4.1 Introduction

In the previous chapter we have learnt to compute expectation values of Wilson loops that contain a wealth of information about various properties of quark-gluon plasma. We have learnt to relate the expectation values to various heavy quark observables like the bound state interaction potential, the screening length and the jet quenching parameter. In this chapter, we examine the first of our anisotropic models - the thermal non-commutative Yang-Mills theory at strong 't Hooft coupling and large number of colors\(^*\). The purpose of studying non-commutative gauge theory is 3-fold\(^†\). Firstly, inNCYM plasma the presence

\(^*\)The present chapter is based on [78, 79].

\(^†\)Space-time non-commutativity is an old idea introduced first by Heisenberg and Pauli [80] in order to evade the infinities in quantum field theory before renormalization was successful. It was Snyder [81] and then Connes who took the idea seriously. Connes along with Chamseddine [82] even introduced non-commutative geometry as a generalization to Riemannian geometry and obtained gauge theory as a companion to general relativity giving rise to a true geometric unification. In this framework parameters of the
of non-commutativity reduces the symmetry of the theory from $SO(3)$ to $SO(2)$ rendering the theory anisotropic. Hence, NCYM theory can serve as an interesting playground for exploring the effects of anisotropy. The presence of non-commutativity singles out a particular direction in space respecting a remnant $SO(2)$ symmetry in the transverse, non-commutative plane. In the context of heavy ion collisions, this particular direction may be thought of as the beam direction. Secondly, NCYM is interesting in its own right since it arises quite naturally in string theory \([83–85]\) and M-theory \([86]\) and it is of interest to see how non-commutativity affects the different observables. Thirdly, a consistent gauge theory can indeed be formulated in non-commutative space-time. Even though, so far, its existence has not been detected in low energy, one can not rule out the possibility that its effect may be manifested at extremely high energy scale, where the fabric of space-time itself may be modified. The experimental lower bound on the non-commutativity scale reported in the literature \([87]\) usually gives a very small effect and is hard to detect. So, it is desirable to search for its effect in alternative channels. High energy heavy ion collision offers one such arena and it may be worthwhile to look whether it can provide a better window for the effect of non-commutativity to be observed\(^1\). Driven by these motivations we perform a similar type of computation of Wilson loops in thermal NCYM plasma. The plan of the present chapter is as follows: In §4.2 we explain the dual string theory background that we shall use for performing the computations. §4.3 is devoted to the computation of the jet quenching parameter $\hat{q}_{\text{NCYM}}$ in strongly coupled NCYM plasma in $(3 + 1)$-dimensions using light-cone coordinates. The effects of non-commutativity upon $\hat{q}_{\text{NCYM}}$ are studied

\(^1\)One might wonder how would space-time non-commutativity appear in heavy ion collision in the first place? It is known that one of the mechanisms for the appearance of spatial non-commutativity is the presence of an intense magnetic field in the background. It has been shown in both analytic calculations \([88]\) and numerical simulations \([89]\) that such an intense magnetic field is indeed possible in heavy ion collision in RHIC (or in LHC). So, it may be quite relevant to consider such a possibility in the present context.
for both small and large values of $\theta$, the non-commutativity parameter, and attempt is made to connect the results to the recent collider data by giving some numerical estimates. In §4.4 we find out the potential of heavy quarkonia using holographic techniques with the velocity $v$ and the non-commutativity $\theta$ as parameters. The results are compared with the known commutative case. An analytic expression for the screening length is obtained in a restricted domain of the parameter space. The limit $v \to 1$ is considered from which the expression for the jet quenching parameter $\hat{q}_{\text{NCYM}}$ is extracted. Finally, we conclude in §4.5 with a summary of the results obtained.

### 4.2 Gravity Dual to Thermal NCYM Plasma

A particular form of anisotropy is manifested in non-commutative gauge theories. In this chapter we consider the 4-dimensional maximally supersymmetric $SU(N_c)$ Yang-Mills theory living on $\mathbb{R}^{1,1} \times \mathbb{R}^2_{\theta}$. The non-commutativity parameter is non-vanishing only in the $\mathbb{R}^2_{\theta}$-plane which defined by the Moyal algebra,

$$[x^2, x^3] = i\theta \quad (4.1)$$

where $x^2, x^3$ define the coordinates along the non-commutative gauge theory directions.

The gravity dual to NCYM theory is given by a particular decoupling limit $[84, 85]$ of non-extremal (D1,D3) bound state of type IIB string theory. (D1,D3) bound state $[90, 91]$ contains a non-zero $B$-field that becomes asymptotically very large in the decoupling limit and sources space-space non-commutativity $[83]$. The non-extremal (D1,D3) bound state solution of type IIB string theory is given by the following metric (in the string frame), the
dilaton $\phi$, the NS-NS $B$-field and the R-R form fields \[91\],

\[
d s^2 = H^{-\frac{1}{2}} \left( -f (d t)^2 + (d x^1)^2 + \frac{H}{F} (d x^2)^2 + (d x^3)^2 \right) + H^{\frac{1}{2}} \left( \frac{d r^2}{f} + r^2 d \Omega_5^2 \right)
\]

\[
e^{2\phi} = g_s^2 \frac{H}{F}, \quad B_{23} = \frac{\tan \alpha}{F}
\]

\[
A_{01} = \frac{1}{g_s} (H^{-1} - 1) \sin \alpha \coth \varphi, \quad A_{0123} = \frac{1}{g_s} \frac{1 - H}{F} \cos \alpha \coth \varphi + \text{T.T.} \quad (4.2)
\]

where the various functions appearing above are,

\[
f = 1 - \frac{r_0^4}{r^4}, \quad H = 1 + \frac{r_0^4 \sinh^2 \varphi}{r^4}, \quad F = 1 + \frac{r_0^4 \cos^2 \alpha \sinh^2 \varphi}{r^4}. \quad (4.3)
\]

