\bar{n} : \mathcal{H}_1 \times \Omega \to \mathcal{H}_1, N_1 : \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \to \mathcal{H}_1, N_2 : \mathcal{H}_1 \times \mathcal{H}_1 \times \Lambda \to \mathcal{H}_2\) and \(T : \mathcal{H}_1 \times \mathcal{H}_2 \times \Lambda \to \mathcal{H}_2\) be single-valued mappings. Let \(\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \times \Omega \to \mathcal{H}_1, \eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \times \Lambda \to \mathcal{H}_2\) be Lipschitz continuous mappings with Lipschitz constants \(r_1 > 0\) and \(r_2 > 0\), respectively, \(K_1 : \mathcal{H}_1 \to \mathcal{H}_1\) be \((r_1, \eta_1)\)-strongly monotone with \((G - \bar{n})(\mathcal{H}_1 \times \Omega) \cap \text{dom} K_1 \neq \emptyset\), and \(K_2 : \mathcal{H}_2 \to \mathcal{H}_2\) be \((r_2, \eta_2)\)-strongly monotone. Let \(M_1 : \mathcal{H}_1 \times \mathcal{H}_2 \times \Omega \to 2^{\mathcal{H}_1}\) be multi-valued mapping such that for each given \((z_1, \omega) \in \mathcal{H}_2 \times \Omega\), \(M_1(., z_1, \omega)\) is \((K_1, \eta_1)\)-monotone mappings with constant \(m_1\) and \((G - \bar{n})(\mathcal{H}_1 \times \Omega) \cap \text{dom} M_1(., z_1, \omega) \neq \emptyset\), and \(M_2 : \mathcal{H}_2 \times \mathcal{H}_1 \times \Lambda \to 2^{\mathcal{H}_2}\) be multi-valued mapping such that for each given \((z_2, \lambda) \in \mathcal{H}_2 \times \Lambda\), \(M_2(., z_2, \lambda)\) is \((K_2, \eta_2)\)-monotone mapping with constant \(m_2\). Suppose the following conditions are satisfied:

(i) \(A, B, C, D, E, F, G, Q, S, \bar{n}, K_1, K_2\) are all Lipschitz continuous in the first argument with Lipschitz constants \(l_A, l_B, l_C, l_D, l_E, l_F, l_G, l_Q, l_S, l_m, l_{K_1}, l_{K_2} > 0\), respectively;

(ii) \(G\) is \(\alpha_G\)-strongly monotone in the first argument;

(iii) \(N_1(., ., ., .)\) is \((l_1, l_2)\)-mixed Lipschitz continuous in the first and second arguments, \((l_3, l_4)\)-mixed Lipschitz continuous in the third and fourth arguments;

(iv) \(N_2(., ., .)\) is \(k_1\)-Lipschitz continuous in the first argument, and \(k_2\)-Lipschitz continuous in the second argument.
(v) $T$ is $(\ell_1, \ell_2)$-mixed Lipschitz continuous in the first and second arguments.

Further assume that there exist $\sigma, \delta > 0$ satisfying

$$
\|J_{\rho, K_1}^{M_1(e, \omega)}(x) - J_{\rho, K_1}^{M_1(e, \omega)}(x)\| \leq \sigma \|e - \bar{e}\|, \quad \forall (e, \bar{e}, \omega) \in \mathcal{H}_2 \times \mathcal{H}_2 \times \Omega, \quad x \in \mathcal{H}_1. \quad (5.3.6)
$$

$$
\|J_{\rho, K}^{M_2(s, \lambda)}(y) - J_{\rho, K}^{M_2(s, \lambda)}(y)\| \leq \delta \|s - \bar{s}\|, \quad \forall (s, \bar{s}, \lambda) \in \mathcal{H}_1 \times \mathcal{H}_1 \times \Lambda, \quad y \in \mathcal{H}_2. \quad (5.3.7)
$$

If

$$
\begin{align*}
\theta_1 &= \sqrt{1 - 2\alpha G + \ell_2^2 + \ell_m}; \\
\Gamma_1 &= l_1l_A + l_2l_B, \quad \Gamma_2 = l_3l_C + l_4l_D; \\
v_2 &= \frac{\tau}{r_2 - \mu n_2}(\mu(k_1l_F + k_2l_Q) + \ell_1) + \delta l_S, \quad t_2 = \frac{\tau}{r_2 - \mu n_2}(l_2 + \ell_2); \\
v_1 &= \theta_1 + \frac{\tau}{r_1 - \mu n_1}(l_1 + l_m) + \rho l_1, \quad t_1 = \frac{\rho}{r_1 - \mu n_1}\Gamma_2 + \sigma l_B; \\
v_2 + v_1 < 1, \quad t_2 + t_1 < 1;
\end{align*}
$$

Then for each $(w_1, \omega) \in \mathcal{H}_1 \times \Omega$ and $(w_2, \lambda) \in \mathcal{H}_2 \times \Lambda$, the solution set $\text{sol} (\omega, \lambda)$ of SPGMQVI(5.3.1) is nonempty and closed in $\mathcal{H}_1 \times \mathcal{H}_2$.

