Chapter 3

The anti-reflexive solutions of the

Matrix equation $AXB = C$ in

Minkowski Space $\mathbb{M}$

3.1 Introduction

Chen and Sameh [17], H.C.Chen [18] has introduced the following two special classes of matrices in $C^{n \times n}$ as

$$C_{r}^{n \times n}(P) = \{ A \in C^{n \times n} : A = PAP \},$$

$$C_{a}^{n \times n}(P) = \{ A \in C^{n \times n} : A = -PAP \},$$

which are generalized reflexive and anti-reflexive matrices and exploited their properties, where $P \in C^{n \times n}$ is a generalized reflection matrix in complex. Cvetkovic [20] studied the reflexive solutions of the matrix equation $AXB = C$. 

37
In this chapter we have introduced the anti-reflexive solutions of the matrix equation $AXB = C$ in Minkowski space $\mathcal{M}$. With respect to the Minkowski inner product (using definition(1.2.20)) the adjoint of a matrix $P \in C^{n\times n}$ is given by $P^* = GP^*G$, where $P^*$ is the usual Hermitian adjoint.

Many results of this chapter form the content of the paper entitled ” The anti-reflexive solutions of the matrix equation $AXB = C$ in Minkowski space $\mathcal{M}$ ” which was published in the International Journal of Research and Reviews in Applied Sciences, ” Vol.5, (2013) 221-227.

3.2 Preliminaries

**Definition 3.2.1** [10] A$^g$ is said to be a generalized inverse(g-inverse) of $A$ if

$$AA^gA = A.$$  \hfill (3.1)

**Definition 3.2.2** [69] A matrix $P \in C^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ and $P^2 = I$.

**Definition 3.2.3** [24] A matrix $X \in C^{n\times m}$ is said to be reflexive with respect to $P$ if

$$X \in C^{n\times m}_{rp} = \{X/PX = X, X \in C^{n\times m}\},$$

and a matrix $X \in C^{n\times m}$ is said to be anti-reflexive with respect to $P$ if

$$X \in C^{n\times m}_{ap} = \{X/PX = -X, X \in C^{n\times m}\}.$$
Definition 3.2.4 A matrix $P \in C^{n \times n}$ is called a generalized reflection anti-symmetric matrix in Minkowski space $\mathcal{M}$ if $GPG = -P^*$ and $(GPG)^2 = I$.

In section 3.3 we have discussed some lemmas which yields a necessary and sufficient condition for $X$ to be in $C_{a}^{n \times n}(P^\sim)$, where $P^\sim$ is a generalized reflection anti-symmetric matrix in Minkowski space $\mathcal{M}$.

Further we discussed about the anti-reflexive solutions of the matrix equation $AXB + CYD = E$ in $\mathcal{M}$.

Here we introduce the following two special classes of subspaces in $C^{n \times n}$ in Minkowski space as

$$C_r^{n \times n}(P^\sim) = \{A \in C^{n \times n} : A = P^\sim AP^\sim\},$$

$$C_a^{n \times n}(P^\sim) = \{A \in C^{n \times n} : A = -P^\sim AP^\sim\},$$

where $P^\sim \in C^{n \times n}$ is a generalized reflection anti-symmetric matrix in Minkowski space $\mathcal{M}$ and $P^\sim = GP^*G$.

From definition (3.2.4) $GPG = -P^*$, and hence $GP^*G = -P$. implies $P^\sim = -P$.

The basic aim is to find the necessary and sufficient conditions of $E, F$ for the existence of a solution to the matrix equation

$$AX = B. \quad (3.2)$$

such that $X$ belongs to some special class of matrices.
In this section we will consider the anti-reflexive solutions of the matrix equation

\[ AXB = C. \]  

(3.3)

From now on, by anti-reflexive solution we mean anti-reflexive with respect to a generalized reflection anti-symmetric matrix \( P^\sim \). The generalization from the equation (3.2) to (3.3) is non-trivial.

### 3.3 Main Results

For generalized reflection anti-symmetric matrix \( P^\sim \) there exists unitary matrix \( U = [U_1 \ U_2] \), where \( U_1 \in \mathbb{C}^{n \times r} \), \( U_2 \in \mathbb{C}^{n \times n-r} \) such that

\[ P^\sim = U \begin{bmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} U^*. \]  

(3.4)

Throughout this section we will assume that a generalized reflection anti-symmetric matrix \( P^\sim \) is represented by (3.4). The next lemma gives a necessary and sufficient condition for \( X \) to be in \( C_{a}^{n \times n}(P^-) \).

