CHAPTER 2

BASICS OF RF ACCELERATORS

INTRODUCTION

From the Lawson-Woodward theorem it follows that an electromagnetic wave cannot accelerate a charged particle if the field is in vacuum, without boundaries, and the region of interaction is infinite [32, 33]. One way to violate the conditions of the Lawson-Woodward theorem is to use guided structures that impose boundaries. Even in this case, if we want to accelerate a beam of charged particles moving in a certain direction, two conditions must be satisfied. Firstly, the electromagnetic wave must have an electric field component in the direction of particle motion, and, secondly, the wave and the particle must travel with same velocity or in other words the wave must be synchronized with the particle. The simplest guided structure is the uniform waveguide which supports the propagation of waves that have a longitudinal component of the electric field, and can hence satisfy the first condition. The second condition, however, cannot be fulfilled by a wave propagating in a uniform waveguide as the phase velocity of the wave is higher than the velocity of light, while the particle velocity is always less than velocity of light. The solution is to slow down the wave velocity in the waveguide to match with the velocity of the charged particle. Classically, the velocity of any object can be reduced by putting an obstacle or perturbation in its path and can be manipulated to the desired value by putting obstacles repeatedly. Using the same analogy it is also possible to slow down the wave velocity by introducing periodic perturbations in the uniform waveguide. The reduction in wave
velocity depends upon the perturbations and by employing a suitable design of perturbations the wave velocity can be matched with the particle velocity. Therefore a waveguide with periodic perturbations along the length can be used to accelerate a charged particle. Looking rigorously the velocity of the wave reduces due to its reflection from each perturbation. The reflected part of the wave travels in the opposite direction and again gets reflected from the previous perturbation and this way the wave starts oscillating between two consecutive perturbations, the pair of them therefore acting like a resonating cavity. Since the perturbations are periodic in nature the whole structure can be described as an array of coupled cavities. As the photocathode RF gun is also a coupled cavity structure, the understanding of cavity basics and behaviour of an array of coupled cavities is important in the development of a photocathode RF gun. Therefore the basics of an RF cavity and coupled cavity structure are discussed in detail in this chapter before move towards the design of a photocathode RF gun.

2.1 PILLBOX CAVITY BASICS

The simplest resonating structure can be formed by shorting the ends of a uniform waveguide with conducting plates, thus forming a uniform cylindrical cavity or a pillbox cavity, which serves as fundamental building block in accelerating structures. The boundary conditions imposed at the cavity walls (transverse as well as longitudinal) allow only certain discrete field patterns known as cavity modes (sometimes electromagnetic modes of the cavity) and can be derived by solving Maxwell’s equations. They can be divided in two categories based on the characteristics satisfied by the longitudinal component of the electric or magnetic field: (1) Transverse Magnetic (TM) for those with \( B_z = 0 \) everywhere in the structure and (2) Transverse Electric (TE) for those with \( E_z = 0 \) everywhere in the structure. To describe any particular mode
three indices are necessary and modes are specified by TM<sub>mnp</sub> or TE<sub>mnp</sub>. Here subscript ‘m’ (m=0,1,2,...) is the number of full period variation of field in Φ direction, ‘n’(n=1,2,3...) is the number of zeros of the axial field component in the radial direction in the range 0 < r ≤ R<sub>c</sub> (radius of cavity), excluding r = 0 and ‘p’(p = 0,1,2...) is the number of half–period variations of the field along the z-axis [34, 35]. As the TE modes, by definition, have no electric field in the ‘z’ direction they are not suited for acceleration in this geometry, while TM modes have non-zero E<sub>z</sub> components and hence can be used for acceleration. The resonant frequencies for the TM<sub>mnp</sub> modes in a pillbox of radius R<sub>c</sub> and length L are given by

\[ f_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \left( \frac{\chi_{mn}}{R_c} \right)^2 + \left( \frac{p\pi}{L} \right)^2, \tag{2.1} \]

where ε and μ are the permittivity and permeability of the medium inside the cavity and \( \chi_{mn} \) is the \( n^{th} \) root of the \( m^{th} \) order Bessel function with \( J_m(\chi_{mn}) = 0 \). The lowest TM mode is TM<sub>010</sub> which has an electric field parallel to the z-axis while the magnetic field circles around it. The field patterns are rotationally symmetric and have no z dependence as shown in Fig. 2.1. Furthermore, the resonant frequency is independent of the length of the pillbox. This mode is used in almost all electric field (E-type) accelerators. Its resonant frequency and fields are given by:

\[ f_{010} = \frac{\chi_{01}}{2\pi\sqrt{\mu\varepsilon R_c}}, \tag{2.2} \]

\[ \chi_{01} = 2.405, \tag{2.3} \]
\[ E_z(r,t) = E_0 J_0 \left( \frac{Z_{01} r}{R_c} \right) e^{j\omega t}, \]  

