Chapter 4

Gluon Radiation off gluons

So far, we have been engaged into the general discussion on radiation and the corresponding energy loss. We have tried to relate the facts that we can either calculate the Poynting’s vector or can find out the radiation distribution in order to estimate the power dissipation. The latter is the approach taken in our calculations (and in many other pQCD calculations) to find out the radiation distribution.

The present chapter will discuss about the non-eikonal correction to the Gunion-Bertsch radiation distribution formula obtained from the $gg \to ggg$ scattering. This process is nothing but the radiative energy loss of high energy gluons in gluonic plasma.

First of all, is purely gluonic plasma an idealization? It may be opportune to discuss the point here. We have already discussed that a charge, when Lorentz boosted, becomes dressed by Weizsäcker-Williams (WW) virtual quanta. After the head-on encounter of two nuclei in heavy-ion collision, the virtual quanta are excited and after some time, they become real quarks and gluons. Now, the WW virtual quanta are also known as ‘sea’ partons\(^1\) and from the variation of the distribution of sea-partons with the momentum fraction $x$ of the nucleon they carry, we

\(^1\)and the Lorentz boosted charge is nothing but the valence parton, which we termed as the *leading parton* in section 2.3

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see that the sea quanta distribution becomes larger and larger as $x \to 0$. In addition to that, gluon distribution dominates over the sea quarks at very small $x$ (see Fig. 4.1). So, the gluons outnumber the quarks for low-$x$ (equivalently, high energy transfer) region which is the case for, at least, the heavy-ion collision experiments at Large Hadron Collider (LHC) at CERN and Relativistic Heavy-Ion Collider (RHIC) at BNL. Hence, we may very well treat the matter produced in the central rapidity region as dominantly gluonic.

It will be interesting to take the help of Eq. 2.8 which might be able to provide an alternative explanation to the increase of parton distribution with energy. We observe that with increasing energy (equivalently, $\gamma$), the upper limit of energy in frequency spectrum of virtual particles associated with the charge ($\omega_{\text{max}} \sim 1/\Delta t$) increases. So, the number of associated particles increase and hence the density of them goes up with increasing energy. Also, there are possibilities of gluons fluctuating into quark-antiquark pairs which contribute to the sea-quark distribution coupled with three-gluon as well as four-gluon vertices. But due to gluon-gluon color factor, the sea-gluons are expected to dominate.

After all these, why $gg \to ggg$? First of all, it is more important process than other similar (i.e. $2\to3$) processes involving one or more quarks. Second of all, inelastic, number non-conserving processes help maintain the chemical equilibrium of the system. In the present discussion we will employ the 2-gluon→3-gluon process for revisiting the Gunion-Bertsch (GB) distribution formula widely used in transport models [1, 2, 3, 4, 5, 6] keeping in mind the recent trend of the similar efforts observed in Refs. [7, 8, 9].
4.1 The radiation spectrum off a gluon in gluonic plasma

The square of the invariant amplitude for the process $g(k_1) + g(k_2) \rightarrow g(k_3) + g(k_4) + g(k_5)$ can be written elegantly as [11]:

$$|M_{gg \rightarrow ggg}|^2 = \frac{1}{2} g^6 \frac{N_c^3}{N_c^2 - 1} \frac{N}{D} \times [(12345) + (12354) + (12435) + (12453) + (12534)$$
$$+ (12543) + (13245) + (13254) + (13425) + (13524) + (14235)$$
$$+ (14325)] \quad (4.1)$$
where

\[ N = (k_1.k_2)^4 + (k_1.k_3)^4 + (k_1.k_4)^4 + (k_2.k_4)^4 + (k_2.k_5)^4 + (k_3.k_4)^4 + (k_4.k_5)^4 \]

\[ + (k_3.k_4)^4 + (k_3.k_5)^4 + (k_4.k_5)^4 \quad (4.2) \]

\[ D = (k_1.k_2)(k_1.k_3)(k_1.k_4)(k_1.k_5)(k_2.k_3)(k_2.k_4)(k_2.k_5)(k_3.k_4)(k_3.k_5)(k_4.k_5) \quad (4.3) \]

and

\[ (ijklm) = (k_i.k_j)(k_j.k_k)(k_k.k_l)(k_l.k_m)(k_m.k_i) \quad (4.4) \]

\( N_c (= 3) \) is the number of colors, \( g = \sqrt{4\pi \alpha_s} \) is the color charge, and \( \alpha_s \) is the strong coupling.

