Chapter 1

$H$-Closedness in Closure and Monotone Spaces

Introduction

In this chapter we introduce the concept of $H$-closedness in closure and monotone spaces and investigate its properties. Also we study the relations to the $H$-closedness of the associated topological space.

Absolutely closedness or $H$-closedness was introduced in 1929 by Alexandroff and Urysohn (for the definition and details, see [P;W]). E. Čech defined closure spaces (cf. [CE]) and T. A. Sunitha [S] discussed relations between closure spaces and the associated topological spaces. In [C;M], C. Calude and M. Malitza defined a different notion of Čech spaces which are now called monotone spaces and in [S], T. A. Sunitha discussed relations between monotone spaces, associated closure spaces and associated
topological spaces. These motivated the study of \( H \)-closedness in closure and monotone spaces.

In section 1.1, we introduce denseness, adherence and \( H \)-closedness in closure spaces and call them respectively \( c \)-denseness, \( c \)-adherence and \( cH \)-closedness analogous to the corresponding notions in topological spaces. Here we study some of their properties; also we obtain relations between these properties and similar properties in the associated topological spaces.

In section 1.2, we discuss the inheritance in closure spaces, by subsets of properties like \( H \)-closed and \( cH \)-closed. Section 1.3 and section 1.4 are similar to section 1.1 and section 1.2 respectively in which closure space is replaced with monotone space. Here we also find the relationship between monotone space, associated closure space and the associated topological space with respect to the properties like \( H \)-closed and \( cH \)-closed.

### 1.1 \( c \)-denseness, \( c \)-adherence and \( cH \)-closedness

In this section we introduce the concept of denseness, adherence and \( H \)-closedness in closure space and they are called \( c \)-denseness, \( c \)-adherence and \( cH \)-closedness respectively. Also we study some of their properties

**Definition 1.1.1.** A set \( A \) in a closure space \((X, c)\) is said to be \( c \)-dense in \( X \) if \( cA = X \).
**Result 1.1.2.** If \( cA = X \) then \( cl A = X \) for any \( A \subseteq X \); that is, a set \( A \subseteq X \) is \( c \)-dense in \( X \) implies \( A \) is dense in \( X \).

**Definition 1.1.3.** Let \((X, c)\) be a closure space, \( \mathcal{I} \) be a filter on \( X \) then the set \( \cap \{cF : F \in \mathcal{I}\} \) is called the \( c \)-adherence of \( \mathcal{I} \) and is denoted by \( a^c(\mathcal{I}) \).

**Note 1.1.4.** \( a^c(\mathcal{I}) \subseteq a(\mathcal{I}) \), the adherence of \( \mathcal{I} \) where \( a(\mathcal{I}) = \cap \{cl F : F \in \mathcal{I}\} \) where \( cl F \) is the closure of \( F \) in the associated topological space.

**Definition 1.1.5.** A closure space \((X, c)\) is said to be \( cH \)-closed if every open filter on \( X \) has non-void \( c \)-adherence.

**Note 1.1.6.** The above definition is analogous to the characterisation of \( H \)-closedness as given in [P;W], namely a topological space \((X, \tau)\) is \( H \)-closed if and only if every open filter on \( X \) has a nonempty adherence. We say that \((X, c)\) is \( H \)-closed if \((X, \tau)\) is \( H \)-closed where \( \tau \) is the associated topology for \( c \).

**Proposition 1.1.7.** If a closure space \((X, c)\) is \( cH \)-closed, then it is \( H \)-closed (by this we mean that the associated topological space is \( H \)-closed).

**Proof.** We know that [P;W] a topological space \((X, \tau)\) is \( H \)-closed if and only if every open filter has non-void adherence, that is, if and only if for any open filler \( \mathcal{I} \), \( \cap \{cl F : F \in \mathcal{I}\} \neq \emptyset \). Given \((X, c)\) is \( cH \)-closed. So if \( \mathcal{I} \) is any open filler then \( \cap \{cF : F \in \mathcal{I}\} \neq \emptyset \). So \( \cap \{cl F : F \in \mathcal{I}\} \neq \emptyset \) since \( cF \subseteq cl F \). Hence \((X, c)\) is \( H \)-closed. \( \square \)
Proposition 1.1.8. A closure space \((X, c)\) is \(H\)-closed if and only if every open cover \(\mathcal{V}\) of \((X, c)\) has a finite subfamily whose union is \(c\)-dense in \(X\).