(2.4)

\[ B_\phi(r,t) = -j \sqrt{\frac{\varepsilon}{\mu}} E_0 J_1 \left( \frac{Z_{01} r}{R_c} \right) e^{j\omega t}, \]  

(2.5)

where \( E_0 \) is the peak electric field, \( J_0 \) and \( J_1 \) are the 0\(^{th}\) and 1\(^{st}\) order Bessel function and \( \omega = 2\pi f \). Since the resonant frequency of the TM\(_{010}\) mode is independent of the cavity length, this length can be tuned for optimum acceleration. For a charge entering a pillbox cavity with speed \( v \), the length for maximum energy gain is \( \pi L/\beta\lambda \), where \( \beta = v/c \) [34].

Fig. 2.1: Pillbox cavity with an electromagnetic field in the TM\(_{010}\) mode. The graphs show the amplitude of \( E_z \) as function of (a) radius and (b) length of pillbox.

To accelerate a charged particle, the pillbox cavity needs to be modified by providing ports for entry and exit of the beam. The modified geometry of the pillbox cavity with entry and exit ports is shown in Fig. 2.2. In this modified geometry \( E_z \) should be zero at the locations of ports to satisfy the boundary condition as shown in Fig. 2.2, and the modified field is given by

\[ E_z(r,t) = E_0 J_0 \left( \frac{Z_{01} r}{R_c} \right) \cos(K_z z)e^{j\omega t}, \]  

(2.6)
where constant \( K_z = (2N+1)\pi / L_{\text{eff}} \) and \( L_{\text{eff}} \) is effective length of the cavity, which is the accelerating gap plus length of ports up to which the field is non-zero [34].

![Fig. 2.2: Pill box cavity with the ports for beam entry and exit, and variation in \( E_z \) with \( z \).](image)

As the electromagnetic fields are oscillating with time, when a charged particle passes through the cavity it see a time-varying electric field and the energy gained by the particle, ignoring the effect of the ports, is given by [34]

\[
\Delta W = qV = q \int_{-L/2}^{L/2} \overline{E_z(z,t)} e^{j\omega t} dz ,
\]

(2.6)

\[
\Delta W = qE_0 TL \cos(\phi_0),
\]

(2.7)

where \( 0 < T \leq 1 \) is the transit time factor defined as the ratio of energy gained in the time varying field to the that gained in a DC field of the same peak value. The maximum attainable field in any resonant structure is ultimately limited by electrical breakdown [35].
2.2 COUPLED PILLBOX STRUCTURE OR COUPLED CAVITY STRUCTURE: SPACE HARMONICS

In the case of a pillbox cavity the maximum on-axis field, for pulses of few µs duration, is typically ~100 MV/m for the S-band frequency, which gives a maximum energy gain of ~3-4 MeV; therefore for higher energies more than one pillbox cavity is required. Having several independent pillbox cavities requires independent RF feeds which is expensive, and the requirement of synchronization between them further complicates the system and adds to the expense. The alternate option is to connect the multiple pillbox cavities such that the fields in individual cavities can be coupled. This type of structure gives high average field gradients and is known as a coupled cavity structure. In principle when RF power is fed into one cavity (normally called the coupling cell) it is transferred to the other cavities also and sets up an electromagnetic field in all of them. The coupling of RF power between two cavities (or cells) is known as inter-cell coupling. When power is coupled by making off-axis slots in the disc separating two pillbox cavities, it is mainly the magnetic field that is coupled and this is known as B-type coupling, while when power is coupled by on-axis apertures it is mainly the electric field that is coupled and this is known as E-type coupling. The presence of off-axis slots in B-type coupling breaks the rotational symmetry of the cavity which scatters the electromagnetic energy into higher-order harmonics which spoils the transverse emittance of the beam; hence magnetic coupling is not preferred where lower emittance is needed like in photocathode RF guns. On the other hand, in E-type coupling rotational symmetry is maintained, and it is therefore used exclusively in high brightness photocathode RF guns.
An E-coupled cavity accelerating structure is shown in Fig. 2.3. It is a periodic structure with periodicity ‘d’ and according to the Floquet theorem it supports an infinite number of traveling waves having the same frequency but different wave numbers called space harmonics [37]. The space harmonics have a constant amplitude of the electric field $E_n$ independent of $z$. The wave number of the $n^{th}$ space harmonic is given by

