CHAPTER 4

SECOND ORDER RUNGE-KUTTA METHOD WITH HARMONIC MEAN FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

4.1 INTRODUCTION

In this chapter, the numerical solution of FDE given by

\[ y' = f(t, y(t)), \quad t \in [a, b], \quad y(a) = y_0 \]  \hspace{1cm} (4.1)

is discussed by Second order Runge–Kutta method in which new parameters are taken in \( k_i \)'s and Harmonic mean of \( k_i \)'s is used in the main formula in order to increase the order of accuracy of the solution. The efficiency of the method is illustrated by solving some linear and non-linear fuzzy differential equations.

4.2 SECOND ORDER RUNGE-KUTTA METHOD WITH HARMONIC MEAN

Consider the Runge-Kutta method with Harmonic mean of \( k_i \)'s

\[ y(t_{n+1}) = y(t_n) + \frac{2k_1k_2}{k_1 + k_2} \]  \hspace{1cm} (4.2)

Where,

\[ k_1 = hf(t_n, y(t_n)) \]  \hspace{1cm} (4.3)
\[ k_2 = hf\{t_n + c_{21}h, y(t_n) + a_{21}k_1 + h a_{22} f_y(t_n, y(t_n))k_1 + h f(t_n, y(t_n))\}\] \hspace{1cm} (4.4)

Utilizing the Taylor’s series expansion techniques and Binomial algorithm in Equation (4.2), the parameters are obtained as follows:

\[ c_{21} = a_{21}, a_{22} = \frac{1}{3}c_{21}a_{22} = \frac{1}{3} \]

Solving this system of equations,

\[ c_{21} = a_{21} = 1 \text{ and } a_{22} = \frac{1}{3}. \]

Runge-Kutta method of order two is given by:

\[ y(t_{n+1}) = y(t_n) + \frac{2k_1k_2}{k_1 + k_2} \] \hspace{1cm} (4.5)

Where

\[ k_1 = hf(t_n, y(t_n)) \] \hspace{1cm} (4.6)

\[ k_2 = hf\{t_n + h, y(t_n) + k_1 + h \frac{1}{2} (f_y(t_n, y(t_n))k_1 + h f(t_n, y(t_n)))\}\] \hspace{1cm} (4.7)

**Theorem 4.1.** Let \( f(t, y) \) belong to \( C^2[a, b] \) and its partial derivatives be bounded and let us assume that there exist positive constants \( L, M \), such that

\[ |f(t, y)| < M, \left| \frac{\partial^{i+j}f}{\partial t^i \partial y^j} \right| < \frac{L^i+j}{M^{i-1}}, \quad i + j \leq m, \]

then in the second order Runge-Kutta method given by Equations (4.5) to (4.7), \( y(t_{i+1}) - y_{i+1} \approx O(h^3) \).
4.3 SECOND ORDER RUNGE–KUTTA METHOD WITH HARMONIC MEAN FOR FIVPS

Case (i)

Assume that \( y'(t; r) \) given in Equation (4.1) is (i) differentiable.

Let the exact solution of the FIVP given by the Equation (4.1) \( [Y(t)]_r = \left[ Y(t; r), \bar{Y}(t; r) \right] \) be approximated by some \( [y(t)]_r = \left[ y(t; r), \bar{y}(t; r) \right] \).

From Equations (4.5) to (4.7), it is defined that

\[
\begin{align*}
\bar{y}(t_{n+1}; r) &= \bar{y}(t_n; r) + 2 \frac{k_1(t_n, y(t_n; r)) \, k_2(t_n, y(t_n; r))}{k_1(t_n, \bar{y}(t_n; r)) + k_2(t_n, \bar{y}(t_n; r))} \\
\bar{Y}(t_{n+1}; r) &= \bar{Y}(t_n; r) + 2 \frac{k_1(t_n, y(t_n; r)) \, k_2(t_n, y(t_n; r))}{k_1(t_n, \bar{Y}(t_n; r)) + k_2(t_n, \bar{Y}(t_n; r))} \\
k_1(t, y(t; r)) &= \min \{ h, f(t, u) \mid u \in \left[ y(t, r), \bar{y}(t, r) \right] \} \\
k_2(t, y(t; r)) &= \max \{ h, f(t, u) \mid u \in \left[ y(t, r), \bar{y}(t, r) \right] \} \\
\bar{k}_1(t, y(t; r)) &= \min \{ h, f(t, u) \mid u \in \left[ \bar{y}_1(t, y(t; r)), \bar{Z}_1(t, y(t; r)) \right] \} \\
\bar{k}_2(t, y(t; r)) &= \max \{ h, f(t, u) \mid u \in \left[ \bar{y}_1(t, y(t; r)), \bar{Z}_1(t, y(t; r)) \right] \}
\end{align*}
\]

