Chapter 3

Radiation Spectrum and energy loss due to scattering: A general perspective

3.1 Potential picture of single scattering

In the previous chapter we have established that finding out the radiation distribution of an energetic particle will enable us calculate the radiative dissipation. Since the degrees of freedom in the present case are the fundamental coloured objects like quarks and gluons, the dynamics will be dictated by the perturbative techniques for dynamics of colours–perturbative Quantum Chromodynamics (pQCD). The basic process we are going to study is the QCD gluon bremsstrahlung (means ‘breaking radiation’). But the gluon bremsstrahlung off coloured particles is not much different, under certain approximation, from the photon emission off (electrically) charged particles. We can see [1] that the approximated radiation four-current has nothing to do with the underlying process it has been formed through. The current depends,
in stead, on the four momenta. The four current has a classical nature and is derivable from the classical electrodynamics by considering the potential induced by charge of an electromagnetic current due to scattering. So, often we will be able to draw the analogies between the radiation spectrum obtained for particles moving with different kinematic constraints in electrodynamics with those in QCD. We will also notice that these kinematic constraints are going to play a major role in deciding the shape of the emission spectrum.

3.1.1 Advantage of being ‘soft’

We have mentioned that ‘under certain approximation’, the electrodynamic radiation current resembles colour radiation\(^1\). That approximation is the ‘soft radiation approximation’. To see how this approximation really helps draw the analogy, let us consider the photon bremsstrahlung off a charged particle (mass \(m\)) induced by a static electromagnetic source. If we write down the quantum amplitudes for the two Feynman diagrams (see Fig. 3.1) representing the scattering, we see that in the soft approximation, \(\i.e.\) when the energy of the emitted radiation \(\omega\) is much less than that of the parent partons \(E\), we can write the radiation amplitude in terms of a product of elastic part times the soft radiation current. Writing down the Feynman amplitudes from Fig. 3.1,

\[
\begin{align*}
\mathcal{M}_{\text{pre}}^\mu &= e \bar{u}(k_3) V \frac{m + k_1 - \bar{k}}{m^2 - (k_1 - k)^2} \gamma^\mu u(k_1) \\
\mathcal{M}_{\text{post}}^\mu &= e \bar{u}(k_3) \gamma^\mu V \frac{m + k_3 + \bar{k}}{m^2 - (k_3 + k)^2} u(k_1)
\end{align*}
\]

\(\text{(3.1)}\)

\(^1\)Then why do we need to do ‘soft radiation QCD’ in stead of classical calculations? We discuss it in Sec. 3.3
Figure 3.1: Feynman diagrams of photon radiation off a charged particle scattered by a static source with interaction $V$

where ‘pre’ and ‘post’ denote the pre-emission (i.e. emission from $k_1$) and post-emission (i.e. emission from $k_3$) respectively. Neglecting $k/\gamma$ with respect to $k_1/\gamma$ and putting $k_i\gamma^\rho = -\gamma^\rho k_i^\rho + 2k_i^\rho$ we get,

\[
(m + k_1^\rho)\gamma^\alpha u(k_1) = (\gamma^\alpha [m - k_1^\rho] + 2k_1^\alpha)u(k_1) = 2k_1^\alpha u(k_1)
\]

\[
\bar{u}(k_3)\gamma^\beta (m + k_3^\rho) = \bar{u}(k_3)([m - k_3^\rho]\gamma^\beta + 2k_3^\beta) = 2k_3^\beta \bar{u}(k_3)
\]

(3.2)

with the help of Dirac’s equation for fermions. Considering on-shell ($k^2 = 0$) gluon emission we obtain the total amplitude as:

\[
\mathcal{M}_{rad} = e j^\mu \times \mathcal{M}_{el}, \quad \text{where} \quad j^\mu = \frac{k_{1\mu}}{k_{1\cdot k}} - \frac{k_{3\mu}}{k_{3\cdot k}}
\]