$$k_n = k_0 + \frac{2\pi n}{d}, \quad n = -\infty...0..+\infty,$$

and the electric field (for TM$_{010}$ like mode) can be expanded in a Fourier series of different space harmonic as

$$E_n^\pm(r, z, t) = \sum_{-\infty}^{+\infty} E_n e^{j(\omega t + k_n z)}.$$ (2.9)

The waves with index $n \geq 0$ travel in the $+z$ direction and are known as forward waves, denoted by $E_n^+$, while the waves with $n < 0$ travel in the $-z$ direction and are known as backward waves, denoted by $E_n^-$. The phase velocity of the $n^{th}$ space harmonics is given by

$$v_{ph} = \frac{\omega}{k_n} = \frac{c}{1 + \frac{2\pi n}{k_0 d}} = \frac{c}{1 + \frac{n\lambda}{d}}.$$ (2.10)
As discussed earlier for acceleration the phase velocity of the wave must be equal to the particle velocity and such a wave is known as a synchronous wave. From Eq. (2.10), it is clear that space harmonics with \( n < 0 \) have phase velocity \( > c \) and are hence not suitable for acceleration. The space harmonics with \( n \geq 0 \), have phase velocity \( \leq c \) and can hence be used for acceleration. As different space harmonics have different phase velocities only one, usually the fundamental \( (n = 0) \) harmonic, is synchronous with the particle and contributes to the energy gain. The effects of the non-synchronous space harmonics on the particle average to zero (in some cells they accelerate while in some cells they decelerate and so the net energy gain is zero) and do not contribute to energy gain at the end of the structure.

![Dispersion curve of the lowest pass band of an infinite periodic structure.](image)

Fig.2.4: Dispersion curve of the lowest pass band of an infinite periodic structure.

### 2.2.1 Coupled Cavity Structure: Normal Modes

In a coupled cavity structure, for each electromagnetic mode of the individual cavity, there is a family of modes that have their own frequencies different from the resonant frequency of a
single cavity, and with a characteristic phase advance from one cavity to next cavity. These are known as normal modes. For an infinite string of coupled cavities, each space harmonic has associated with it an infinite number of normal modes within a certain frequency band, known as the pass-band, and those are the only waves that can propagate in the structure. Figure 2.4 shows the lowest pass-band corresponding to the TM$_{010}$ mode of a single cavity. For $n = 0$, space harmonics normal modes correspond to a phase advance per cavity $0 < kzd < \pi$, and having frequency from $f_0$ to $f_\pi$. Practically, a structure is made by a coupling finite number of cavities and such a structure supports a finite number of discrete normal modes. An accelerating structure made of $N$ coupled cavities, supports $N$ normal modes with phase shift per cavity given by

$$\phi_q = \frac{q\pi}{N-1}, \quad q = 0,1,2...N \ .$$

(2.11)

For an example if an accelerator is composed of two coupled cavities ($N = 2$) then it supports two normal modes with phase shifts of $\phi_q = 0$ and $\pi$ with resonant frequencies $f_0$, $f_\pi$, and if an accelerator composed of four coupled cavities ($N = 4$) then it supports four normal modes with phase shifts of $\phi_q = 0$, $\pi/3$, $2\pi/3$ and $\pi$ with resonant frequencies of $f_0$, $f_{\pi/3}$, $f_{2\pi/3}$, $f_{\pi}$. Each normal mode has a unique phase velocity which depends upon the individual cavity length (or more specifically it depends upon the separation between two cavities) and strength of inter-cell coupling. Therefore, for efficient acceleration, the structure has to be designed considering a particular normal mode as the synchronous mode. The mode-separation or frequency difference between different normal modes is also an important design parameter because if the modes are not well separated, then a small perturbation in the structure (due to RF heating or beam loading or any other reason) may switch the mode of operation, and in such a case when RF power is
coupled to the structure to excite the synchronous normal mode, it may excite a non-synchronous mode, which does not accelerate the beam. Therefore while designing a coupled cavity accelerating structure mode separation should be kept sufficiently high or greater than the bandwidth of the RF source being used to excite the structure to prevent undesired mode switching.