Proof. We know that \([P;W]\), a topological space \((X, \tau)\) is \(H\)-closed if and only if every open cover \(\mathcal{V}\) of \((X, \tau)\) has a finite subfamily whose union is dense in \(X\). Let \(\mathcal{V}\) be an open cover of \(X\) such that for each finite set \(A \subseteq \mathcal{V}\), \(X \neq c(\bigcup A)\). Let \(\mathcal{I} = \{U : U\) open and \(U \subseteq X \setminus c(\bigcup A)\) for some finite set \(A \subseteq \mathcal{V}\}\). Clearly \(\mathcal{I}\) is nonempty and \(\mathcal{I}\) is an open filter on \(X\) and \(a(\mathcal{I}) = \cap \{\text{cl } U : U \in \mathcal{I}\}\)

\[
\subseteq \cap \{\text{cl } (X \setminus c(\bigcup A)) : A \subseteq \mathcal{V} \text{ is finite}\}
\subseteq \cap \{\text{cl } (X \setminus c(V)) : V \in \mathcal{V}\}
\subseteq X \setminus \bigcup (\mathcal{V}) = \emptyset \text{ since } \bigcup (\mathcal{V}) = X.
\]

Thus \((X, c)\) is not \(H\)-closed.

Conversely, \(c\)-dense implies dense. Hence \((X, c)\) is \(H\)-closed. \(\Box\)

Proposition 1.1.9. If a closure space \((X, c)\) is \(cH\)-closed then every open cover \(\mathcal{V}\) of \((X, c)\) has a finite subfamily whose union is dense in \(X\).

Proof. \(cH\)-closed implies \(H\)-closed (by proposition 1.1.7) and by the proposition 1.1.8 we have the result. \(\Box\)

Note 1.1.10. \(H\)-closed does not imply \(cH\)-closed.

Eg:- Let \(X = N\), the set of natural numbers define \(cA = A \cup (A - 1)\) for
A \subseteq N \text{ where } A - 1 = \{x - 1 | x \in A\} \text{ and } 1 - 1 = 0 \text{ is not considered. Then }
\{N, \{2, 3, 4, \ldots\}, \{3, 4, 5, \ldots\}, \ldots\} \text{ is an open filter base for the associated topology. Then } (X, c) \text{ is } H\text{-closed and not } cH\text{-closed. Thus converse of the proposition 1.1.9 is not true.}

**Proposition 1.1.11.** If a closure space \((X, c)\) is \(cH\)-closed then every open cover \(\mathcal{V}\) of \((X, c)\) has a finite subfamily whose union is \(c\)-dense in \(X\).

**Proof.** \(cH\)-closed implies \(H\)-closed and by proposition 1.1.8 we have the result. \(\square\)

**Note 1.1.12.** Converse of the above proposition is not true in general, since \(H\)-closed does not imply \(cH\)-closed by note 1.1.10.

### 1.2 Inheritance of properties in \(cH\)-closed spaces

In this section, we discuss the inheritance by subsets of properties like \(H\)-closed and \(cH\)-closed.

**Proposition 1.2.1.** Let \((X, c)\) be \(cH\)-closed and \(U \subseteq X\) be open. Then \(cl(U)\) is \(cH\)-closed.

**Proof.** Let \(U\) be open in \((X, c)\) and \(\mathcal{I}\) be an open filter on \(cl(U) = A\). Then \(\{F \cap U : F \in \mathcal{I}\}\) is an open filter base on \(X\). Let \(G = \{W \subseteq X : W\) is open in \(X\) and \(W \supseteq F \cap U\) for some \(F \in \mathcal{I}\}\). Then \(G\) is an open filter on \(X\). Given \((X, c)\) is \(cH\)-closed. So \(\phi \neq a^c_X(G) \subseteq \bigcap \{c_X(F \cap U) :
Thus \( \text{cl}(U) \) is \( cH \)-closed.

**Corollary 1.2.2.** Let \( (X, c) \) be \( cH \)-closed and \( U \subseteq X \) be open. Then \( \text{cl}(U) \) is \( H \)-closed.

**Proof.** By Proposition 1.2.1 \( \text{cl}(U) \) is \( cH \)-closed. But by 1.1.7 \( cH \)-closed implies \( H \)-closed. Thus the corollary. \( \Box \)

**Remark 1.2.3.** Let \( (X, c) \) be \( H \)-closed and \( U \subseteq X \) be open. Then \( \text{cl}(U) \) need not be \( cH \)-closed.

**Proposition 1.2.4.** Let \( (X, c) \) be \( cH \)-closed and \( U \subseteq X \) be open. Then \( c(U) \) is \( cH \)-closed.