Where

\[
\begin{align*}
\bar{z}_1(t, y(t; r)) &= \bar{y}(t, r) + \bar{k}_1(t, y(t; r)) + \frac{1}{3}(a + b) \\
\bar{Z}_1(t, y(t; r)) &= \bar{y}(t, r) + k_1(t, y(t; r)) + \frac{1}{3}(|a + b|),
\end{align*}
\]
\[ a = \min \{ h \cdot f_j(t,u) : v \backslash u \in [y(t,r), \overline{y}(t,r)] & v \in \left[ k_1(t,y(t;r)), k_1(t,y(t;r)) \right] \} \]

\[ \bar{a} = \max \{ h \cdot f_j(t,u) : v \backslash u \in [y(t,r), \overline{y}(t,r)] & v \in \left[ k_1(t,y(t;r)), k_1(t,y(t;r)) \right] \} \]

\[ b = \min \{ h \cdot f_i(t,u) \backslash u \in [y(t,r), \overline{y}(t,r)] \} \]

\[ \bar{b} = \max \{ h \cdot f_i(t,u) \backslash u \in [y(t,r), \overline{y}(t,r)] \} \]

Define,

\[ F[t,y(t;r)] = 2 \frac{k_1(t,y(t;r))}{k_1(t,y(t;r)) + k_2(t,y(t;r))} \]

\[ G[t,y(t;r)] = 2 \frac{k_2(t,y(t;r))}{k_1(t,y(t;r)) + k_2(t,y(t;r))} \] (4.14)

\[ G[t,y(t;r)] = 2 \frac{k_2(t,y(t;r))}{k_1(t,y(t;r)) + k_2(t,y(t;r))} \] (4.15)

The exact and approximate solutions at \( t_n, 0 \leq n \leq N \) are denoted by

\[ [y(t_n)]_r = [y(t_n; r), \overline{y}(t_n; r)] \] and \[ [y(t_n)]_r = [y(t_n; r), \overline{y}(t_n; r)] \], respectively.

The solution is calculated by grid points at

\[ a = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{n+1} - t_n. \]

Therefore, we have

\[ Y(t_{n+1}; r) = Y(t_n; r) + F[t_n, Y(t_n; r)] \] (4.16)

\[ \overline{Y}(t_{n+1}; r) = \overline{Y}(t_n; r) + G[t_n, Y(t_n; r)] \] (4.17)
\[ y(t_{n+1}; r) = y(t_n; r) + F[t_n, y(t_n, r)] \]
(4.18)

\[ \overline{y}(t_{n+1}; r) = \overline{y}(t_n; r) + G[t_n, y(t_n, r)] \]
(4.19)

The following lemmas will be applied to show the convergences of these approximates i.e., \( \lim_{n \to 0} y(t, r) = Y(t, r) \) and \( \lim_{n \to 0} \overline{y}(t, r) = \overline{Y}(t, r) \).

**Lemma 4.2.1** Let a sequence of numbers \( \{W_n\}_{n=0}^N \) satisfy \( |W_{n+1}| \leq A|W_n| + B, 0 \leq n \leq N - 1 \), for some given positive constants A and B, then \( |W_n| \leq A^n|W_0| + B \frac{A^n-1}{A-1}, 0 \leq n \leq N - 1 \).

The proof of Lemma (4.2.1) follows Lemma 1 of Ming Ma et al (1999).