2.3 RF ACCELERATING STRUCTURE FIGURES OF MERIT

The RF parameters of an accelerating structure play an important role in the acceleration of the beam and affect the beam parameters. Therefore it is important to define and understand the RF parameters of an accelerating structure. Description of some of the important and crucial RF parameters is given below.

2.3.1 Quality Factor

The quality factor, $Q_0$ of an accelerating structure is a measure of the energy stored in the structure for unit power dissipation and is defined as the ratio of the time averaged stored energy $U$ in the structure to the power dissipated per radian of the RF cycle,

$$Q_0 = \frac{\omega_0 U}{P_c},$$

(2.12)

where $\omega_0$ is the angular frequency of the RF wave and $P_c$ is the power dissipated in the structure. For a practical accelerating structure, the field decays with time due to power dissipation in the walls of the structure, and to maintain the field gradient RF power needs to be fed into the structure. In this configuration the quality factor of the cavity is different from $Q_0$ due to
reflections at the entrance to the accelerating structure. One therefore defines the loaded quality factor $Q_L$,

$$Q_L = \frac{Q_0}{(1 + \beta)},$$

(2.13)

where $\beta$ is the coupling coefficient of the structure to the RF transmission line, as defined later, in section 2.3.5.

### 2.3.2 Shunt Impedance

The shunt impedance of an accelerator structure is a figure of merit measuring the energy transfer efficiency from the RF field to the charged particles. For an axial electric field $E_z(z)$ over the structure length $L$, the shunt impedance, $R_{sh}$ per unit length is defined as,

$$R_{sh} = \frac{1}{L} \left[ \int_0^L E_z(z) dz \right]^2$$

(2.14)

where $dP_c/dz$ is the power dissipation per unit length of the structure. High shunt impedance is desirable since it means high accelerating field for a given power dissipation per unit length of the structure.

### 2.3.3 Transit Time Factor

The energy gained by a charged particle in a time varying accelerating field is less than that from a corresponding static field because the particle takes some time to cross the accelerating structure and it consequently sees an average field which is always less than the peak value. The ratio between these energies is known as the transit time factor, and is defined by,
\[ T = \left| \int E_z(z)e^{ikz}dz \right| \left( \int E_z(z)dz \right), \quad (2.15) \]

where \( k = \omega c \) is the propagation constant. For a pillbox cavity with beam ports, as shown in Fig. 2.2, the transit time factor for a particle with velocity \( \beta \) (and assuming that this does not change during the acceleration process) is given by [34]

\[ T = \frac{\sin \left( \frac{\pi L_{\text{gap}}}{\beta \lambda} \right)}{\left( \frac{\pi L_{\text{gap}}}{\beta \lambda} \right)}, \quad (2.16) \]

where \( L_{\text{gap}} \) is the length of accelerating gap (total length of cavity minus length of ports). Although from above equation it is clear that to maximize \( T \) the accelerating gap should be minimum, for a given accelerating gradient the total energy gain in the cavity reduces with accelerating gap as given in equation (2.7). Therefore, in spite of the transit time factor the length should be optimized for energy gain, which is maximum for cavity length of \( \beta \lambda /2 \) [34].

Sometimes it is preferred to include the transit time factor in the calculation of the shunt impedance. Then, one defines an effective shunt impedance \( R = R_{sh}T^2 \). For higher efficiency of the accelerator, the effective shunt impedance should be higher.