(3.3)

and $e$ is the coupling. The radiation current $j^\mu$ is exactly that obtained from the classical calculations also. This clearly shows an universality in the radiation current which, upon squaring, yields the radiation distribution. So, to find out the radiation distribution, we may proceed by calculating the radiative amplitude from which the elastic part is separable by virtue of the soft approximation. Then the remaining part will yield the desired spectrum. The soft approximation is, thus, a very convenient one which implies inability of the emitted gluon to
probe the elastic part of the process. Also, the form of radiation current \( j^\mu \) in Eq. 3.3 shows that the radiation current is dependent only on the momenta of the particles. No reference to the process they undergo in the elastic part is necessary. So, in the same spirit of discussion about the classical nature of the soft radiation, it will really be interesting to see (in chapter 5) how the soft radiation spectrum off a heavy quark takes after that of a heavy classical charged particle or dipole under certain kinematic constraints.

### 3.1.2 Softness ‘burns’?

We have already discussed that the square of the radiation current provides us the radiation spectrum. The radiation probability can be written multiplying the (Lorentz invariant) phase space factor as below:

\[
dW_{\text{rad}} = \sum_{\beta=1,2} |e_\beta^\alpha j^\mu|^2 \frac{d^3k}{2\omega(2\pi)^3} dW_{\text{el}}
\]  

where \( e^\beta_\mu \) is the polarization of the emitted radiation. Using the Feynman gauge\(^2\), for which \( \sum_{\beta=1,2} e_\beta^\alpha e_\beta^\nu \Rightarrow -g^\mu\nu \), we get

\[
dN = \frac{dW_{\text{rad}}}{dW_{\text{el}}} \sim (j^\mu)^2 \frac{d^3k}{2\omega(2\pi)^3}
\]

\[
\sim \frac{d\omega d\Omega}{\omega^2} \frac{(1 - \cos \theta_{13})}{(1 - \cos \theta_1)(1 - \cos \theta_3)}
\]  

from Eq. 3.3, where \( \theta_{13} \) stands for the angle between incoming and the scattered particle, \( i.e. \) the angle between \( k_1 \) and \( k_3 \). \( \theta_1(\theta_3) \) is the angle the emission, with momentum \( k \), makes with the particle with momentum \( k_1(k_3) \). We also neglect the term \((m/E)^2\) (energy of parent

\(^2\)In the next section, we will repeat the same calculation in a different gauge, called the light cone gauge, for some added advantages.
particles are $\sim E$) with respect to the terms involving angles. Now let us assume that the emitted gluon is almost collinear with the incoming particle so that $\theta_{13} \approx \theta_3$, then

$$dN = \frac{dW_{\text{rad}}}{dW_{\text{el}}} \sim \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{1}{(1 - \cos \theta_1)}$$  \hspace{1cm} (3.6)$$

assuming the z direction to be along that of the incoming particle, we get, for $\theta_1 \to 0$,

$$dN \sim \frac{d\omega}{\omega} \frac{\sin \theta_1 d\theta_1}{(1 - \cos \theta_1)} \sim \frac{d\omega}{\omega} \frac{d\theta_1^2}{\theta_1^2}$$  \hspace{1cm} (3.7)$$

Treating the emission to be i) soft ii) collinear (with incoming particle) and iii) treating $m^2 << E^2$, we get a double logarithm [2] distribution for emission spectrum. For $\omega, \theta \to 0$ there exists divergences (called soft and collinear divergence respectively). The soft divergence exists if we consider the radiative bremsstrahlung processes only$^3$. However, for the physical scenarios, these divergences are not present because there the radiative process cannot be separated. The collinear divergence, i.e. emission of huge number of photons/gluons almost grazing the trajectory of the parent quark, is regulated once we consider mass of emitting particles. It can also be shown from the classical electrodynamics [4] that there exists a conical region around the direction of motion of a massive particle, whose velocity is parallel to acceleration, where radiation is negligible. This region is called the ‘dead cone’ region and is a classical phenomenon.