**Proof.** Proof is exactly similar to that of Proposition 1.2.1 with \( \text{cl}(U) \) replaced with \( c(U) \). \( \Box \)

**Proposition 1.2.5.** Let \( (X, c) \) be \( cII \)-closed and \( U \subseteq X \) be open. Then \( c(U) \) is \( H \)-closed.

**Proof.** By proposition 1.2.4, \( c(U) \) is \( cII \)-closed. And we know \( cH \)-closed implies \( H \)-closed. Hence the result. \( \Box \)

**Remark 1.2.6.** Let \( (X, c) \) be \( H \)-closed and \( U \subseteq X \) be open. Then \( c(U) \) need not be \( cH \)-closed.

**Proposition 1.2.7.** If \( (Y, c') \) is a \( c'II \)-closed subspace of a Hausdorff closure space \( (X, c) \) where \( c' = c \) restricted to \( Y \), then \( Y \) is closed in \( (X, c) \).
Proof. Given \((Y, c')\) is \(c'H\)-closed implies \((Y, c')\) is \(H\)-closed. Thus \(Y\) is closed in \(X\).

\[\square\]

1.3 \textit{\(m\)-denseness, \(m\)-adherence and \(mH\)-closedness}

In this section we introduce the concept of denseness, adherence and closedness in monotone space and they are called \(m\)-denseness, \(m\)-adherence and \(mH\)-closedness respectively. Also we investigate their properties and relationship between them and the same in the associated closure space and the associated topological space.

\textbf{Definition 1.3.1.} A set \(A\) in a monotone space \((X, m)\) is said to be \(m\)-dense in \(X\) if \(mA = X\).

\textbf{Remark 1.3.2.} [S] If \((X, m)\) is a monotone space and \(\text{cl}\) is the closure operation in the associated topological space. Then \(\text{cl} \leq m\), that is, \(\text{cl} A \supseteq mA\) for all \(A \subseteq X\).

\textbf{Result 1.3.3.} If \(mA = X\) then \(\text{cl} A = X\) where \(A \subseteq X\), that is, a set \(A \subseteq X\) is \(m\)-dense in \(X\) implies \(A\) is dense in \(X\).

\textit{Proof.} Trivial using remark 1.3.2.

\[\square\]

\textbf{Definition 1.3.4.} Let \((X, m)\) be a monotone space, \(\mathcal{I}\) be a filter on \(X\), then the set \(\bigcap \{mF : F \in \mathcal{I}\}\) called \(m\)-adherence of \(\mathcal{I}\) and is denoted by \(a^m(\mathcal{I})\).
Note 1.3.5. Clearly \( a^m(\mathcal{I}) \subseteq a(\mathcal{I}) \) where \( a(\mathcal{I}) = \cap \{\text{cl} F : F \in \mathcal{I}\} \) is the adherence of \( \mathcal{I} \) in the associated topology.

Definition 1.3.6. A monotone space \((X, m)\) is said to be \(mH\)-closed if every filter on \(X\) of open sets in the associated topology has non void \(m\)-adherence.

Note 1.3.7. The above definition is motivated by the characterisation of \(H\)-closedness given in [P;W], namely a topological space \((X, \tau)\) is \(H\)-closed if and only if every open filter on \(X\) has a nonempty adherence and the definition of \(H\)-closedness in closure spaces given in section 1.1. Here we say that \((X, m)\) is \(cH\)-closed if it is \(H\)-closed in the associated closure space that is \(cH\)-closed in \((X, c)\) where \(c\) is the associated closure operator, and we say \((X, m)\) is \(H\)-closed if it is \(H\)-closed in the associated topology.

Proposition 1.3.8. If a monotone space \((X, m)\) is \(mH\)-closed then it is \(cH\)-closed and \(H\)-closed.

Proof. Given \((X, m)\) is \(mH\)-closed. So if \(\mathcal{I}\) is any open filter then \(\cap \{mF : F \in \mathcal{I}\} \neq \phi\). Thus \(\cap \{cF : F \in \mathcal{I}\} \neq \phi\). Hence \((X, m)\) is \(cH\)-closed.

Also we have \(cH\)-closed implies \(H\)-closed. Thus we have the proposition.