**2.3.4 Characteristic Impedance**

The characteristic impedance, \( R/Q_0 \) is a geometry dependent parameter and measures the efficiency of acceleration per unit stored energy at a given frequency. It is given by
\[
R/Q_0 = \left[ \int E_z(z)dz \right]^2 / \alpha_0 W.
\]  \hspace{1cm} (2.17)

2.3.5 Waveguide to Cavity Coupling Coefficient (\(\beta\))

To power the accelerating structure RF power is transported from the RF source to the accelerating structure using a transmission line (for high frequencies this is usually a waveguide) and is coupled into the cavity by using a loop or through a small coupling slot located on its outer wall. The efficiency of RF power coupling to the structure is determined by the waveguide to cavity coupling coefficient (\(\beta\)) which is defined as

\[
\beta = \frac{P_{\text{ext}}}{P_c} = \frac{Q_0}{Q_{\text{ext}}},
\]  \hspace{1cm} (2.18)

where \(P_c\) is the power dissipated inside the cavity and \(P_{\text{ext}}\) is the power lost outside of the cavity, or, more precisely, in the matched load of the waveguide when RF source is turned off [38]. The amount of RF power coupled to the cavity is given by

\[
P_c = \frac{4\beta}{(1 + \beta)^2} P_+, \]  \hspace{1cm} (2.19)

where \(P_+\) is the input RF power. When \(\beta = 1\) the power transferred to the cavity is maximum.

2.3.6 Filling Time

When RF power is fed into an accelerating structure, part of the RF power dissipates in the structure and part of it goes towards setting up the electromagnetic field in the cavity. The filling time of a standing wave RF cavity is the time needed to build the electromagnetic field in the structure up to \((1 - 1/e) = 0.632\) times its steady-state value, and mathematically can be written as,
\[ t_f = \frac{2Q_L}{\omega_0} = \frac{2Q_0}{(1 + \beta)\omega_0}. \] (2.20)

For a traveling wave structure, the electromagnetic field travels along the length of the structure and is dumped at the end in a matched load. In this case the filling time is defined as

\[ t_f = \frac{L}{v_g}. \] (2.21)

where \( L \) is the total length of the structure and \( v_g \) is the group velocity of the traveling wave in the structure.

To achieve the maximum and uniform energy gain the beam is injected into the structure after a few filling times for a standing wave and for one filling time for a traveling wave linac. Cavities with large quality factor take longer to be filled with energy; for example, in superconducting cavities which have \( Q_0 \) of the order of \( 10^{10} \) the fill time is of the order of \( ms \) to \( s \) while for normal conducting structures with \( Q_0 \) of the order of \( 10^4 \), the fill time is of the order of few hundreds of ns to few \( \mu s \).

### 2.4 LOW POWER RF CHARACTERIZATION THEORY AND MEASUREMENTS

An RF accelerating structure is characterized by its RF properties, viz. resonant frequency, quality factor, shunt impedance, characteristic impedance, variation in on-axis accelerating field, waveguide to cavity coupling coefficient and fill time, as discussed in the earlier section. Therefore it is important to understand the methods used to measure the RF properties of an RF accelerating structure. In this section, we discuss the theory and methods to measure the RF...
properties of an RF accelerating structure at low power, which are also known as *cold test* measurements.

### 2.4.1 RF Measurement Theory

For DC applications a network can be characterized completely by measuring the voltage and current; however it is not possible to measure the voltage and current at high frequencies as they vary rapidly in time and space and such fast detectors are not typically available. Also, the measurement of voltage and current at two different times at same position or at two different positions at same time, gives different results. Therefore, at high frequencies measurement of the reflected and transmitted waves with respect to the incident wave is employed to characterize a network. When a RF signal is fed to a network a fraction of it reflected back, a fraction is transmitted and the remaining gets dissipated in the network. As the energy of the incident wave is scattered into reflected and transmitted waves the ratios of the reflected and transmitted waves to the incident wave are known as *scattering parameters* or *S-parameters* and are the basics building blocks for RF measurements. If a network has more than one port then the signal at any port is represented as a linear combination of the reflected and transmitted signals due to different ports. For example, for a two-port network, when RF power is fed at port 1 and port 2 is matched, then the ratio of the reflected wave to the input wave at port 1 is defined as $S_{11}=\frac{V_{-1}}{V_{+1}}$ where $V_{-1}$ is the reflected wave and $V_{+1}$ is the incident wave at port 1, and the ratio of RF signal at port 2 ($V_{+2}$) when the input port is matched is defined as $S_{21}=\frac{V_{+2}}{V_{+1}}$. Similarly when the RF signal is fed at port 2 two more S parameters, $S_{22}$ and $S_{12}$ are defined. Now, the signal at port 1 is represented as $V_{1-}=S_{11}V_{1+} + S_{12}V_{2+}$ and the signal at port 2 is given by $V_{2-}=S_{21}V_{1+} + S_{22}V_{2+}$. From the above it is clear that a network can be represented completely by a matrix of S
parameters which is known as the scattering matrix [39, 40]. For an ‘n’ port RF device the scattering matrix is given by