$^3$for regularization of soft divergence see [3]
3.1.3 Potential picture of single scattering in a different gauge

Single scattering is the building block for a multiple-scattering scenario; and so we must understand how single radiative scattering amplitudes can be calculated. But, we have done so just in the last section, and what makes us redo the same calculation? As hinted in the footnote [2], light cone gauge is preferred to the Feynman gauge, used for treating the single scattering and for understanding the divergences therein, because the light cone gauge calculations allow one to neglect the Feynman diagrams arising due to the radiation off the target partons as the amplitudes corresponding to the said diagrams are kinematically suppressed. So, much complexities (i.e. number of diagrams) can be avoided while treating the multiple scattering with the help of the single scattering.

Now, let us consider the quark($q$)-quark radiative (i.e. radiation of gluons ($g$)) scattering in scalar QCD (i.e. spin neglected). The scattering amplitudes, however, are calculated [5] in the soft-eikonal limit and the radiation distribution, i.e. number of gluons ($n_g$) per unit transverse momentum of emission ($k_\perp$) and per unit rapidity ($\eta$) is obtained as below:

$$\left[ \frac{dn_g}{d^2k_\perp d\eta} \right]_{GB} = \frac{C_A \alpha_s}{\pi^2} \frac{q_\perp^2}{k_\perp^2 (k_\perp - q_\perp)^2}$$

(3.8)

where $q_\perp$ is the transverse momentum transfer, $C_A$ is the Casimir factor and $\alpha_s$ is the strong coupling. The above result is the celebrated Gunion-Bertsch formula which was derived by the authors of [6] who found out the Eq. 3.8 for the emitted gluon emitted making a substantial angle ($\sim \pi/2$, i.e. the gluon mid-rapidity, $\eta \sim 0$) with the incoming or the scattered particle.

Now, there are two singularities in the formula – the noted soft singularity for the radiative processes and the singularity at $k_\perp = q_\perp$. It can be shown [7] that for $k_\perp << q_\perp$, the contribution from the diagrams with three-gluon (3g) vertices can be neglected compared to that of the other diagrams. The corresponding radiative matrix element can be written as:
\[ \mathcal{M}^\mu_{rad} = \frac{\vec{k}_{\perp}}{k_{\perp}^2} \mathcal{M}_{el} + (\text{a term, constant as } k_{\perp} \to 0) \] (3.9)

So we have the Gribov’s bremsstrahlung theorem which states that there exists a factorization of elastic and radiation current provided \( k_{\perp} \ll q_{\perp} \). The corresponding limit is called the Gribov limit [1].

But the inequality \( k_{\perp} \geq q_{\perp} \) makes the diagrams with 3g vertices important. So while considering the corresponding amplitude, we must shield the \( k_{\perp} = q_{\perp} \) singularity by the thermal mass of gluon.

### 3.2 Potential picture of multiple scattering: The energy loss model

After a brief introduction to the single particle scattering by a potential, we encounter a very similar situation where multiple scatterings off static centres occur in a colour neutral ensemble. The medium partons are considered to be static at the positions \( x_i = (z_i, \vec{x}_{\perp i}) \) such that \( z_{i+1} > z_i \) and the inter-scatterer longitudinal distance is much larger compared to the color screening length. Then we can model the potential offered by the scatterer as a static Debye screening potential:

\[ V_{i,AA'}^{\alpha}(\vec{q}) = \frac{gT_{i,AA'}^{\alpha}}{q^2 + \mu^2} \] (3.10)

for the \( i \) th scatterer with colors \( A, A' \) and \( T_i \) is the \( d_i \) dimensional generator of the representation corresponding to the target parton at the \( i \) th position. In each scattering the amount of the transferred momentum is, on the average small compared to the incident parton energy, \( E \). Also, in the high temperature limit (\( i.e. \ g \ll 1 \)), it can be shown [8] that the energy transfer
is $g$ times smaller than the transverse momentum transfer, thereby providing a justification of the use of the potential model in which energy transfer must be negligible. The Born amplitudes, neglecting the spin of the particles are written down with the *soft, eikonal and collinear approximations*, where soft approximation enables to neglect the energy of the emission with respect to the incident particle, eikonality allows to neglect the transverse momenta of the scattered/radiated particles with respect to the energies of the scattered/radiating particle; and collinearity makes us assume that radiation almost grazes the parent particle. With these three assumptions, the multiple scattering and hence the jet energy loss model calculations are done. It may be a good opportunity to discuss the physical scenarios, though general, of the Gyulassy-Wang Potential Model (GWPM) with which we have computed the energy loss for a gluonic jet in a gluonic plasma in the next chapter.