Finally, we have the theorem which gives the continuity of the solution set of SPGMQVI(5.3.1).

**Theorem 5.3.2[33].** Under the hypotheses of Theorem 5.3.1, further assume that

(i) $A, B, C, D, E, F, G, Q, S, \bar{n}$, are all Lipschitz continuous in their second arguments with Lipschitz constants $L_A, L_B, L_C, L_D, L_E, L_F, L_G, L_Q, L_S, L_m$, respectively;

(ii) $N_1$ and $N_2$ are all Lipschitz continuous in the fifth and third argument with Lipschitz constants $L_5$ and $L_3$, respectively;

(iii) $T$ is Lipschitz continuous in the third argument with Lipschitz constant $l_3$;

(iv) There exist $\xi_1, \xi_2 > 0$, such that for any $(e, s) \in \mathcal{H}_2 \times \mathcal{H}_1$, $(\omega, \lambda) \in \Omega \times \Lambda$, such that

$$
\|J_{\rho, K_1}^{M_1(e, \omega)}(x) - J_{\rho, K_1}^{M_1(e, \omega)}(x)\| \leq \xi_1 \|\omega - \bar{\omega}\|, \quad (5.3.9)
$$

$$
\|J_{\rho, K_2}^{M_2(s, \lambda)}(x) - J_{\rho, K_2}^{M_2(s, \lambda)}(x)\| \leq \xi_2 \|\lambda - \bar{\lambda}\|. \quad (5.3.10)
$$

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Then the solution set sol(ω, λ) of SPGMQVI (5.3.1) is Lipschitz continuous with respect to the parameter (ω, λ) ∈ Ω × Λ.

**Remark 5.3.2.** If each mappings in Theorem 5.3.2 is assumed to be continuous with respect to the parameter (ω, λ) ∈ Ω × Λ. By using similar arguments as above, we show that the solution set sol(ω, λ) of SPGMQVI(5.3.1) is continuous with respect to the parameter (ω, λ) ∈ Ω × Λ.

### 5.4 SYSTEM OF PARAMETRIC MIXED QUASI-VARIATIONAL INCLUSIONS

We consider a system of parametric mixed quasi-variational inclusions with H-accretive mappings in q-uniformly smooth Banach space. Let Ω and Λ be two nonempty open subsets of $E$ in which the parameters ω and λ take values; $A, B : E × Ω → E$ and $C, D : E × Λ → E$ be nonlinear single-valued mappings. Let $M : E × Ω → 2^E$ and $N : E × Λ → 2^E$ be multi-valued mappings such that for each given $(ω, λ) ∈ Ω × Λ$, $M(., ω)$ and $N(., λ) : E → 2^E$ are H-accretive mappings. For each fixed $(ω, λ) ∈ Ω × Λ$, the system of parametric mixed quasi-variational inclusions with H-accretive mappings (in short, SPMQVI) is: Find $(x, y) ∈ E × E$ such that

\[
0 ∈ H(x) - y + p(A(y, ω) + B(y, ω)) + pM(x, ω),
\]

\[
0 ∈ H(y) - x + j(C(x, ω) + D(x, λ)) + jN(y, λ),
\]

where $p > 0$ and $γ > 0$ are constants.

Now, we give the following definitions.

**Definition 5.4.1.** A multi-valued mapping $M : E → 2^E$ is said to be

(i) **accretive** if

\[
⟨u - v, J_q(x - y)⟩ ≥ 0, \quad ∀x, y ∈ E, u ∈ M(x), v ∈ M(y);
\]

(ii) **H-accretive** if $M$ is accretive and $(H + λM)(E) = E$ for all $λ > 0$. 

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If $H = I$, the identity mapping, then Definition 5.4.1 reduces to the definition of $m$-accretive mapping.

**Definition 5.4.2.** A mapping $A : E \times \Omega \to E$ is said to be

(i) **$\delta$-strongly accretive** with respect to the first argument, $\delta \in (0, 1)$, if

$$\langle A(x, \omega) - A(y, \omega), J_q(x - y) \rangle \geq \delta \|x - y\|^q, \quad \forall x, y \in E;$$

(ii) **$\lambda_A$-Lipschitz continuous** with respect to the first argument if there exists a constant $\lambda_A > 0$ such that

$$\|A(x, \omega) - A(y, \omega)\| \leq \lambda_A \|x - y\|, \quad \forall (x, y, \omega) \in E \times E \times \Omega.$$

If $H : E \to E$ be a strictly accretive mapping and $M : E \to 2^E$ be an $H$-accretive mapping, then for a constant $\lambda > 0$, the resolvent operator associated with $H$ and $M$ is defined by

$$R_{H, \lambda}(x) = (H + \lambda M)^{-1}(x), \quad \forall x \in E.$$

It is well known that $R_{H, \lambda}$ is a single-valued mapping [3].