The D3-branes span $x^1, x^2$ and $x^3$ directions while the D1-branes lie along $x^1$. $\alpha$ measures the relative number of D1 and D3 branes through $\cos \alpha = N/\sqrt{N^2 + M^2}$, with $N$ being the number of D3-branes and $M$ the number of D1-branes per unit codimension two surface transverse to D1-branes \[92\]. $\varphi$ is the boost parameter, $r_0$ denotes the horizon of the non-extremal (D1,D3) bound state and $g_s$ is the string coupling constant. $A_{01}$ and $A_{0123}$ are R-R form fields corresponding to D1-brane and D3-brane respectively. T.T. denotes a term, involving transverse part of the brane to make the field-strength self-dual, whose explicit form is not required for our discussion. $B_{23}$ is the NS-NS form responsible for the appearance of non-commutativity in the decoupling limit. The NCYM decoupling limit is a low energy limit for which we zoom into the region \[84\],

\[
r_0 < r \sim r_0 \sqrt{\sinh \varphi \cos \alpha} \ll r_0 \sqrt{\sinh \varphi}. \quad (4.4)
\]

In this limit $\varphi$ is a large parameter and $\alpha$ is close to $\pi/2$ so that,
\[ H \approx \frac{r_0^4 \sinh^2 \varphi}{r^4}, \quad \frac{H}{F} \approx \frac{1}{\cos^2 \alpha (1 + a^4 r^4)} \equiv \frac{h}{\cos^2 \alpha} \] (4.5)

where we have defined
\[ h \equiv \frac{1}{1 + a^4 r^4}, \quad \text{with} \quad a^4 \equiv \frac{1}{r_0^4 \sinh^2 \varphi \cos^2 \alpha}. \] (4.6)

From Eq. 4.2 we notice that asymptotically the \( B \)-field becomes very large in the decoupling limit. The non-vanishing component of the \( B \)-field is \( B_{23} \) which gives rise to a magnetic field in the D3-brane world-volume and is responsible for making \( x^2 \) and \( x^3 \) directions non-commutative \[93\]. Using Eq. 4.5, we rewrite the metric in Eq. 4.2 as,
\[ ds^2 = \frac{r^2}{r_0^2 \sinh \varphi} (-f dt^2 + (dx^1)^2 + h [(dx^2)^2 + (dx^3)^2]) + \frac{r_0^2 \sinh \varphi}{r^2} \left( \frac{dr^2}{f} + r^2 d\Omega_5^2 \right). \] (4.7)

where we have scaled \( x^{2,3} \to \cos \alpha x^{2,3} \). The metric along with the other fields (Eq. 4.2) in the decoupling limit is the gravity dual of \((3 + 1)\)-dimensional thermal NCYM theory.

Before proceeding further, let us also make some comments about the geometry. First note that if we set \( r_0 = 0 \), the geometry reduces to the familiar \( AdS_5 \times S^5 \) case. With some hindsight let us also note that unlike the \( AdS_5 \) case, now the boundary (ultra-violet) is not located at \( r \to \infty \) but rather at \( r = r_0 \Lambda \) which is taken to be very large but finite. When we send \( \Lambda \to \infty \), we have \( h \to 0 \) and the geometry degenerates. Hence, we need to impose \( r < r_0 \Lambda \). Also note that the background above can be obtained as a chain of T-duality transformations on the \( AdS_5 \times S^5 \) geometry. The non-trivial \( B \)-field and the dilaton are generated due to this sequence of T-duality transformations. Thus, one can view the \( \{x^2, x^3\} \) directions as a 2-torus \( T^2_\theta \cong \mathbb{R}_\theta^2 / \mathbb{Z}_2 \). The limit \( \Lambda \to \infty \) can thus be considered as a degeneration of this 2-torus.
4.3 Jet Quenching Parameter in Thermal NCYM Plasma

In this section we first compute the jet quenching parameter, which measures the radiative energy loss of an energetic parton, in NCYM plasma. We have already seen in the previous chapter, that this is furnished by the expectation value of a light-like Wilson loop. Hence, it proves convenient to recast the space-time metric (Eq. 4.7) in light-cone coordinates,

\[
\frac{r^2}{r_0^2 \sinh \varphi} \left[ -(1 + f) dx^+ dx^- + \frac{1}{2} (1 - f) \left( (dx^+)^2 + (dx^-)^2 \right) 
+ h \left[ (dx^2)^2 + (dx^3)^2 \right] \right] + \frac{r_0^2 \sinh \varphi dr^2}{f} + \frac{r_0^2 \sinh \varphi d\Omega_5^2}
\equiv G_{MN} dx^M dx^N \tag{4.8}
\]

where we have defined \( x^\pm = (t \pm x^1) / \sqrt{2} \).

By the AdS/CFT dictionary, the light-like Wilson loop is related via Eq. 3.46 to the extremized action \( S(C) \) of the string world-sheet \( \Sigma \) whose boundary \( \partial \Sigma \) is the mentioned loop \( C \) \cite{74, 75}. The Nambu-Goto action is easily calculated from Eqs. 2.10 and 2.11 with \( G_{MN} \) obtained from Eq. 4.8. We set \( \tau = x^- \) and \( \sigma = x^2 \). The length of the rectangular loop \( C \) along \( x^2 \) and \( x^- \) are \( L \) and \( L^- \) respectively and we assume \( L^- \gg L \). As a result the surface is invariant under \( \tau \)-translation and we have \( x^M(\tau, \sigma) = x^M(\sigma) \). Furthermore, the Wilson loop lies at \( x^+ = \) constant and \( x^3 = \) constant. Note that one of the sides of the rectangular Wilson loop is chosen along a non-commutative direction \((x^2)\) so that \( \hat{q}_{\text{NCYM}} \) evaluated from this Wilson loop will carry the effect of non-commutativity. The radial coordinate \( r(\sigma) \) gives the string embedding and we impose the condition that the world-sheet has \( C \) as its boundary, i.e., \( r(\pm L/2) = r_0 \Lambda \), for some finite \( \Lambda \). The configuration is shown clearly in Fig. 4.1. The action (Eq. 2.11) now reduces to,
\[ S = \frac{\sqrt{2} L}{2\pi\alpha'\sinh\varphi} \int_0^{L/2} d\sigma \left[ \frac{1}{1 + a^4 r^4} + \frac{r_0^4 \sinh^2 \varphi (r')^2}{r^4 - r_0^4} \right]^{\frac{1}{2}} \]  