**Lemma 3.3.1** The matrix \( X \in C_{a}^{n \times n}(P^-) \) if and only if \( X \) can be expressed as \( X = U \begin{bmatrix} 0 \\ M \\ 0 \end{bmatrix} U^* \) where \( M \in C^{r \times (n-r)} \), \( N \in C^{(n-r) \times r} \).

**Proof:**

Let \( X = U \begin{bmatrix} E \\ M \\ F \end{bmatrix} U^* \in C_{a}^{n \times n}(P^-) \),

(3.5)

where \( M \in C^{r \times (n-r)} \) and \( N \in C^{(n-r) \times r} \).

We claim that \( P^-XP^- = -X \).
Chapter - 3 The anti-reflexive solutions of the Matrix equation $AXB = C$ in Minkowski Space.

$P^{-\lambda}XP^{-\lambda} = U \begin{bmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} U^* U \begin{bmatrix} E & M \\ N & F \end{bmatrix} U^* U \begin{bmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} U^* $ 

$= U \begin{bmatrix} E & -M \\ -N & F \end{bmatrix} U^*.$

i.e., $P^{-\lambda}XP^{-\lambda} = U \begin{bmatrix} E & -M \\ -N & F \end{bmatrix} U^*.$

It follows that $\begin{bmatrix} E & -M \\ -N & F \end{bmatrix} = \begin{bmatrix} E & M \\ N & F \end{bmatrix}$ which implies that $M = 0$ and $N = 0$. The other direction is trivial.

**Lemma 3.3.2** The matrix $X \in C_r^{n \times n}(P^{-\lambda})$ if and only if $X$ can be expressed as

$$X = U \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} U^*,$$

where $M \in C_r^{r \times r}, N \in C^{(n-r) \times (n-r)}$ and $U$ is the same as (3.4). Without loss of generality we will suppose that the matrices $A \in C^{m \times n}, B \in C^{n \times p}$ and $C \in C^{m \times p}$ have the following decompositions

$$A = U \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} U^*, \quad B = U \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U^*, \quad C = U \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} U^* \quad (3.6)$$

where $A_1, B_1, C_1 \in C_r^{r \times r}$ and $A_4 \in C^{(m-r) \times (n-r)}, B_4 \in C^{(n-r) \times (p-r)}, C_4 \in C^{(m-r) \times (p-r)}.$

**Theorem 3.3.1** The matrix equation $AXB = C$ has a solution $X \in C_a^{n \times n}(P^{-\lambda})$ if and only if the following system of the matrix equation has a solution

$$A_2NB_1 + A_1MB_3 = C_1, \quad A_2NB_2 + A_1MB_4 = C_2,$$

$$A_4NB_1 + A_3MB_3 = C_3, \quad A_4NB_2 + A_3MB_4 = C_4. \quad (3.7)$$
In this case, the solution is represented by

\[ X = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} U^*. \]

Proof: First suppose that the matrix equation (3.3) has a solution \( X \in \mathbb{C}^{n \times n} \). By lemma 3.3.1 there exists \( M \in \mathbb{C}^{r \times (n-r)}, N \in \mathbb{C}^{(n-r) \times r} \), such that

\[ X = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} U^*. \]

Using (3.6) and the decompositions of the matrices \( A, B \) and \( C \), we get

\[
\begin{bmatrix}
A_2NB_1 + A_1MB_3 & A_2NB_2 + A_1MB_4 \\
A_4NB_1 + A_3MB_3 & A_4NB_2 + A_3MB_4
\end{bmatrix} = \begin{bmatrix}
C_1 & C_2 \\
C_3 & C_4
\end{bmatrix}
\]

If the system of the matrix equation (3.6) has a solution then

\[ X = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} U^* \in \mathbb{C}^{n \times n} \) and \( AXB = C. \)

**Lemma 3.3.3** The matrix equation \( AXB = C \) has a solution \( X \in \mathbb{C}^{n \times n} \) if and only if the following system of the matrix equation has a solution

\[ A_1MB_1 + A_2NB_3 = C_1, \quad A_1MB_2 + A_2NB_4 = C_2, \]
\[ A_3MB_1 + A_4NB_3 = C_3, \quad A_3MB_2 + A_4NB_4 = C_4. \]

In this case, the solution is represented by

\[ X = U \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} U^*. \]