### 3.2.1 Multiple scattering in potential model: the radiation distribution

So, the question we may ask is: ‘is multiple scattering just like adding the probabilities of the single scatterings or we have to add the amplitudes, not the probabilities?’ This is a general concern of basic quantum mechanics where we add the probabilities when we know that two processes are independent whereas we add the amplitudes when the processes are not independent. In the present scenario what we mean by this is: the answer of the question, just asked, depends on how frequently the scatterings are taking place and how much time does the radiation take to form, the formation time (see section 2.1 for a discussion on the formation time). Actually, when the scattering centres are well separated so that the radiation gets ample time to be formed, then the single scattering amplitudes are independent and it suffices to add the squared amplitudes or the probabilities (the cross-sections). But, because of a scattering which has taken place before the radiation is formed, the amplitudes interfere and then the basic quantum mechanics tells us to add the quantum mechanical amplitudes and
not the probabilities. This is, as already mentioned in the previous chapter, the famous issue of Landau-Pomeranchuk-Migdal (LPM) suppression where the two time scales, the formation time $\tau_f$ and the time between two scatterings $\tau_m$ become important. $\tau_f \gg \tau_m$ gives the factorization limit where the resulting radiation distribution is not just the addition of all the single scattering radiation distribution patterns. The other limit dictated by $\tau_f \ll \tau_m$ is the Bethe-Heitler (BH) limit in which the scattering centres act independently. We may write down a relation between the multiple collision differential radiation distribution and the corresponding single scattering one as below:

\[
\frac{dn_s^{(m)}}{d^2k_\perp d\eta} = C_m(k) \frac{dn_s^{(1)}}{d^2k_\perp d\eta},
\]  

(3.11)

where ‘1’ stands for single scattering and ‘m’ stands for multiple scattering. $C_m$ is called radiation formation factor characterizing the interference pattern due to multiple scattering. $\eta$ is the gluon rapidity related with its emission angle $\theta$ with respect to the emitting particle as:

\[
\eta = -\ln \left( \tan \frac{\theta}{2} \right)
\]

(3.12)

Naturally, in the BH limit, $C_m \approx m$, i.e. the scatterings add up to give the resultant intensity with no interference pattern. On the other hand, the factorization limit gives [7, 8],

\[
C_m(k) \approx \frac{8}{9} [1 - (-1/8)^m] \quad \text{for quarks}
\]

\[
\approx 2(1 - 1/2^m) \quad \text{for gluons}
\]

(3.13)

Eq. 3.13 shows that the interference effect due to many multiple scatterings for quarks leaves corresponding radiation spectrum a factor of $\sim 8/9$ of that due to single scattering. It can also be checked that the gluon intensity radiated by gluon jet is $9/4$ times higher than that
radiated by quark jets in multiple scattering. Thus, the LPM effect in QCD depends on colour representation due to non-abelian nature of the problem under discussion.