(4.9)

where \( r' = \partial_{\sigma} r \). Defining new dimensionless variables \( y = r/r_0 \), \( \tilde{\sigma} = \sigma/(r_0 \sinh \varphi) \) and \( \ell = L/(r_0 \sinh \varphi) \), we can rewrite the action as,

\[ S = \frac{\sqrt{2} L - r_0}{2\pi\alpha'} \int_0^{\ell/2} d\tilde{\sigma} \left[ \frac{1}{1 + a^4 r_0^4 y^4} + \frac{(y')^2}{y^4 - 1} \right]^{\frac{1}{2}}. \]  

(4.10)

(Note that we have omitted the ‘tilde’ from \( \sigma \)) from which equation of motion follows,

\[ y' = \left[ 1 - q_0^2 (1 + a^4 r_0^4 y^4) \right]^{\frac{1}{2}} \frac{\sqrt{y^4 - 1}}{q_0 (1 + a^4 r_0^4 y^4)} \]  

(4.11)

where \( q_0 \) is an integration constant. From the first factor in Eq. 4.11 we have \( q_0 < 1/(1 + a^4 r_0^4 y^4)^{\frac{1}{2}} \) for all values of \( y \). In fact, \( q_0 \) has more stringent restriction to be mentioned later.
4.3. JET QUENCHING PARAMETER IN THERMAL NCYM PLASMA

The above equation has a solution\(^5\) where \(y\) starts from \(\Lambda\) coming all the way down to the turning point at \(y = 1\) with \(y' = 0\) and goes back again to \(\Lambda\). Integration of Eq. 4.11 yields,

\[
\ell = 2 \int_0^{\ell/2} d\sigma = 2q_0 \int_1^\Lambda dy \frac{1 + a^4 r_0^4 y^4}{\sqrt{(y^4 - 1)} [1 - q_0^2 (1 + a^4 r_0^4 y^4)]}.
\]

(4.12)

Since \(\ell = L/(r_0 \sinh \varphi)\) is very small compared to any other length scale of the problem, it implies from Eq. 4.12 that \(q_0\) must be very small, i.e., \(q_0 \ll 1/\sqrt{1 + a^4 r_0^4 \Lambda^4}\) and so, we can expand Eq. 4.12 in powers of \(q_0\) and from there we formally obtain its value as,

\[
q_0 = \frac{\ell}{2} \left[ \int_1^\Lambda dy y^4 \frac{1 + a^4 r_0^4 y^4}{\sqrt{y^4 - 1}} \right]^{-1}.
\]

(4.13)

Substituting Eq. 4.11 in Eq. 4.10 and expanding in powers of \(q_0\), we obtain

\[
S - S_0 = \frac{\sqrt{2L} - r_0 q_0^2}{4\pi \alpha'} \int_1^\Lambda dy \frac{1 + a^4 r_0^4 y^4}{\sqrt{y^4 - 1}} = \frac{\sqrt{2L} - r_0 \ell^2}{16\pi \alpha'} \left[ \int_1^\Lambda dy y^4 \frac{1 + a^4 r_0^4 y^4}{\sqrt{y^4 - 1}} \right]^{-1}
\]

(4.14)

where use has been made Eq. 4.13. \(S_0\) denotes the action for the world-sheet of two free strings (or the self-energy of the quark-antiquark pair). The integral in square brackets in Eq. 4.14 diverges if we take the boundary (\(\Lambda\)) where the NCYM theory lives, to \(\infty\). The evaluation of the action here differs from the commutative case. In the commutative version the action, after subtracting the self-energy of the quarks, becomes finite. This is evident if we put \(a^4 r_0^4\), which is a measure of non-commutativity (to be discussed later), to zero. However, for the non-commutative case, the action in Eq. 4.14 is still divergent if we put \(\Lambda \rightarrow \infty\). This is because in the non-commutative case the gauge theory does not

\(^5\)Here we discard another solution at UV corresponding to the surface at infinity. Since \(\hat{q}\) is a property of the thermal medium and does not describe UV physics, the surface at infinity is not physically relevant [75].
live at $r = \infty$, the usual boundary of the $AdS_5$-space, but rather lives on a surface which is at a finite distance. Instead of directly evaluating this distance we shall, instead, first evaluate the integral in Eq. 4.14 for finite $\Lambda$ and then subtract the divergent part obtained by letting $\Lambda \to \infty$. This way we regularize the integral in order to give any meaning to the extremized action\footnote{This is implicit in the quark-antiquark potential calculation done in [84] (see also [94]). There it was not possible to fix the position of the string at infinity since a small perturbation would change it violently. So, the calculation was performed by going to a conjugate ‘momentum’ variable and the energy was found to be divergent. A finite answer was obtained only after subtracting the divergent part. This, in turn, implies that the boundary screen is not at infinity but at a finite radial distance [94].}. Once the subtraction is made the NCYM theory can be considered to be living effectively at $r = \infty$. So, we first evaluate the integral for finite $\Lambda$ as follows,

$$
\int_{1}^{\Lambda} dy \frac{1 + a^4 r_0^4 y^4}{\sqrt{y^4 - 1}} = -\Lambda \sqrt{\Lambda^4 - 1} + \frac{1}{3} (3 + a^4 r_0^4) \sqrt{\pi \Gamma \left( \frac{5}{4} \right)} \Gamma \left( \frac{3}{4} \right) + \frac{1}{3 (3 + a^4 r_0^4)} \Lambda^3 \ {}_2 F_1 \left( -\frac{3}{4}, \frac{1}{2}, \frac{1}{4}; \frac{1}{\Lambda^4} \right) \quad (4.15)
$$

where ${}_2 F_1 (a, b; c; 1/\Lambda^4)$ is a hypergeometric function. For large $\Lambda$ it has an expansion

$$
{}_2 F_1 \left( a, b; c; \frac{1}{\Lambda^4} \right) = 1 + \frac{ab}{c} \frac{1}{\Lambda^4} + \frac{a(a + 1)b(b + 1)}{2c(c + 1)} \frac{1}{\Lambda^8} + \cdots \quad (4.16)
$$

Using the above expansion in Eq. 4.15 and finally setting $\Lambda \to \infty$, we find that apart from a finite part the above integral has a single divergent piece of the form $(a^4 r_0^4/3) \Lambda^3$ and all other terms vanish. So, removing the divergent part we get the regularized integral as,

$$
\int_{1}^{\infty} dy \frac{1 + a^4 r_0^4 y^4}{\sqrt{y^4 - 1}} = \left( 1 + \frac{a^4 r_0^4}{3} \right) \frac{\sqrt{\pi \Gamma \left( \frac{5}{4} \right)} \Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{5}{4} \right)}. \quad (4.17)
$$