**Lemma 4.2.2** Let the sequence of numbers \( \{W_n\}_{n=0}^N, \{V_n\}_{n=0}^N \) satisfy \( |W_{n+1}| \leq |W_n| + Amax\{|W_n|, |V_n|\} + B, \) \( |V_{n+1}| \leq |V_n| + Amax\{|W_n|, |V_n|\} + B \), for some given positive constants A and B, and denote \( U_n = |W_n| + |V_n|, 0 \leq n \leq N \). Then \( U_n \leq A^nU_0 + B \frac{A^n-1}{A-1} \), \( 0 \leq n \leq N \), where \( A = 1 + 2A \) and \( B = 2B \).

The proof of Lemma (4.2.2) follows Lemma 2 of Ming Ma et al (1999).

Let \( F(t, u, v) \) and \( G(t, u, v) \) be obtained by substituting \( [y(t)]_r = [u, v] \) in the Equations (4.14) & (4.15),

\[ F[t, u, v] = 2 \frac{K_1[t, u, v]K_2[t, u, v]}{K_1[t, u, v] + K_2[t, u, v]}, \quad G[t, u, v] = 2 \frac{K_1[t, u, v]K_2[t, u, v]}{K_1[t, u, v] + K_2[t, u, v]} \]
The domain where \( F \) and \( G \) are defined is therefore

\[
K = \{(t, u, v) \mid 0 \leq t \leq T, -\infty < v < \infty, -\infty < u \leq v\}.
\]

**Theorem 4.2.1**: let \( F(t, u, v) \) and \( G(t, u, v) \) belong to \( C^2(K) \) and let the partial derivatives of \( F \) and \( G \) be bounded over \( K \). Then, for arbitrary fixed \( r, 0 \leq r \leq 1 \), the approximate solutions given in Equations (4.18) & (4.19) converge to the exact solutions \( \underline{Y}(t; r) \) and \( \overline{Y}(t; r) \) uniformly in \( t \).

The proof of Theorem (4.2.1) follows Theorem 1 of Ming Ma et al (1999).

**Case (i)**

Assume that \( y'(t; r) \) given in Equation (4.1) is (ii)-differentiable.

Let the exact solution of the FIVP given by the Equation (4.1) \[ Y(t) \] be approximated by some \[ \?U (t) \] and \( \overline{Y}(t) \).

From the Equations (4.5) to (4.7), it is defined that

\[
y(t_{n+1}; r) = y(t_n; r) + 2 \left( \frac{k_1(t_n,y(t_n;r)) k_2(t_n,y(t_n;r))}{k_1(t_n,y(t_n;r)) + k_2(t_n,y(t_n;r))} \right) \tag{4.20}
\]

\[
\overline{y}(t_{n+1}; r) = \?U (t_n; r) + 2 \left( \frac{k_1(t_n,y(t_n;r)) k_2(t_n,y(t_n;r))}{k_1(t_n,y(t_n;r)) + k_2(t_n,y(t_n;r))} \right) \tag{4.21}
\]

Where \( k_1, k_2, \overline{k}_1, \overline{k}_2 \) and \( \overline{k}_2 \) are given by Equations (4.10) to (4.13)

The exact solution of FIVP given in the Equation (4.1) is given by

\[
Y(t_{n+1}; r) = Y(t_n; r) + G[t_n; Y(t_n, r)] \tag{4.22}
\]
\[
\overline{Y}(t_{n+1}; r) = \overline{Y}(t_n; r) + F[t_n, Y(t_n; r)]
\] (4.23)

And the approximate solution is given by

\[
\underbar{y}(t_{n+1}; r) = \underbar{y}(t_n; r) + G[t_n, y(t_n; r)]
\] (4.24)

\[
\overline{y}(t_{n+1}; r) = \overline{y}(t_n; r) + F[t_n, y(t_n; r)]
\] (4.25)

As in case(i), it can be shown that the approximate solutions \( \underbar{y}(t_{n+1}; r) \) and \( \overline{y}(t_{n+1}; r) \) approach the exact solutions \( \overline{Y}(t; r) \) and \( \overline{Y}(t; r) \) respectively.

### 4.4 NUMERICAL EXAMPLES

The method discussed in this chapter is illustrated using numerical examples. Manual computation is tedious and hence Java version 1.5 is used for computation and the data are imported into Matlab and represented graphically.