\[
\begin{bmatrix}
V_1^- \\
\vdots \\
V_n^{-}
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} & \cdots & S_{1n} \\
S_{21} & S_{22} & \cdots & S_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
S_{n1} & S_{n2} & \cdots & S_{nn}
\end{bmatrix}
\begin{bmatrix}
V_1^+ \\
\vdots \\
V_n^+
\end{bmatrix},
\]

(2.22)

where the \(ij\)th element of the S matrix is defined as

\[
S_{ij} = \frac{V_j^-}{V_i^+} \text{ for } i \neq j.
\]

(2.23)

\(V_j^+\) is the amplitude of the incident wave at port \(j\) and \(V_i^-\) is the amplitude of the reflected wave coming out from port \(i\). The S parameters of an RF device can be measured by using a multiport Vector Network Analyzer (VNA). [41].

An RF accelerating structure with an RF coupling port can be considered as two port device and by measuring \(S_{11}\) and/or \(S_{21}\), RF parameters of the accelerating structure can be determined [17]. The determination of RF parameters using \(S_{11}\) measurement is known as the reflection method while measurement using \(S_{21}\) is known as the transmission method. The methods of measuring the different RF parameters are discussed below.

### 2.4.2 Resonant Frequency Measurement

As an RF accelerating structure acts like an oscillator, at resonance the energy stored in it is maximum and reflection is minimum [34]. Therefore, the resonant frequency of an RF accelerating structure can be determined by measuring \(S_{11}\) while varying the frequency of the
input signal. As the energy stored in the structure is maximum at resonance the resonance frequency can also be measured by measuring the $S_{21}$ which gives the strength of the field in the cavity as a function of frequency. A typical plot of $S_{11}$ and $S_{21}$ with frequency measured by using a VNA is shown in Fig.2.5.

![Fig.2.5: Variation of $S_{11}$ and $S_{21}$ of an accelerating structure, with frequency.](image)

### 2.4.3 Quality Factor and Waveguide to Cavity Coupling Coefficient Measurement

#### 2.4.3.1 Transmission and VSWR Technique

The quality factor can be calculated by measuring the -3dB down points (half power points) on the $S_{21}$ graph and is given by,

$$Q_0 = \frac{f_r}{|f_2 - f_1|}.$$  

(2.24)
where $f_1$ and $f_2$ are the frequencies of the -3dB down power points with respect to the resonant frequency ($f_r$). The measurement of $Q_0$ employing above the technique gives wrong results as loss in coupling port also contributes in the measurement. So, instead $Q_0$ it is the loaded quality factor $Q_L$ that is actually measured [39].

The waveguide to cavity coupling coefficient is directly related to the Voltage Standing Wave Ratio (VSWR) which is directly measurable in modern VNA's [17, 39], and is given by

$$
\beta = \begin{cases} 
\frac{\text{VSWR}}{\text{VSWR}}, & \text{if over coupled} \\
\frac{1}{\text{VSWR}}, & \text{if under coupled} 
\end{cases} \tag{2.25}
$$

Clearly, the value of $\beta$ will be wrong if we do not know whether the cavity is over coupled or under coupled.

2.4.3.2 Smith Chart Technique

Smith Chart is a representation of the normalized impedances (or admittances) on the reflection plane [42]. In a Smith Chart the reflection coefficient is converted in terms of constant resistance and reactance circles, and impedance of a device or network at a particular frequency is represented by a single point, and for a frequency band it represents the loci of the impedances.