\footnote{There are two ways to describe the finiteness of the integral in the action (Eq. 4.14). Either we take $\Lambda$ to be finite in which case the integral is obviously finite (in this case the integral can be evaluated only if we know the exact position of the boundary) or we take $\Lambda$ to be infinite and subtract the unique divergent part (as explicitly calculated below) of the integral and obtain a finite result. In the former case the boundary is at a finite radial distance, but for the latter case it is at infinity. But, effectively, they describe the same thing. Here we have adopted the second approach.}
4.3. JET QUENCHING PARAMETER IN THERMAL NCYM PLASMA

Substituting Eq. 4.17 in Eq. 4.14 yields,

\[ S - S_0 = \frac{\sqrt{2} L - r_0 \ell^2}{16\pi \alpha'} \frac{\Gamma \left( \frac{3}{4} \right)}{\sqrt{\pi} \Gamma \left( \frac{5}{4} \right)} \left( 1 + \frac{a^4 r_0^4}{3} \right)^{-1}. \] (4.18)

Now to extract \( \hat{q}_{\text{NCYM}} \) we invoke Eq. 3.46 whence, we obtain

\[ \hat{q}_{\text{NCYM}} = \frac{r_0}{\pi \alpha' r_0^2 \sinh^2 \varphi} \frac{\Gamma \left( \frac{3}{4} \right)}{\sqrt{\pi} \Gamma \left( \frac{5}{4} \right)} \left( 1 + \frac{a^4 r_0^4}{3} \right)^{-1}. \] (4.19)

where we have reinserted \( \ell = L / (r_0 \sinh \varphi) \). Since \( \hat{q}_{\text{NCYM}} \) is a gauge-theoretic quantity, we replace all the parameters of string theory appearing in Eq. 4.19 by the corresponding gauge theory parameters making use of the gauge/string dictionary [84]. The temperature of the non-extremal (D1,D3) bound state, which by the gauge/string duality is the temperature of the NCYM theory, can be obtained from Eq. 4.2,

\[ T = \frac{1}{\pi r_0 \cosh \varphi} \approx \frac{1}{\pi r_0 \sinh \varphi} \] (4.20)

where in the last expression we have used the fact that in the decoupling limit (Eq. 4.4), \( \varphi \) is large. Also from the charge of the D3-brane we have

\[ r_0^4 \sinh^2 \varphi = 2\hat{\lambda} \alpha'^2. \] (4.21)

Here \( \hat{\lambda} = g_Y^2 N_c \) is the 't Hooft coupling of NCYM theory and \( g_Y \) is the NCYM coupling.

The NCYM 't Hooft coupling is related to the ordinary 't Hooft coupling by \( \lambda = (\alpha'/\theta)\hat{\lambda} \). Here \( \theta \) is a finite parameter and in the decoupling limit as \( \alpha' \to 0 \), \( \hat{\lambda} \) remains finite. Using Eqs. 4.20 and 4.21 we obtain,
\[
\sinh \varphi = \frac{1}{\pi^2 \sqrt{2\lambda T^2 \alpha'}} \quad \text{and} \quad r_0 = \pi \sqrt{2\lambda T \alpha'}.
\] (4.22)

Also we have
\[
a^4 r_0^4 = \frac{1}{\sinh^2 \varphi \cos^2 \alpha} = \pi^4 (2\hat{\lambda}) T^4 \theta^2.
\] (4.23)

In the above we have used the decoupling limit \( \cos \alpha = \alpha' / \theta \) and as \( \alpha' \to 0, \alpha \to \pi / 2 \) as we mentioned earlier. Also, from Eq. 4.23 we notice that since \( a^2 r_0^2 \) is proportional to \( \theta \), therefore, \( ar_0 \) is a measure of non-commutativity. Now using Eqs. 4.22 and 4.23 in Eq. 4.19 we find that for small non-commutativity \( (a^2 r_0^2 \sim \theta \ll 1) \)
\[
\hat{q}_{\text{NCYM}} = \frac{\pi^2 \Gamma \left( \frac{3}{4} \right)}{\Gamma \left( \frac{7}{4} \right)} \sqrt{\hat{\lambda} T^3} \left[ 1 - \frac{\pi^4 \hat{\lambda} T^4 \theta^2}{3} + O(\theta^4) \right].
\] (4.24)

As expected, by setting \( \theta = 0 \), we recover the SYM result. In this case the NCYM 't Hooft coupling \( \hat{\lambda} \) equals the ordinary 't Hooft coupling \( \lambda \) and also in writing Eq. 4.24 we have replaced \( 2\hat{\lambda} \) by \( \hat{\lambda} \) to match the commutative results in [75]. This difference in a factor of 2 is just a convention as mentioned in [74]. In the presence of non-commutativity the jet quenching parameter gets reduced from its commutative value and the reduction gets enhanced with temperature as \( T^7 \), keeping other parameters fixed. This reduction in radiative energy loss for the non-commutative case can be intuitively understood as non-commutativity introduces a non-locality in space due to space uncertainty and there is no point-like interaction among the partons. So, the parton energy loss would be less in this case. We can try to estimate the correction (the second term in Eq. 4.24) arising due to non-commutativity from the experimental bound on the non-commutativity scale. In the literature various disparate experimental bounds on \( \theta \) have been obtained from various physical considerations. The bound on \( \theta \) has been claimed to be \( \sim (1-10 \text{ TeV})^{-2} \) in [87],
4.3. JET QUENCHING PARAMETER IN THERMAL NCYM PLASMA