3.2.2 Energy loss models: general considerations and approximations

We have already discussed about the radiation distribution due to multiple scattering off an energetic particle inside a thermal bath considered to be consisting of static scattering centres. But for ‘not-so-high’ momenta \( i.e. \sim 2 \text{ GeV} \) there is collisional loss also and the partonic energy loss in QGP Considering elastic partonic interaction with thermal quarks and gluons (see Fig. 3.2a) was estimated by Bjorken [9]. The energy loss per unit length can be shown to be:

\[
\frac{dE}{dx} = \frac{g^4}{6\pi} \left(1 + \frac{N_f}{6}\right) T^2 \ln \left(\frac{q_{\text{max}}}{q_{\text{min}}}\right)
\]  

(3.14)

where \( g \) is the strong coupling, \( q_{\text{max}}(q_{\text{min}}) \) is the maximum(minimum) momentum transfer, \( T \) is the bath temperature and \( N_f \) is the number of quark flavors. The energy loss taking into account the plasma effects has also been calculated in Ref. [10] and is shown to be:

\[
\frac{dE}{dx} = \frac{g^4}{12\pi} \left(1 + \frac{N_f}{6}\right) T^2 \ln \left(\frac{aE_T}{m_D^2}\right)
\]  

(3.15)

where \( m_D \) is the Debye screening mass and \( a \) is a constant \( (O(1)) \). Also, Ref. [11] calculates the partonic energy loss for both hard and soft momentum exchange for two-body going to two-body \( (2 \rightarrow 2) \) scattering. Cancellation of the intermediate energy scale \( q^* \) below(above) which the momentum transfer is considered to be soft(hard) while summing the energy dissipation from two sectors is also shown. They exhibit the existence of a cut-off energy \( E_c \) below which the collisional energy loss dominates. The closed form of the energy loss is given by:
\[
\frac{dE}{dx} = \frac{\nu g^2}{48\pi} \omega_p^2 \ln \left( \frac{E}{g^2T} \right)
\]  
(3.16)

where \(\nu\) is the statistical degeneracy factor and \(\omega_p\) is the plasma frequency.

![Diagram of energy loss models](image)

Figure 3.2: (a) Collisional and (b) radiative energy loss of high energy particles.

However, as already depicted in Fig. 3.2b, the multiple scattering of the incident particle with the soft \((i.e.\ \text{energy } \sim T)\) medium particles has to be incorporated and to meet that end there are a handful of other energy loss models\(^4\) like those by Baier-Dokshitzer-Mueller-Peigne-Schiff (the BDMPS formalism)[13], Gyulassy-Levai-Vitev (the GLV formalism) [14], Armesto, Salgado, Wiedemann (the ASW formalism) [15], Arnold, Moore and Yaffe (AMY) (the thermal perturbative approach) [16], Higher twist approach [17] etc. The energy loss models make use of the following three approximations:

- **soft** \(i.e.\) energy of radiation, \(\omega\), is much smaller compared to that of the parent parton, \(E\). Only AMY formalism, however, refrains from using this limit.

- **eikonal** \(i.e.\) no recoil of the leading parton due to scattering and radiation \(i.e.\) if \(q_\perp\) is the transverse momentum transfer and \(k_\perp\) is the transverse momentum of radiation, then, \(k_\perp, q_\perp \ll E\), where \(E\) is the order of energy values of the radiating/ scattered particle;

\(^4\)For a detailed comparison of all the energy loss models see [12]
and

- collinear i.e. gluons/photons almost graze the trajectory of the parent particle. From the on shell condition for the radiated particle,

\[ k^2 = 0, \quad \Rightarrow \omega^2 - k^2_\perp - k^2_z = 0 \]
\[ \Rightarrow \omega = k_\perp \sin \theta, \quad k_z = k_\perp \cos \theta \]

(3.17)

if we parametrize \( \omega \) and \( k_z \) in terms of the radiation emission angle \( \theta \). Hence \( \omega/k_\perp << 1 \) implies the emission angle is very small. Hence the condition for collinearity boils down to \( \omega << k_\perp \).