The quantity, \( |M_{gg \to ggg}|^2 \) after simplifying up to \( \mathcal{O}(t^3/s^3) \) and \( \mathcal{O}(1/k_{\perp}^2) \) can be written as[12] (calculation given in Appendix A also):

\[
\frac{|M_{gg \to ggg}|^2}{|M_{gg \to gg}|^2} = 12g^2 \frac{1}{k_{\perp}^2} \left[ \left( 1 + \frac{t}{2s} + \frac{5t^2}{2s^2} - \frac{t^3}{s^3} \right) - \left( \frac{3}{2s\sqrt{s}} + \frac{4t}{s\sqrt{s}} - \frac{3t^2}{2s^2\sqrt{s}} \right) k_{\perp} \right] \left( \frac{5}{2s} + \frac{t}{2s^2} + \frac{5t^2}{s^3} \right) k_{\perp}^2 \quad (4.5) \]

where \( |M_{gg \to ggg}|^2 = (9/2)g^4s^2/t^2 \), Mandelstam variables: \( s = (k_1 + k_2)^2 \), \( t = (k_1 - k_3)^2 \), \( u = (k_1 - k_4)^2 \), \( k_{\perp} \) is the magnitude of the transverse momentum of the radiated gluon.

The Mandelstam variable \( t \approx -q_{\perp}^2 \) and \( q_{\perp}^2 \), the square of the transverse momentum transfer, is replaced by the corresponding average value:

\[
\langle q_{\perp}^2 \rangle = \frac{1}{\sigma_{el}} \int_{m_B^2}^{s} dq_{\perp}^2 \frac{d\sigma_{el}}{dq_{\perp}^2} q_{\perp}^2 \quad (4.6)
\]
where

\[
\sigma_{el} = \int_{m_D^2}^{s} dq_{\perp}^2 \frac{d\sigma_{el}}{dq_{\perp}^2}
\]  

\[(4.7)\]

For dominant small-angle scattering \((t \to 0)\),

\[
\frac{d\sigma_{el}}{dq_{\perp}^2} = C_i \frac{2\pi\alpha_s^2}{q_{\perp}^4}
\]  

\[(4.8)\]

\(C_i\) is 9/4, 1 and 4/9 for gg, qg, and qq scattering. \(\langle q_{\perp}^2 \rangle\) is then obtained as,

\[
\langle q_{\perp}^2 \rangle = \frac{sm_D^2}{s - 4m_D^2} \ln \left( \frac{s}{4m_D^2} \right)
\]  

\[(4.9)\]

\(m_D = \sqrt{2\pi\alpha_s(T)(C_A + \frac{N_F}{2})/3\ T}\), is the thermal mass of the gluon [13], \(N_F\) is the number of flavors contributing in the gluon self-energy loop, \(C_A = 3\) is the Casimir invariant for the SU(3) adjoint representation. With the replacement as in Eq. 4.9, the emission distribution can be obtained by the following steps [14]:

\[
\int dn_g = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2) \left| \frac{M_{gg \to ggg}}{M_{gg \to gk}} \right|^2 \\
= \int \frac{d^3k}{(2\pi)^3} dk_0 \delta(k_0^2 - |\vec{k}|^2) \left| \frac{M_{gg \to ggg}}{M_{gg \to gk}} \right|^2 \\
= \int \frac{d^3k}{(2\pi)^3} dk_0 \delta(k_0^2 - k_\perp^2 - k_z^2) \left| \frac{M_{gg \to ggg}}{M_{gg \to gk}} \right|^2
\]  

\[(4.10)\]

Where \(k_0(k_z)\) is the energy(longitudinal momentum) of the emitted gluon. But if we parametrize \(k_0 = k_\perp \cosh \eta\) and \(k_z = k_\perp \sinh \eta\) in terms of the gluon rapidity \(\eta\), then \(k_z = 0\) at \(\eta = 0\). So, we can write Eq. 4.10 as,
\[
\int dn_g = \frac{12g^2}{2(2\pi)^3} \int \frac{1}{k_{\perp}^2} d^2 k_{\perp} d\eta D^{(1)}
\]

\[
\frac{dn_g}{d^2 k_{\perp} d\eta} = \frac{C_A \alpha_s}{\pi^2} \frac{1}{k_{\perp}^2} D^{(1)}
\]

\[
\approx \frac{C_A \alpha_s}{\pi^2} \frac{q_{\perp}^2}{k_{\perp}^2 (k_{\perp} - q_{\perp})^2} D^{(1)}
\]

\[
= \left[ \frac{dn_g}{d^2 k_{\perp} d\eta} \right]_{GB} D^{(1)}
\]

in the limit, transverse momentum transfer \( q_{\perp} >> k_{\perp} \) (see Eq. 3.8 and the discussions thereafter). In the light of the discussion of Sec. 3.2.2 about the eikonal approximation due to scattering where \( q_{\perp} << \sqrt{s} \), we identify Eq. 4.5 as the non-eikonal correction to the noted GB radiation distribution. Though it seems that \( O(k_n^2), n = 0, 1, 2 \) terms introduce corrections beyond the ‘soft approximation’, they are actually divided by the factor \( 1/k_{\perp}^2 \) in front, and hence are the most dominant terms in the soft approximation limit. Other terms for \( n > 2 \) will have a \( k_{\perp} \) in the numerator and so those will tend to zero in the soft limit. So Eq. 4.5 refers to the more general modification (compared to those in Refs.[7, 8]) to the Gunion-Bertsch distribution within \( O(t^3/s^3) \).