**Remark 5.4.1.** Since $M$ is an $H$-accretive mapping with respect to the first argument, for any fixed $\omega \in \Omega$, we define

$$R_{H, (\cdot, \cdot), \lambda}(x) = (H + \lambda M(\cdot, \omega))^{-1}(x), \quad \forall x \in D(M),$$

which is called the **parametric resolvent operator** associated with $H$ and $M(\cdot, \omega)$.

Next, we give the following lemmas.

**Lemma 5.4.1[43].** Let $H : E \to E$ be a strongly accretive mapping and $M : E \to 2^E$ be an $H$-accretive mapping. Then the resolvent operator $R_{H, \lambda}^H : E \to E$ is Lipschitz continuous with constant $\frac{1}{s}$, i.e.,

$$\|R_{H, \lambda}^H(x) - R_{H, \lambda}^H(y)\| \leq \frac{1}{s} \|x - y\|, \quad \forall x, y \in E.$$

The following lemma gives the equivalence between SPMQVI(5.4.1) and a fixed-point problem.
Lemma 5.4.2[61]. For all fixed \((\omega, \lambda) \in \Omega \times \Lambda\), \((\tilde{x}(\omega, \lambda), \tilde{y}(\omega, \lambda))\) is a solution of SPMQVI(5.4.1) if and only if for some given \(\rho, \gamma > 0\), the mapping \(F : E \times \Omega \times \Lambda \to E\) defined by

\[
F(x, \omega, \lambda) = R^H_{(\omega, \lambda)}(R^H_N(\omega, \lambda)\gamma(x - \gamma(C + D)(x, \lambda)) - \rho(A + B)(R^H_N(\omega, \lambda)\gamma)(x - \gamma(C + D)(x, \lambda), \omega)
\]

has a fixed point \(\tilde{x}\).

The following theorem shows the existence and Lipschitz continuity of solution of SPMQVI(5.4.1).

Theorem 5.4.1[61]. Let \(H : E \to E\), \(A, B : E \times \Omega \to E\), \(C, D : E \times \Lambda \to E\) be five mappings, and \(M : E \times \Omega \to 2^E\), \(N : E \times \Omega \to 2^E\) be two multi-valued mappings satisfying the following conditions:

(i) \(H\) is \(s\)-strongly accretive;
(ii) \(A\) is \(\lambda_A\)-Lipschitz continuous with respect to the first argument;
(iii) \(B\) is \(\delta\)-strongly accretive and \(\lambda_B\)-Lipschitz continuous with respect to the first argument;
(iv) \(C\) is \(\lambda_C\)-Lipschitz continuous with respect to the first argument;
(v) \(D\) is \(\alpha\)-strongly accretive and \(\lambda_D\)-Lipschitz continuous with respect to the first argument;
(vi) \(M\) and \(N\) are \(H\)-accretive with respect to the first argument.

Suppose that there exist \(\rho > 0\) and \(\gamma > 0\) such that

\[
\begin{align*}
1 - \alpha q \gamma + c_q \gamma^q \lambda_D^q < (s - \gamma \lambda_C)^q, \\
1 - q \rho \delta + c_q \rho^q \lambda^q < (s - \rho \lambda_A)^q.
\end{align*}
\]

(5.4.3)

Then

(i) the mapping \(F : E \times \Omega \times \Lambda \to E\) defined by (5.4.2) is a uniform \(\theta\)-contractive mapping with respect to \((\omega, \lambda) \in \Omega \times \Lambda\).
(ii) for each \((\omega, \lambda) \in \Omega \times \Lambda\), SPMQVI(5.4.1) has a nonempty solution set \(S(\omega, \lambda)\) and \(S(\omega, \lambda)\) is a closed subset of \(E\).

**Theorem 5.4.2**[61]. Under the hypothesis of Theorem 5.4.1, further assume that for any \(x, y \in E\), the mappings \(\omega \mapsto A(x, \omega)\), \(\omega \mapsto B(x, \omega)\), \(\lambda \mapsto C(y, \lambda)\) and \(\lambda \mapsto D(y, \lambda)\) are Lipschitz continuous with constants \(l_A, l_B, l_C, l_D\), respectively. Suppose that for any \((t, \omega, \omega) \in E \times \Omega \times \Omega\) and \((z, \lambda, \lambda) \in E \times \Lambda \times \Lambda\)

\[
\begin{align*}
\| R_{M(\omega,\omega),\rho}^H(t) - R_{M(\omega,\omega),\rho}^H(t) \| & \leq \mu \| \omega - \omega \|, \\
\| R_{N(\lambda,\lambda),\gamma}^H(z) - R_{N(\lambda,\lambda),\gamma}^H(z) \| & \leq \tau \| \lambda - \lambda \|, 
\end{align*}
\]

(5.4.4)

where \(\mu > 0\) and \(\tau > 0\) are two constants. Then the solution \((x(\omega, \lambda), y(\omega, \lambda))\) of SPMQVI(5.4.1) is Lipschitz continuous.