### 3.2.3 Energy loss from the radiative distribution

All the above kinematic constraints are generally imposed while obtaining the radiation distribution and once we have the radiation distribution, the radiative energy loss can be obtained by integrating the distribution function multiplied by the energy of each gluon (\( \omega \)) over the transverse momentum (\( k_\perp \)) and the rapidity (\( \eta \)) of the gluon. Hence we can write the additive energy loss for each collision after the first one as:
\[ \Delta E_{\text{rad}} = \frac{E_{m+1} - E_m}{m+1-m} \]

\[ = \int d^2k_\perp d\eta \frac{d_{n_g}}{d^2k_\perp d\eta} \frac{C_{m+1} - C_m}{m+1-m} \theta(E - k_\perp \cosh \eta) \]

\[ = \int d^2k_\perp d\eta \frac{d_{n_g}}{d^2k_\perp d\eta} \frac{dC_m}{dm} \theta(E - k_\perp \cosh \eta) \]

\[ \sim \int d^2k_\perp d\eta \frac{d_{n_g}}{d^2k_\perp d\eta} \omega \theta(\tau_m - \tau_F) \theta(E - k_\perp \cosh \eta) \quad (3.18) \]

where \( \frac{dC_m}{dm} \) has been approximated as a \( \theta \)-function following [7] and the second \( \theta \)-function imposes the constraint that the emission cannot have energy more than that of the emitting particle. The differential energy loss per unit length can be obtained by multiplying the energy loss per collision \( \Delta E_{\text{rad}} \) with the scattering rate \( \Lambda \).

### 3.3 ‘Our aims and expectations’ revisited

Now, as pledged in the section 1.5, after general discussion on energy loss phenomenon and its estimation in the radiative domain, we return to review our ‘aims and expectations’ again and we can modify the statement made there that the present thesis is about counting the number of emitted radiation. The reason is, we have already established that counting is same as finding out the radiation distribution. The two issues related to calculating the distribution which have been addressed are:

- Keeping in mind the kinematic approximations used in the energy loss models we endeavour to calculate the radiation distribution off energetic particles like gluon or heavy quark relaxing them (the eikonal approximation due to scattering, to be specific) at the level of single scattering. During this course we have calculated the non-eikonal gluon distribution off gluon jets and expect to find out non-eikonal corrections to the Gunion-Bertsch
distribution. Non-eikonal corrections to the heavy quark radiation distribution has also been found out with a view to re-explore the noted ‘dead-cone’ spectrum off heavy quarks which we have mentioned about in the beginning of this chapter.

The important messages we will get from these parts are:

– The non-eikonal corrections to the Gunion-Bertsch distribution create significant differences in the energy loss of gluons specifically in the lower temperature region. The non-eikonal corrections become important while considering the chemical equi-libration of the gluons in thermal bath, too.

– The non-eikonal corrections to the heavy quark radiation spectrum helps display the absence of the ‘dead-cone’ region along the direction of propagation of the heavy quark.

• The second issue is a bit different because this problem deals with the radiation distribution off quarks whose virtuality due to the acceleration received in heavy-ion collision is taken in to account.

We have already discussed the Weizsäcker-Williams (WW) picture of the energy loss where the WW gluons associated with a parton are detached from its parent due to acceleration and there is radiation in the form of these detached quanta. Now, along with the acceleration received by an incoming particle by the medium particles, there exists already a huge acceleration imparted upon them at the beginning when we collide two heavy nuclei to produce these jet particles, and, much later, the QGP. When comes the context of radiation distribution due to scattering inside medium, there is also a question whether the radiation is same as the radiation due to off-shellness, or a mix. A rigorous field theoretic technique of their interplay is necessary. The radiation off virtual quarks has been addressed and distribution of radiation given off by them has been calculated.
This part of thesis infers that the radiation distribution off virtual quarks, heavy or light, are similar and it is only after they become real that the difference between their spectra become distinguishable.

In all these calculations just mentioned above, it is not surprising to see the radiation distributions resembling with those obtained from the Classical Electrodynamics. We will see it most vividly in chapter 5 while dealing with the radiation distribution off the heavy quarks. But we should keep in mind that the dynamics is after all dictated by QCD and while calculating the energy loss we have the asymptotically free strong coupling in our calculation. So, though the radiation current is same as the classical current, the QCD plays its part in the radiation distribution where the property of asymptotic freedom becomes important.
Bibliography


