Chapter 3

Properties of Product Closure Spaces

Introduction

In this chapter we study some properties of closure spaces and product closure spaces. Also we discuss some separation properties.


In section 3.1 we find some properties of mappings into product closure spaces with respect to associated topological spaces [S]. The properties are proved using the result proved in [CE] namely, ‘a mapping $f$ of a space $X$ into the product-space $X = \prod X_a$ is continuous if and only if the mapping $\pi_a \circ f$ is continuous for each $a$’. Also we find some separation properties in product closure spaces.

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In section 3.2 we discuss some separation properties involving zero sets. In this section we say a closure space \((X, c)\) is \(c\)-completely regular if it is completely regular in closure space, \(c\)-Hausdorff if it is Hausdorff in closure space. Also \((X, c)\) is completely regular if it is completely regular in the associated topological space. Similarly \((X, c)\) is Hausdorff if it is Hausdorff in the associated topological space.

3.1 Some properties in product closure spaces

In this section we find some properties of product closure spaces. For this we use properties of mapping between product closure spaces with respect to associated topological spaces.

Remark 3.1.1. It is known from [CE] that a mapping \(f\) of a closure space with closure operator \(c\) into the product closure space with closure operator \(c'\) is \(c \rightarrow c'\) morphism if and only if the mapping \(\pi_a \circ f\) is \(c \rightarrow c'\) morphism for each \(\pi_a\) (in our terminology).

Note 3.1.2. Using the above remark and the following conditions,

(i) \(c \rightarrow c'\) morphism \(\Rightarrow c \rightarrow \text{cl}'\) morphism (Trivial using definitions)

(ii) continuity \(\Rightarrow c \rightarrow \text{cl}'\) morphism (Result 2.3.2)

(where \(f : (X, c) \rightarrow (Y, c')\) and \(\text{cl}, \text{cl}'\) are closure operators on associated topological space of \((X, c), (Y, c')\) respectively) we have the following results.
**Result 3.1.3.** (i) A mapping $f$ of a closure space with closure operator $c$ into the product closure space with closure operator $c'$ is $c - cl'$ morphism if the mapping $\pi_a \circ f$ is $c - c'$ morphism for each $a$ where $\pi_a$ is the projection function.

(ii) If $f$ is $c - c'$ morphism then the mapping $\pi_a \circ f$ is $c - cl'$ morphism for each $a$.

(iii) If the mapping $\pi_a \circ f$ is continuous for each $a$ then $f$ is $c - cl'$ morphism.

(iv) If $f$ is continuous then the mapping $\pi_a \circ f$ is $c - cl'$ morphism for each $a$.

**Remark 3.1.4.** It is known that [W] an arbitrary product space is completely regular if and only if each factor space is completely regular and we know completely regular implies $c$-completely regular [S]. Hence we have the following results.

**Result 3.1.5.** An arbitrary product space is $c$-completely regular if each factor space is completely regular.

*Proof.* Given each factor space is completely regular and by remark (3.1.4) we have, the arbitrary product space is completely regular. Also by [S] completely regular implies $c$-completely regular. Thus arbitrary product space is $c$-completely regular. \qed
**Result 3.1.6.** An arbitrary product space is completely regular then each factor space is $c$-completely regular.

*Proof.* Given arbitrary product space is completely regular, so by remark (3.1.4) we have each factor space is completely regular. By [S] completely regular implies $c$-completely regular. Thus each factor space is $c$-completely regular. □

**Result 3.1.7.** Every subspace of a $c$-Hausdorff space is $c$-Hausdorff.

*Proof.* Let $A$ be a subspace of a $c$-Hausdorff space $(X, c)$. Let $x$ and $y$ be two distinct points of $A$. Since $(X, c)$ is $c$-Hausdorff by [S] there exist disjoint neighbourhoods $U$ and $V$ containing $x$ and $y$ respectively. Then $U \cap A$ and $V \cap A$ are disjoint neighbourhoods of $x$ and $y$ in $A$. Thus $A$ is $c$-Hausdorff. □

**Remark 3.1.8.** It is known that [W] a non-empty product space is Hausdorff if and only if each factor space is Hausdorff and we know Hausdorff implies $c$-Hausdorff [S]. Hence we have the following results.

**Result 3.1.9.** A non-empty product space is $c$-Hausdorff if each factor space is Hausdorff.

*Proof.* Given each factor space is Hausdorff and by above remark (3.1.8) we have, product space is Hausdorff. Also by [S] Hausdorff implies $c$-Hausdorff. Thus product space is $c$-Hausdorff. □
Result 3.1.10. If a non-empty product space is Hausdorff then each factor space is $c$-Hausdorff.

Proof. Given product space is Hausdorff and by remark (3.1.8) we have each factor space is Hausdorff. Also by [S] Hausdorff implies $c$-Hausdorff. Hence each factor space is $c$-Hausdorff. 

Remark 3.1.11. It is known that [W] a non-empty product space is regular if and only if each factor space is regular and we know regular implies $c$-regular [S]. Hence we have the following results.

Result 3.1.12. (i) A non-empty product space is $c$-regular if each factor space is regular
(ii) If a non-empty product space is regular then each factor space is $c$-regular.

Proof. Proof is trivial using remark 3.1.11. 

3.2 Some separation properties in closure spaces

In this section we discuss some separation properties involving zero set.

Notation 3.2.1. We denote $C_1(X) = \{ f : X \to R | f \text{ is } c - \text{cl morphism} \}$ where $c$ is the closure operator on $X$ and cl is the Kuratowski closure operator on $R$ and the zero set of $X$ is denoted by $Z_1(X) = \{ Z_1(f) : f \in C_1(X) \}$ and $Z_1(f) = \{ x \in X | f(x) = 0 \}$. 

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Theorem 3.2.2. A $c$-Hausdorff space $X$ is $c$-completely regular if and only if the family $Z_1(X) = \{Z_1(f) : f \in C_1(X)\}$ is a base for the closed sets in the associated topology of $X$.

Proof. Suppose $X$ is $c$-completely regular. Then whenever $F$ is closed set and $x \in F'$ there exist $f \in C_1(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$. Then $Z_1(f) \supseteq F$ and $x \not\in Z_1(f)$. Thus $Z_1(X)$ is a base. □

Conversely, suppose $Z_1(X)$ is a base. So if $F$ is closed set and $x \in F'$ then there exist $g \in C_1(X)$ such that $Z_1(g) \supseteq F$ and $x \not\in Z_1(g)$. Let $r = g(x)$, then $r \neq 0$ and let $f = gr^{-1} \in C_1(X)$ and $f(x) = 1$, $f(F) = \{0\}$ so that Hausdorff space $X$ is completely regular. But by [S] completely regular implies $c$-completely regular. Hence the theorem.

Remark 3.2.3. It is known that [CHA] ‘A Hausdorff space $X$ is completely regular if and only if $\{Z(f) : f \in C^*(X)\}$ forms a base for the closed set. Also continuity implies $c - cl'$ morphism by 2.3.2.

Remark 3.2.4. A Hausdorff space $X$ is completely regular then $Z_1(X) = \{Z_1(f) : f \in C_1(X)\}$ is a base for the closed sets in the associated topology.

Proof. Trivial using remark (3.2.2). □
3.3 Conclusion

In this chapter we found some properties in product closure space using morphisms. Also we find some properties of complete regularity in closure space.