**Lemma 3.3.4** [10] (i) If \( B_1 = 0 \) and \( B_2 = 0 \), then the matrix equation \( AXB = C \) has a solution \( X \in \mathbb{C}^{n \times n} \) if and only if \( A^tA^gC'B'B^g = \)

42
Chapter - 3 The anti-reflexive solutions of the Matrix equation $AXB = C$ in Minkowski Space

$C'$, where $A' = [A_1 \ A_3]^T$, $B' = [B_3 \ B_4]$, $U^*CU = C'$. In this case the general solution is represented by

$$X = U \begin{bmatrix} 0 & M \\ A' C' B' + Y - A' Y B' & 0 \end{bmatrix} U^*,$$

where $M \in C^{r \times (n-r)}$ and $Y \in C^{(n-r) \times (n-r)}$ are arbitrary matrices.

(ii) If $B_2 = 0$ and $B_3 = 0$, then the matrix equation $AXB = C$ has a solution $X \in C_a^{n \times n}(P^-)$ if and only if $A' A'^T C' B' B'^T = C'$ and $A'' A''^T C'' B'' B''^T = C''$, where $A' = [A_2 \ A_4]^T, A'' = [A_1 \ A_3]^T, C' = [C_2 \ C_4]^T, C'' = [C_1 \ C_3]^T$. In this case the general solution is represented by

$$X = \begin{bmatrix} 0 & A' C' B' + Y - A' Y B' & 0 \\ A'' C'' B'' + W - A'' Y B'' & 0 \end{bmatrix} U^*,$$

where $Y \in C^{r \times r}$ and $W \in C^{(n-r) \times (n-r)}$ are arbitrary matrices.

**proof (i):** Let $B_1 = 0$, $B_2 = 0$, and $X \in C_a^{n \times n}(P^-)$ the solution of equation (3.3). We can assume that $X$ is represented by (3.5). Now from $AXB = C$ it follows that

$$\begin{bmatrix} A_2 NB_1 + A_1 MB_4 \\ A_2 NB_2 + A_1 MB_4 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

Here $B_1 = 0, B_2 = 0$ implies that

$$\begin{bmatrix} A_1 MB_3 \\ A_3 MB_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_3 \\ C_4 \end{bmatrix}$$

$$[A_1 \ A_3]^T M [B_3 \ B_4] = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

It is denoted by

$A' = [A_1 \ A_3]^T, B' = [B_1 \ B_3] \text{ and } U^*CU = U^*U \begin{bmatrix} C_1 \\ C_3 \\ C_4 \end{bmatrix} U^*U = C'$.

That is $C' = \begin{bmatrix} C_1 \\ C_3 \\ C_4 \end{bmatrix}$, Therefore $U^*CU = C'$. 

43
It is well known that equation $A'MB' = C'$ has a solution if and only if $A'A^gC'B^gB' = C'$. In this case the general solution is represented by $M = A'^gC'B^g + Y - A'^gA'YB'B^g$ for arbitrary $Y \in C^{(n-r)\times(n-r)}$.

**proof (ii):** Suppose that $B_1 = 0$, $B_2 = 0$, and $X \in C_{a\times n}^n(P)$ is the solution of equation (3.3). Then $X$ is represented by (3.5). From $AXB = C$, we obtain that $$\begin{bmatrix} A_2 \\ A_4 \end{bmatrix} NB_1 = [C_1 \ C_3], \quad [A_2 A_4]^T NB_1 = [C_1 \ C_3],$$ $$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} MB_4 = [C_2 \ C_4] \quad \text{and} \quad [A_1 A_3]^T MB_4 = [C_2 \ C_4].$$

Let $A' = [A_2 A_4]^T$, $A'' = [A_1 A_3]^T$, $C' = [C_1 \ C_3]$, $C'' = [C_2 \ C_4]^T$, solutions of these equation exists if and only if $A'A'^gC'B^gB_1 = C'$ and $A''A''^gC''B^gB_4 = C''$.

$$A'NB_1 = C'A'^gA'NB_1B^g_1 = A'^gCB^g_1.$$ 

Thus $N = A'^gCB^g_1$.

$i.e,$, $N = A'^gCB^g_1 + Y - A'^gA'YB_1B^g_1$.

$$A''MB_4 = C''.$$ 

Then $M = A''A'^gC''B^g_4$.

$$i.e,$, $M = A''A'^gC''B^g_4 + W - A''A''WB_4B^g_4.$

In this case the general solutions are represented by $$N = A'^gC'B^g + Y - A'^gA'YB'B^g, \quad M = A''A'^gC'B^g + W - A''A''WB_4B^g.$$ for arbitrary $Y \in C^{(n-r)\times(n-r)}$ and $W \in C^{(n-r)\times(n-r)}$.