**Example 4.3.1** Second order Runge-Kutta method with harmonic mean (RKH2) is illustrated in this example.

Consider the fuzzy initial value problem,

\[
y'(t) = -y(t) + t + 1, t \in [0,1],
\]

\[
y(0) = (0.75 + 0.25 r, 1.125 - 0.125 r), 0 < r \leq 1
\] (4.26)

Case (1): The exact solution of (4.26) under (i)-differentiability is given by

\[
Y(t; r) = [t + (0.985 + 0.015 r)e^{-t} - (1 - r)0.025e^t,
\]

\[
t + (0.985 + 0.015 r)e^{-t} + (1 - r)0.025e^t], 0 < r \leq 1
\]
Figure 4.1 Errors by RKH2, CRK2 and Euler methods

The errors at $t=1$ with $h=0.1$, between exact and Euler’s approximation (Euler), between exact and CRK2 method and between Exact and RKH2 method are plotted in Figure 4.1. The Figure 4.1 shows that the proposed method gives better result than the Euler methods.

Figure 4.2 Errors for various $h$ values

The errors at $t=0.1$ for different $h$ values are plotted in Figure 4.2. It is evident that the approximations are better when the step size is small.
Case 2: The exact solution of (4.26) under (ii)-differentiabiltiy is given by

\[ Y(t; r) = [t + (0.75 + 0.25r)e^{-t}, t + (1.125 - 0.125r)e^{-t}] . \]

The errors between exact and Euler’s approximation (Euler), between exact and CRK2 method and between Exact and NRK2 method at \( t=1 \) with \( h=0.1 \) are plotted in Figure 4.3. It is seen from the Figure 4.3 that, there is a high fluctuation in the error by RKH2 method. There is a gradual reduction in the error as the r-levels increases in CRK2 and gradual increase in the error by Euler method. The standard deviation of the error by RKH2 is 0.00081631, that of CRK2 is 0.00012716 and of Euler method is 0.000021292. Therefore, Euler method is consistent for this problem.

![Figure 4.3 Errors in (ii)-differentiabiltiy](image)

**Example 4.3.2**

Consider the fuzzy initial value problem

\[ y''(t) = cy^2(t) + d, y(0) = 0 \text{ where } c \text{ and } d \text{ are triangular fuzzy numbers given by } c = [0.5 + 0.5r, 1.5 - 0.5r] \text{ and } d = [0.75 + 0.25r, 1.125 - 0.25r], 0 \leq r \leq 1 \]  

(4.27)

The exact solution of (4.27) is given by
\[ Y(t; r) = \left[ l_1(r) \tan(w_1(r)t), l_2(r) \tan(w_2(r)t) \right], \text{where} \]

\[ l_1(r) = \frac{d_1(r)}{\sqrt{c_1(r)}}, \quad l_2(r) = \frac{d_2(r)}{\sqrt{c_2(r)}}, \quad w_1(r) = \sqrt{c_1(r)d_1(r)}, \quad w_2(r) = \sqrt{c_2(r)d_2(r)}, \quad [c]_r = [c_1(r), c_2(r)] \quad \text{and} \quad [d]_r = [d_1(r), d_2(r)]. \]

Figure 4.4 Errors by RKH2 and CRK2

Errors by RKH2 and CRK2 at different r-levels at=1 with h=0.001 are plotted in Figure 4.4. It is seen from the Figure 4.4 that error by CRK2 is smaller up to levels 0.4 and is same at level 0.6 and is higher at levels 0.8 and 1 when compared with the error by RKH2. The standard deviation of errors by NRK2 is 0.0146 and that of CRK2 is 0.0040. For this problem CRK2 is better than RKH2.

4.5 CONCLUSION

In this chapter, the numerical solution of FDEs have been discussed by Second order Runge–Kutta method in which new parameters are taken in \( k_i \)'s and Harmonic mean of \( k_i \)'s is used in the main formula in order to increase the order of accuracy of the solution. The proposed formula gives better approximation than the standard Euler method. The approximations by the proposed method are better whenever the step size \( h \) is small. From the examples, it is inferred that under both (i) and (ii)- differentiability's the approximation by CRK2 is better. Also in the numerical example 4.3.2, it is seen that CRK2 is better than RKH2.