### 4.2 Reaction rate of \( gg \to ggg \)

Why are we interested to know the rate of this reaction? The rate is important because the number/particle identity non-conserving processes are important maintain the chemical equilibration of the plasma medium. So processes like \( gg \to ggg \) (number non-conserving) or \( gg \to q\bar{q} \) (particle identity non-conserving) coupled with the reverse processes will significantly contribute to the chemical equilibrium. When the inelastic processes cease, the medium restraints from being chemically equilibrated and the number of gluons gets fixed.
Figure 4.2: Temperature variation of the ratio of the equilibration rate obtained in Ref. [12] (solid line), Ref. [7] (dashed line), and [8] (dot-dashed) normalized by the corresponding value putting GB distribution for the process $gg \rightarrow ggg$.

The reaction rate of $gg \rightarrow ggg$, $R_3$, has been estimated by finding out $\sigma^{gg \rightarrow ggg}$, the corresponding cross-section, and multiplying it with the gluon density $\rho_g \sim T^3$, where $T$ is the bath temperature. The $\sigma^{gg \rightarrow ggg}$ can be found out by integrating the triple differential cross-section

$$\frac{d\sigma^{gg \rightarrow ggg}}{d^2 \vec{q}_\perp d\eta d^2 \vec{k}_\perp} \approx \frac{d\sigma^{gg \rightarrow g}}{d^2 \vec{q}_\perp} \left[ \frac{d\eta}{d^2 \vec{k}_\perp} \right]$$

over $q_\perp$ and the emitted gluon phase space. We find out $2 \rightarrow 3$ reaction rate putting the corrections to the GB distribution obtained in Refs. [7, 8, 12] in Eq. 4.12 and compare their respective ratios to the reaction rate putting GB distribution in Fig. 4.2. We see that the non-eikonal correction has significant contribution to the equilibration rate of gluon.
4.3 Energy loss by energetic gluons

Energy loss of energetic gluon in a gluonic plasma can be obtained with the help of Eq. 3.18. As long as we are in the additive (Bethe-Heitler) region, the energy loss for each collision beyond the first one is given by the formula. Hence the energy loss obtained by Eq. 3.18 yields that over the distance of mean free path of plasma. The mean free path of the plasma is obtainable from the inverse interaction rate which can evaluated using pQCD in the same way as [15]. The $\theta$-functions in the formula constrain the phase space of emitted gluon because the energy loss formula demands $\tau_m > \tau_F$ as well as $E > k_{\perp} \cosh \eta$. We put mean free path $\tau_m \sim \Lambda^{-1}$, where $\Lambda$ is the interaction rate obtained from [15] and $\tau_F \sim \cosh \eta/k_{\perp}$ is the formation time of gluon. So we get $E/\cosh \eta > k_{\perp} > \Lambda \cosh \eta$.

![Figure 4.3](image)

Figure 4.3: Temperature variation of the energy loss of 10 GeV gluon, obtained from Refs. [12] (solid line), [7](dashed), [8](dot-dashed) and scaled by that obtained from Ref. [14]. The ratio is denoted by $\Delta E_R$.

Keeping the above considerations in mind, we find out the energy losses obtained using the distributions of Refs. [7, 8, 12] and scaled by the corresponding value obtained from Ref. [14]. The difference in the scaled energy loss obtained from Ref [12] with those from Refs. [7, 8]
is more prominent in the lower temperature region than the upper temperature realm. So it is expected that the distribution function will have considerable effect in case of energy loss calculations around the RHIC energy (see Fig. 4.3).

The qualitative difference of the $\Delta E_R$ in different cases can be attributed to the fact that while the correction terms in [7, 8] contain terms like $(t/s)^n$ only, the present calculation shows the existence of (at $\mathcal{O}(k_{\perp}^{-2-1})$) terms like $s^{-1/2-1}$ (see Eq. 4.5). Given $s = 18T^2$ in the COM frame of the colliding particles, the temperature variation of $\frac{1}{k_{\perp}s}$ is negligible in comparison with $\frac{1}{k_{\perp}\sqrt{s}}$ (for a given $k_{\perp}$) or $1/s$ — a fact which results in the qualitative difference in the present radiation distribution formula, and hence, in energy loss. So, in conclusion, we can tell that the qualitative difference in the energy loss using modified Gunion-Bertsch formula will compel us rethink the radiation distribution to be used in phenomenological models; as also, this may be seen as a step towards the continuous endeavour of removing the approximations prevailing inside the energy loss calculations.
Bibliography


[10] Taken from a lecture by Prof. Jan-e Alam