whereas, it is $\sim \left(10^{12} - 10^{13}\right)$ GeV$^{-2}$ in [95] or even stronger $\sim \left(10^{15}\right)$ GeV$^{-2}$ in [96]. In theories of gravity it can be of the order of Planck scale $\sim \left(10^{19}\right)$ GeV$^{-2}$ [97]. It is clear that in all these cases except the first one there is no hope of getting a significant correction due to non-commutativity in collider experiments. At RHIC collision energy $\sim 200$ GeV where the temperature attained by QGP is $\sim 300$ MeV, even the first case does not give a significant correction ($\pi^4 \hat{\lambda} T^4 \theta^2 / 3 \sim 4.96 \times 10^{-12}$ taking $\hat{\lambda} = 6\pi$ and $T = 300$ MeV relevant for the Au-Au collision at RHIC and taking $\theta = 1$ TeV$^{-2}$) compared to the leading order term. At LHC where the collision energy would be much higher, the temperature of the QGP may rise and is expected to go up to 1-10 GeV. In that case the correction to the jet quenching due to non-commutativity can be estimated to be $\pi^4 \hat{\lambda} T^4 \theta^2 / 3 \sim (6.12 \times 10^{-6} - 6.12 \times 10^{-10})$, still too low to be detected. Conversely, to get a 10% correction on the jet quenching parameter due to non-commutativity, the temperature of the plasma would have to be $T \sim 200$ GeV. For large non-commutativity ($ar_0 \sim \sqrt{\theta} \gg 1$), on the other hand, the jet quenching parameter in Eq. 4.19 takes the form,

$$\hat{q}_{NCYM} = \frac{3 \Gamma \left(\frac{3}{4}\right)}{\pi^2 \Gamma \left(\frac{3}{4}\right)} \frac{1}{\sqrt{\hat{\lambda} T \theta^2}} \left[1 - \frac{3}{\pi^4 \hat{\lambda} T^4 \theta^2} + \mathcal{O} \left(\frac{1}{\theta^4}\right)\right].$$

(4.25)

We thus find that for large non-commutativity, the jet quenching varies inversely with temperature and also inversely with the square-root of the NCYM ’t Hooft coupling.

As discussed earlier, the presence of non-commutativity singles out the $x^1$ direction so that all the space coordinates are no longer on equal footing. Introduction of non-commutativity alters the Minkowskian boundary space-time $\mathbb{R}^{1,3}$ to $\mathbb{R}^{1,1} \times \mathbb{R}^2_\theta$ where the non-commutativity parameter $\theta$ is non-vanishing only on the Moyal plane $\mathbb{R}^2_\theta$. Thus, we can think of non-

\[**\text{We have taken the ’t Hooft coupling of the NCYM theory to be the same as that of the commutative theory, although there is no concrete reason for this. This is taken just for the estimate. Actually these two couplings are related as given earlier and as } \alpha' \rightarrow 0, \lambda \rightarrow 0, \text{ but } \hat{\lambda} \text{ remains finite. We have taken this finite value to be } 6\pi \text{ for better comparison.}\]
commutativity as the source of anisotropy in the gauge theory and treat \( \theta \) as a measure of anisotropy. Our results suggest that the introduction of anisotropy leads to a suppression in jet quenching whose direct fallout will be a reduction in the suppression of quarkonium states like \( J/\Psi \).

\section{4.4 \( Q\bar{Q} \) Potential in Thermal NCYM Plasma}

In this section we compute the quarkonium bound state potential in hot NCYM plasma in (3+1)-dimensions from gauge/string duality. Since we have already discussed the computation of \( Q\bar{Q} \) potential in the preceding chapter we shall be brief in our discussion here. Using a fundamental open string as a probe we consider its dynamics in the given background. The line joining the end-points of the string or the dipole lie along \( x^2 \), one of the non-commutative directions and move along \( x^1 \) with a velocity \( v \) where \( 0 < v < 1 \) \( ^\dagger\dagger \). We boost to the rest frame \((t', x^1')\) of the dipole through the transformation,

\[
\begin{align*}
    dt & = \cosh \eta dt' - \sinh \eta dx^1' \\
    dx^1 & = -\sinh \eta dt' + \cosh \eta dx^1'.
\end{align*}
\]

The rectangular Wilson loop lies along \( t' \) and \( x^2 \) directions and we denote the lengths along those directions as \( T \) and \( L \) respectively. Eq. 4.7 written in terms of the boosted

\( ^\dagger\dagger \) There are various other possibilities one can consider, for example, the dipole lies along the commutative direction \( x^1 \) and moves along one of the non-commutative directions \( x^2 \) (say) or the dipole lies along one of the non-commutative directions \( x^2 \) and moves along the other non-commutative direction \( x^3 \). The dipole can even have an arbitrary orientation with respect to its motion and the motion can also be in arbitrary direction in the mixed commutative-non-commutative boundary. Here we consider only the simplest case to see the non-commutative effect.
coordinates assumes the form,

\[
ds^2 = -A(r)dt^2 - 2B(r)dtdx^1 + C(r)(dx^1)^2 + \frac{r^2h}{r_0^2 \sinh \varphi} [(dx^2)^2 + (dx^3)^2] \\
+ \frac{r_0^2 \sinh \varphi}{r^2} dr^2 + r_0^2 \sinh \varphi d\Omega^2
\]

\[\equiv \tilde{G}_{MN} dx^M dx^N \quad (4.27)\]

where

\[
A(r) = \frac{r^2}{r_0^2 \sinh \varphi} \left(1 - \frac{r_0^4 \cosh^2 \eta}{r^4}\right),
\]

\[
B(r) = \frac{r_0^2 \sinh \eta \cosh \eta}{r^2 \sinh \varphi},
\]

\[
C(r) = \frac{r^2}{r_0^2 \sinh \varphi} \left(1 + \frac{r_0^4 \sinh^2 \eta}{r^4}\right). \quad (4.28)
\]

Note that since we will be using the ‘primed’ coordinates from now on, we have dropped
the prime for simplicity. Using the space-time metric defined in Eq. 4.27 we evaluate the
Nambu-Goto action employing the static gauge \(\tau = t, \sigma = x^2\), where \(-L/2 \leq x^2 \leq L/2\)
and \(r = r(\sigma), x^1(\sigma), x^3(\sigma) = \text{constant}\), to we get,

\[
S = \frac{T}{2\pi \alpha'} \int_{-L/2}^{L/2} d\sigma \left[ A(r) \left( \frac{r^2h}{r_0^2 \sinh \varphi} + \frac{r_0^2 \sinh \varphi (\partial_r r)^2}{f} \right) \right]^{\frac{1}{2}} \quad (4.29)
\]

with \(A(r)\) as given in Eq. 4.28. Introducing the dimensionless quantities \(y = r/r_0, \tilde{\sigma} = \sigma/(r_0 \sinh \varphi)\) and \(\ell = L/(r_0 \sinh \varphi)\), Eq. 4.29 can be rewritten as,

\[
S = \frac{T r_0}{\pi \alpha'} \int_0^{\ell/2} d\sigma \mathcal{L} = \mathcal{T} \mathcal{T} \sqrt{\lambda} \int_0^{\ell/2} d\sigma \mathcal{L} \quad (4.30)
\]

where
\[
\mathcal{L} = \sqrt{\left( y^4 - \cosh^2 \eta \right) \left( \frac{1}{1 + a^4 r_0^4 y^4} + \frac{y'^2}{y^4 - 1} \right)}. \quad (4.31)
\]