The Smith Chart gives the direct measurement of the quality factor and waveguide to cavity coupling coefficient [39]. The quality factors given by

$$
Q_0 = \frac{f_r}{|f_2 - f_1|}, \tag{2.26}
$$
\[ Q_{\text{ext}} = \frac{f_r}{|f_4 - f_3|}, \]  
\[ (2.27) \]

\[ Q_L = \frac{f_r}{|f_6 - f_3|}, \]  
\[ (2.28) \]

where \( f_r \) and \( f_{1-6} \) are the frequencies determined from the intersection, on the normalized Smith Chart, of impedance \( z = r + jx \) loci Chart and \( x \) lines, as described below. \( f_r \) is the resonant frequency of the RF accelerating structure and can be determined from intersection of \( S_{11} \) circle and the \( x = 0 \) line. Similarly \( f_1 \) and \( f_2 \) are the frequencies determined from the intersection of the \( S_{11} \) circle (or Smith Chart) and \( x = \pm r \) line, \( f_3 \) and \( f_4 \) are frequencies determined from the intersection of the \( S_{11} \) and the \( x = \pm 1 \) line, and \( f_5 \) and \( f_6 \) are frequencies determined from the intersection of the \( S_{11} \) and the \( x = \pm (r \pm 1) \) line.

A typical Smith Chart for an RF cavity and waveguide coupled system is shown in Fig 2.6.

Fig. 2.6: Smith Chart of an RF cavity waveguide coupled system.
After determining the different frequencies from a Smith Chart the quality factors can be measured directly by using Eqs. (2.26-2.28) while $\beta$ can be calculated using Eq. (2.18).

### 2.4.4 Shunt Impedance and Characteristic Impedance Measurement

From Eqs. (2.14) and (2.16) it is clear that the shunt impedance and characteristic impedance of an RF accelerating structure can be determined when the variation in the on-axis accelerating field ($E_z$) is known. The variation in $E_z$ can be determined by employing the bead pull technique as discussed below.

#### 2.4.4.1 Bead Pull Technique

In an RF cavity at resonance, both the magnetic and electric field energies are equal, and any perturbation in the field distribution changes the resonant frequency of the structure. This change depends upon the strength of the field at the location of the perturbation. Therefore, by measuring the variation in the resonant frequency the electromagnetic field can be determined using the Slater perturbation theory [43], according to which the frequency shift in the resonant frequency, $\omega_0$ due to perturbation of the cavity field is given by

$$\frac{\omega^2 - \omega_0^2}{\omega_0^2} = \frac{\int_{\delta V} \left[H^2_\alpha - E^2_\alpha \right] d\tau}{\int_{\infty} \left[\mu_0 H^2 + \varepsilon_0 E^2 \right] dv}.$$  \hspace{1cm} (2.29)

Here, $\mu_0$ and $\varepsilon_0$ are the permeability and permittivity of free space, respectively. $H_\alpha$ and $E_\alpha$ are the perturbed magnetic and electric field vectors at the location of the perturbing object of volume $\delta \tau$. The amplitude of the perturbed fields depends on the bead’s volume, shape, composition and location. $H$ and $E$ are the magnetic and electric field strengths in the unperturbed cavity of volume $dv$. 

52
Therefore, by pulling a small perturbing object (known as bead) through the cavity, the distribution or profile of the electromagnetic field can be measured and this technique is known as the bead pull technique [44].

For the $\text{TM}_{010}$ mode, $H_\alpha=0$ on the axis of the structure, and the denominator of Eq. (2.29) represents the total energy stored in the structure, which is constant. Then, the electric field at each location of the bead is proportional to the frequency shift as

$$E_z(z) \propto \sqrt{\omega_0 - \omega(z)},$$

(2.30)

where $\omega(z)$ is the resonant frequency of the RF cavity when the bead is at location $z$. Using $E_z$ in Eq. (2.17) the characteristic impedance of the structure is given by,

$$\frac{R}{Q_0} = \frac{1}{2\pi f_0^2 \mathcal{S}_z} \left[ \mathcal{L} \left( \sqrt{\left( f_0(z=0) - f(z=z) \right)} \right) \right],$$

(2.31)

where $\mathcal{S}_z$ depends upon the shape and size of the bead and is known as the form factor [45]. Once $R/Q_0$ is known the shunt impedance of the structure can be calculated by using the measured value of $Q_0$ in $R = (R/Q_0) \times Q_0$.

**SUMMARY**

The building block of an accelerating structure is a pillbox cavity operating in the $\text{TM}_{010}$ like mode and a complete accelerating structure is formed when many such cavities are connected together. The basics of a pillbox cavity along with coupled cavity structures and concept of electromagnetic and structure modes have been discussed. The figures of merit of an accelerating structure and techniques to measure them have also been discussed in detail.