Now in the following theorem we will discuss about the anti-reflexive solutions of the matrix equation

$$AXB + CYD = E. \quad (3.8)$$
**Theorem 3.3.2** Given $A, B, C, D, E \in C^{n \times n}$ and a generalized anti-reflection matrix $P^\sim$ of size $n$. Then the following conditions are equivalent.

(i) The matrix equation $AXB + CYD = E$ has the anti-reflexive solutions $X, Y \in C^{n \times n}_{a}(P^\sim)$,

(ii) The following matrix equation has a solution

$$A''X_3B' + A'X_2B'' + C''Y_3D' + C'Y_2D'' = E'$$

where

$$A' = (A_1^t, A_2^t), B' = (B_1, B_2), A'' = (A_2^t, A_4^t), B'' = (B_3, B_4), C' = (C_1^t, C_3^t)$$

$$D' = (D_1, D_2), C'' = (C_2^t, C_4^t), D'' = (D_3, D_4), E' = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}. \quad (3.9)$$

(iii). The following system of matrix equations has a solution

$$A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 = E_1,$$

$$A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_4 = E_2,$$

$$A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 = E_3,$$

$$A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4 = E_4. \quad (3.10)$$

In that case, the anti-reflexive solutions of the matrix equation $AXB + CYD = E$ can be expressed by the following

$$X = U \begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix} U^* \text{ and } Y = U \begin{bmatrix} 0 & Y_2 \\ Y_3 & 0 \end{bmatrix} U^*.$$

**Proof:** First let us show that (i) implies (ii). Since $AXB + CYD = E$ has the anti-reflexive solutions $X, Y \in C^{n \times n}_{a}(P^\sim)$ where $A, B, C$ and $D$ is defined as in (3.6). By partitioning the matrices $A, B, C$ and $D$ we have (3.10). By transforming (3.10) into matrix form we have $A''X_3B' + A'X_2B'' + C''Y_3D' + C'Y_2D'' = E'$. Therefore (i) implies (ii). Next we show that (ii) implies (iii).

$$A''X_3B' + A'X_2B'' + C''Y_3D' + C'Y_2D'' = E'.$$
\[(A'_2, A'_4)'X_3(B_1, B_2) + (A'_1, A'_3)'X_2(B_3, B_4) + (C'_2, C'_4)'Y_3(D_1, D_2)\]
\[+ (C'_1, C'_3)'Y_2(D_3, D_4) = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}.\]
\[A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 = E_1,\]
\[A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_4 = E_2,\]
\[A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 = E_3,\]
\[A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4 = E_4.\]

Thus (ii) implies (iii). (iii) implies (i) is trivial. Now to show that (i) implies (iii).

(i) implies that the matrix equation \(AXB + CYD = E\) has the anti-reflexive solutions \(X, Y \in C_a^{n \times n}(P^-)\). (3.8) implies that \(AXB + CYD = E\). By a lemma, the matrix equation \(A \in C_a^{n \times n}(P^-)\) if and only if \(A\) can be expressed as

\[A = U \begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix} U^*,\]

where \(A_2 \in C^{r \times (n-r)}, A_3 \in C^{(n-r) \times r}\). Suppose that the matrix equation \(AXB + CYD = E\) has the anti-reflexive solutions \(X \in C_a^{n \times n}(P^-)\) and \(Y \in C_a^{n \times n}(P^-)\). By Lemma (3.3.1) there exists \(X_2, Y_2 \in C^{(n-r) \times (n-r)}\) and \(X_3, Y_3 \in C^{(n-r) \times (n-r)}\) such that \(X = U \begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix} U^*\) and \(Y = U \begin{bmatrix} 0 & Y_2 \\ Y_3 & 0 \end{bmatrix} U^*\).

Now using the decompositions (3.6) from \(AXB + CYD = E\), we can get

\[
\begin{bmatrix}
A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 & A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_4 \\
A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 & A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4
\end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}.
\]

If the system of matrix equations (3.9) has a solution, then

\[X = U \begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix} U^*, \quad Y = U \begin{bmatrix} 0 & Y_2 \\ Y_3 & 0 \end{bmatrix} U^* \in C_a^{n \times n}(P^-)\] and \(AXB + CYD = E\).