We shall, henceforth, not use the ‘tilde’ on \( \tilde{\sigma} \) anymore. Here \( y' \equiv \frac{dy}{d\sigma} \) and we have used the fact that \( y \) is an even function of \( \sigma \) by symmetry. In writing the second expression in Eq. 4.30 we have made use of the standard gauge/string relations \([84, 85]\), given in Eqs. 4.20, 4.21 and 4.23 now with \( \hat{\lambda} \) replaced by \( 2\hat{\lambda} \). To find the string profile we will compute \( y(\sigma) \) by extremizing the action in Eq. 4.30. Now since the Lagrangian density in Eq. 4.30 does not explicitly depend on \( \sigma \), we have the following constant of motion,

\[
H = \mathcal{L} - y' \frac{\partial \mathcal{L}}{\partial y'} = \frac{y^4 - \cosh^2 \eta}{\left( 1 + a^4 r_0^4 y^4 \right) \sqrt{\left( y^4 - \cosh^2 \eta \right) \left( \frac{1}{1 + a^4 r_0^4 y^4} + \frac{y'^2}{y^4 - 1} \right)}} = q = \text{constant}. \quad (4.32)
\]

As in the commutative theory \([74]\) discussed in chapter 3 we will consider two different regimes: (a) we take \( \sqrt{\cosh \eta} < \Lambda \) and then take \( \Lambda \rightarrow \infty \). The rapidity in this case remains finite, the Wilson loop is time-like and the action is real. We shall compute the \( Q^- \bar{Q} \) potential in this case and also provide an expression of the screening length in a specific case. (b) we take \( \sqrt{\cosh \eta} > \Lambda \) and then take \( \eta \rightarrow \infty \), keeping \( \Lambda \) finite. The Wilson loop in this case is light-like and the action is imaginary. We will take \( \Lambda \rightarrow \infty \) in the end to obtain an expression for \( \hat{q}_{\text{NCYM}} \) in hot NCYM plasma. As we shall shortly, this will match with the expression for the jet quenching parameter found out in the previous section. We consider case (a) in this section and postpone the discussion of case (b) to the next section.

When \( \sqrt{\cosh \eta} < \Lambda \), the action would be real and from Eq. 4.32 \( y' \) can be solved as,

\[
y' = \frac{\sqrt{1 - a^4 r_0^4 q^2}}{q(1 + a^4 r_0^4 y^4)} \sqrt{(y^4 - 1) \left( y^4 - y_0^4 \right)}, \quad (4.33)
\]
where \( y_c^4 = (\cosh^2 \eta + q^2)/(1 - a^4 r_0^4 q^2) > 1 \) denotes the larger turning point where \( y' \) vanishes. Integrating Eq. 4.33 we obtain,

\[
2 \int_0^{\ell/2} d\sigma = \ell(q) = \frac{2q}{\sqrt{1 - a^4 r_0^4 q^2}} \int_{y_c}^{\Lambda} \frac{1 + a^4 r_0^4 y^4}{\sqrt{(y^4 - 1)(y^4 - y_c^4)}} dy.
\]

Observe that if we naively take \( \Lambda \), where the boundary theory is supposed to live, to \( \infty \), the above integral diverges. Here \( \ell \) is related to the dipole length \( L \) by \( \ell = L/(r_0 \sinh \varphi) = \pi LT \) and so the divergence in \( \ell(q) \) is physically meaningless. Note that \( \ell(q) \) in the commutative theory is indeed finite as can be seen from Eq. 4.34 by putting \( a^2 r_0^2 = 0 \). However, for the non-commutative case \( \ell(q) \) is divergent if we take \( \Lambda \to \infty \). The reason why this divergence crops up is the same as that discussed in §4.3 in the discussion of \( \hat{q}_{\text{NCYM}} \).

In the context of Wilson loop calculation, it has been noticed before [98] that the string endpoints for a static string can not be fixed at a finite length at \( \Lambda \to \infty \) in a non-commutative theory. Therefore, the dipole length \( L \) indeed diverges. The reason for this divergence has been argued to be the non-local interaction between the \( Q\bar{Q} \) pair in a magnetic field [99]. To be precise, the interaction point in terms of the center-of-mass coordinate gets shifted by a momentum-dependent term. Thus if the only non-zero component of the \( B \)-field is \( B_{23} \), as in our case, then by placing the dipole along \( x^2 \), it automatically gets a momentum along \( x^3 \). So, if we keep the dipole static along \( x^3 \), the length will diverge at infinity. To compensate the momentum along \( x^3 \), the dipole must move along \( x^3 \) with a particular velocity [100]. In that case, the end-points of the string can be fixed at a finite length on the boundary at infinity and thus the divergence in the dipole length gets removed.

In the following, we, however, take recourse to the same strategy as in the preceding section in the evaluation of \( \hat{q}_{\text{NCYM}} \) to get rid of the divergence, i.e., we first perform the integration for finite \( \Lambda \) and then identify the unique divergent part by allowing \( \Lambda \to \infty \). Then we
subtract this divergent piece from the integral to cure it of the divergence. After regularization, the non-commutative theory may be thought of as living at $\Lambda = \infty$. By inspection it can be seen that the divergent piece is of the form $2qa^4r_0^4\Lambda / \sqrt{1 - a^4r_0^4q^2}$. Removing the divergent part the finite $\ell(q)$ can be written as,

$$\ell(q) = \frac{2q}{\sqrt{1 - a^4r_0^4q^2}} \left[ \int_{y_c}^{\Lambda} \frac{1 + a^4r_0^4y^4}{\sqrt{(y^4 - 1)(y^4 - y_c^4)}} dy - a^4r_0^4\Lambda \right]_{\Lambda \to \infty}. \quad (4.35)$$

The above equation, therefore, gives us the $Q-\bar{Q}$ separation $L(q) = \ell(q)/(\pi T)$ of the bound state as a function of the constant of motion $q$.