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\square
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Proposition 1.3.9. A monotone space \((X, m)\) is \(H\)-closed if and only if every open cover \(\mathcal{V}\) of \((X, m)\) has a finite subfamily where union is \(m\)-dense in \(X\).
Proof. We know that (cf. [P;W]), a topological space \((X, \tau)\) is \(H\)-closed if and only if every open cover \(\mathcal{V}\) of \((X, \tau)\) has a finite subfamily whose union is dense in \(X\). Here given that \((X, m)\) is \(H\)-closed. That is, \((X, t)\) is \(H\)-closed where \(t\) is the associated topology of \((X, m)\). We have to prove that every open cover \(\mathcal{V}\) of \((X, m)\) has a finite subfamily whose union is \(m\)-dense in \(X\). For that we assume the contrary. Let \(\mathcal{V}\) be an open cover of \(X\), such that for each finite set \(A \subseteq \mathcal{V}\), \(X \neq m(\bigcup A)\). Let \(\mathcal{I} = \{U : U\) is nonempty and open and \(U \subseteq X \setminus m(\bigcup A)\) for some finite set \(A \subseteq \mathcal{V}\}\). Clearly \(\mathcal{I}\) is an open filter on \(X\) and adherence of \(\mathcal{I}\), that is, \(a(\mathcal{I}) = \bigcap\{\operatorname{cl} U : u \in \mathcal{I}\} \subseteq \bigcap\{\operatorname{cl} (X \setminus m(\bigcup A)) : A \subseteq \mathcal{V}\text{ finite}\}\) \(\subseteq \bigcap\{\operatorname{cl} (X \setminus m V) : V \in \mathcal{V}\} \subseteq X \setminus \bigcup \mathcal{V} = \emptyset\) since \(\bigcup \mathcal{V} = X\). Thus \((X, m)\) is not \(H\)-closed. \(\square\)

Converse follows from the fact that \(m\)-dense implies dense and by the characterisation of \(H\)-closedness in topological space.

**Proposition 1.3.10.** If a monotone space \((X, m)\) is \(mH\)-closed then every open cover \(\mathcal{V}\) of \((X, m)\) has a finite subfamily where union is \(m\)-dense in \(X\) and hence \(e\)-dense and dense in \(X\).

**Proof.** We know that \(mH\)-closed implies \(H\)-closed and by the proposition 1.3.9 we have the result. \(\square\)
1.4 Inheritance of properties in $mH$-closed spaces

In this section we discuss the inheritance by subsets of properties $H$-closed, $cH$-closed and $mH$-closed.

**Proposition 1.4.1.** Let $(X, m)$ be $mH$-closed and $U \subseteq X$ be open then $cl(U)$ is $mH$-closed and hence $cH$-closed and $H$-closed.

*Proof.* We have to prove $cl(U)$ is $mH$-closed, that is, to prove $a^m(\mathcal{I}) \neq \phi$ with respect to $cl(U)$, where $\mathcal{I}$ is an open filter on $cl(U)$. Let $U$ be open in $(X, m)$ and $\mathcal{I}$ be an open filter on $cl(U) = A$. Then $\{F \cap U : F \in \mathcal{I}\}$ is an open filter base on $X$. Let $G = \{W \subseteq X : W$ is open in $X$ and $W \supseteq F \cap U$ for some $F \in \mathcal{I}\}$. Then $G$ is an open filter on $X$. Given $(X, m)$ is $mH$-closed. So $\phi \neq a^m_X(G) \subseteq \bigcap\{m_X(F \cap U) : F \in \mathcal{I}\} = \bigcap\{m_A(F \cap U) : F \in \mathcal{I}\} \subseteq \bigcap m_A(F) : F \in \mathcal{I}\} = a_m(\mathcal{I})$. Hence the proposition. \(\square\)

**Proposition 1.4.2.** Let $(X, m)$ be $mH$-closed and $U \subseteq X$ be open then $cU$ is $mH$-closed and hence $cH$-closed and $H$-closed.

*Proof.* Proof is exactly similar to that of proposition 1.4.1 with $cl(U)$ replaced with $cU$. \(\square\)

**Proposition 1.4.3.** If $(Y, m_Y)$ is a $m_Y$ $H$-closed subspace of a Hausdorff monotone space $(X, m)$, then $Y$ is closed in $(X, m)$.

*Proof.* Given $(Y, m_Y)$ is $mH$-closed then $(Y, m_Y)$ is $H$-closed. Thus $Y$ is closed in $X$, since $X$ is given to be Hausdorff. Thus the proposition. \(\square\)
1.5 Conclusion

In this chapter we defined denseness, adherence and $H$-closedness in closure and monotone spaces. Using these definitions we prove Kuratowski closure of any open set in monotone space (or closure space) is $H$-closed in monotone space (or closure space) and hence $H$-closed in associated topological space of $(X, m)$. 