Substituting $y'$ from Eq. 4.33 into the action (Eq. 4.30) results in,

$$S(\ell) = \frac{TT\sqrt{\lambda}}{\sqrt{1 - a^4r_0^4q^2}} \int_{y_c}^{\Lambda} \frac{y^4 - \cosh^2\eta}{\sqrt{(y^4 - 1)(y^4 - y_c^4)}} dy. \quad (4.36)$$

As in the commutative case, this action is divergent as it contains contribution from the $Q-\bar{Q}$ self-energy $S_0$

$$S_0 = \frac{TT\sqrt{\lambda}}{1 - a^4r_0^4q^2} \int_{y_c}^{\Lambda} dy. \quad (4.37)$$

So, subtracting $S_0$ from $S(\ell)$ we get,

$$S(\ell) - S_0 = \frac{TT\sqrt{\lambda}}{\sqrt{1 - a^4r_0^4q^2}} \left[ \int_{y_c}^{\Lambda} dy \left\{ \frac{y^4 - \cosh^2\eta}{\sqrt{(y^4 - 1)(y^4 - y_c^4)}} - \sqrt{1 - a^4r_0^4q^2} \left( \frac{y^4}{y_c^4} - 1 \right) \right\} \right]. \quad (4.38)$$

However, owing to the fact that the NCYM theory does not live at $\Lambda \to \infty$ this action still diverges after regularization. Hence, to obtain a well-behaved action, we shall subtract the divergent term $S_{\text{div}}$ from Eq. 4.38 and then take $\Lambda \to \infty$. Doing that we find the finite
quark-antiquark potential in the quarkonium bound state as,

\[
E(\ell) = \frac{S - S_0 - S_{\text{div}}}{T} = \frac{T \sqrt{\Lambda}}{\sqrt{1 - a^4 r_0^4 q^2}} \left[ \int_{y_c}^{\Lambda} dy \left\{ \frac{y^4 - \cosh^2 \eta}{\sqrt{(y^4 - 1)(y^4 - y_c^4)}} - \sqrt{1 - a^4 r_0^4 q^2} \right\} - \sqrt{1 - a^4 r_0^4 q^2 (y_c - 1)} - \left(1 - \sqrt{1 - a^4 r_0^4 q^2}\right) \Lambda \right] \bigg|_{\Lambda \to \infty} \tag{4.39}
\]

where in the above \( S_{\text{div}} = (1 - \sqrt{1 - a^4 r_0^4 q^2}) \Lambda \) with \( \Lambda \to \infty \). Here too it is not possible to perform the integration in Eq. 4.39 in a closed form. So, we will obtain the \( Q-Q \) potential numerically. We first plot \( \ell(q) - q \) using Eq. 4.35 and use it to plot \( E(\ell) - \ell \) from Eq. 4.39 at different fixed values of \( \eta \) and \( ar_0 \). In the ensuing subsection we provide the various plots along with a discussion of the results.

### 4.4.1 Plots and discussion of the results

In this subsection we give and discuss the various plots of \( Q-Q \) separation \( \ell(q) \) as a function of constant of motion \( q \) and the velocity-dependent \( Q-Q \) potential \( E(\ell) \) as a function of the \( Q-Q \) separation length \( \ell \) for various values of the rapidity \( \eta \) as well as the non-commutativity parameter \( ar_0 \sim \sqrt{\theta} \).

In Figures 4.2-4.5, \( \eta \) is fixed at \( \eta = 0.1 \). In Figures 4.2 and 4.3 \( ar_0 \) takes small values starting from 0 (where there is no non-commutativity) to 1.0, whereas, in Figures 4.4 and 4.5 \( ar_0 \) takes fairly large values starting from 2.0 to 10.0. The main difference between the commutative results and the non-commutative results is that in the former case the constant of motion \( q \) can take arbitrarily large values, but in the latter case \( q \) can not exceed certain finite value (\( q_{\text{max}} \)) since beyond this value the \( Q-Q \) separation \( \ell(q) \) becomes negative which is unphysical. The reason behind this cut-off is the regularization of the integral performed...
Figure 4.2: $Q$-$\bar{Q}$ separation $\ell(q)$ as a function of $q$ for different values of $ar_0$ at $\eta = 0.1$

in Eq. 4.35 - the last term in Eq. 4.35 is subtracted to make $\ell(q)$ finite as $\Lambda \to \infty$. However, as $q$ increases, $y_c$ increases which makes the last term dominate over the integral and therefore, $\ell(q)$ becomes negative. Thus this effect is due to the non-commutativity of the underlying boundary theory. We see from Figure 4.2 that as $ar_0$ increases, $\ell(q)$ curve deviates more and more from the commutative curve, the maximum value of $\ell$, i.e., $\ell_{max}$ falls and the peak shifts towards the left (i.e., the maximum occurs at a smaller value of $q$). In particular, the deviation from the commutative case becomes more pronounced after $\ell_{max}$ is reached. However this feature continues upto certain value of $ar_0 \sim 2.0$ and as it is increased further (see Figure 4.4) the $\ell(q)$ curve now deviates more from the commutative case throughout the allowed range of $q$, but the maximum value, $\ell_{max}$, again starts rising and the peak as before shifts further towards left, i.e., towards smaller values of $q$. Figures 4.3 and 4.5 show the plot of the velocity-dependent $Q$-$\bar{Q}$ potential $E(\ell)$ with the $Q$-$\bar{Q}$ separation length $\ell$ for $\eta = 0.1$ with various values of $ar_0$. Each curve has two
4.4. $Q$-$\bar{Q}$ POTENTIAL IN THERMAL NCYM PLASMA

branches corresponding to the two dipole solutions obtained in Figures 4.2 and 4.4. The slight deviation of $\ell(q)$ from the commutative case for small values of $q$, i.e., below the value of $q$ corresponding to $\ell(q) = \ell_{max}$, (see Figure 4.2) is reflected in the fact that in Figure 4.3 the upper branches almost merge with the commutative counterpart whereas the greater deviation in $\ell(q)$ after $\ell_{max}$ is reached leads to a rise in the lower branch of the $E(\ell)$ curve from the commutative case in Figure 4.3. However, as the non-commutativity parameter is increased the overall deviation (particularly in the lower branch) of the $E(\ell)$ curve is more pronounced from its commutative value. In contrast, in Figure 4.5 as the non-commutativity parameter is further increased, $E(\ell)$, in general, dips slightly for both the branches. The feature that the screening length ($\sim \ell_{max}$) initially drops and then rises and correspondingly the lower branch of the potential $E(\ell)$ rises and then drops as we go on increasing $ar_0$ (with the transition occurring at around $a_lr_0 \sim ar_0 = 2.0$), occurs only

![Graph](image)

Figure 4.3: Normalized $Q$-$\bar{Q}$ potential $E(\ell)$ as a function of $\ell$ for the same set of $ar_0$ (as in Figure 4.2) at $\eta = 0.1$
for the smaller value of the rapidity, $\eta = 0.1$. There exists a critical value of $\eta = \eta_c$ above which this transition is not observed. As the rapidity becomes higher than $\eta_c$, its effect starts to dominate and the transition (from falling $\ell_{\text{max}}$ to rising $\ell_{\text{max}}$ as the non-commutativity parameter is increased) is suppressed so that now the screening length continuously drops and the lower branch of the $Q-\bar{Q}$ potential continuously rises. We have seen this to happen for $\eta = 0.5$ and $\eta = 1.0$. That is why we have given those plots only for the smaller values of $ar_0$ in Figures 4.6-4.9. Although the details of these plots are different, the general features remain very similar to those of $\eta = 0.1$ (for small non-commutativity) and hence, we refrain from an elaborate discussion for these cases. Therefore, we shall take the $\eta = 0.1$ case as the prototype and discuss the generic features of the plots. We see from the plots in Figure 4.2 that $\ell_{\text{max}}$ drops as the non-commutativity is increased. This implies that with increase of non-commutativity, the quarkonia bound states will be more vulnerable to dissociation with the consequence that there will be an increase in $J/\Psi$
suppression [101]. On the other hand, from the plots in Figure 4.3 we observe that with increase of non-commutativity the $Q$-$\bar{Q}$ potential rises (the lower curves which correspond to the stable states) in value which means that the quark and the antiquark will be more and more loosely bound and eventually there will be no bound state formation. This may be expected since there is a fuzziness in the direction of the dipole due to non-commutativity. However, in Figure 4.4, when the non-commutativity is large (and the rapidity remains small) we see that around $a_{r_0} = 2.0$, $\ell_{\text{max}}$ starts rising again and so more dipoles can form, but from Figure 4.5 we see (from the lower curve) that in this case the quark-antiquark pair will be very very loosely bound. This does not happen when the rapidity is large (we have not shown the plots for this case with large non-commutativity). In contrast to Figures 4.2-4.9, where we plot $\ell(q)$-$q$ and $E(\ell)$-$\ell$ for fixed values of $\eta$ but with varying values of $a_{r_0}$, in Figures 4.10-4.13, we plot the same functions for fixed values of $a_{r_0}$, but varying

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.5.png}
\caption{Normalized $Q$-$\bar{Q}$ potential $E(\ell)$ as a function of $\ell$ for the same set of $a_{r_0}$ (as in Figure 4.4) at $\eta = 0.1$}
\end{figure}
values of $\eta$. In Figures 4.10 and 4.11, $ar_0$ is fixed to a small value 0.1 whereas in Figures 4.12 and 4.13, it is fixed to a large value 10.0. In Figure 4.10 we find that as the rapidity increases the screening length decreases (which means there will be less dipole formation i.e., more $J/\Psi$ suppression) and the peaks shift towards right, i.e., to a larger value of $q$. This is expected as in the commutative case also there is a decrease in screening length with increase in rapidity. Further note that for large $q$, $\ell(q)$ becomes independent of $q$. This is also manifest in the $E(\ell)-\ell$ plots in Figure 4.11, i.e., the lower branches of the curves merge. Contrast this to the case when $ar_0$ is changed (but still kept small) keeping $\eta$ fixed, when the lower part of the $\ell(q)$ curve (i.e., small $q$) does not exhibit significant deviation, the peak shifts towards left and the upper branch of the $E(\ell)$ curves merge. So we can think of $\eta$ and $ar_0$ as sort of having opposite effects. These features are also evident for large values of $ar_0$ given in Figures 4.12 and 4.13. In Figure 4.12 as the rapidity $\eta$ increases the screening length decreases and the peaks shift towards right, but since now the scale of the
\section*{4.4. $Q$-$\bar{Q}$ POTENTIAL IN THERMAL NCYM PLASMA}

$q$-axis is very much enlarged this is not much visible (this is also due to the fact that $\eta$ now changes by a very small amount). The independence of $\ell(q)$ with $q$ for larger values of $q$ is not evident in this case due to the differences in scale along the $q$-axis in Figures 4.10 and 4.12. Unlike in Figure 4.11, the lower branches of the $E(\ell)$ curves do not merge for different values of $\eta$ as is evident from Figure 4.13. However, the spread is again due to the enlarged (compared to Figure 4.11) scale of the $E(\ell)$-axis which is chosen to show the two branches of the $E(\ell)$ curve distinctly.

So let us summarize our result about the bound state quarkonium potential in NCYM plasma at finite temperature. For our purpose, we can treat the non-commutativity parameter $\theta$ as a measure of anisotropy in the gauge theory. We can then distinguish between different regimes. Firstly, there exists a critical value $\eta_c$ below which the screening length and the $Q$-$\bar{Q}$ potential exhibits interesting phenomena. For $\eta < \eta_c$ there is again two differ-

![Figure 4.7: Normalized $Q$-$\bar{Q}$ potential $E(\ell)$ as a function of $\ell$ for the same set of $ar_0$ (as in Figure 4.6) at $\eta = 0.5$](image-url)
ent regimes depending upon $a r_0 \gtrsim a_t r_0$. For $a r_0 \lessgtr a_t r_0$ as the value of $a r_0$ is increased, the screening length steadily decreases and correspondingly, the potential rises showing that the bound state becomes more and more unstable. In the opposite regime where $a r_0 > a_t r_0$ as the non-commutativity parameter is increased, the screening now witnesses a marked rise while, on the other hand, the potential falls slightly with rising non-commutativity. But, the potential becomes almost flat so that the quark-antiquark pair is now very very loosely bound. On the other hand, in the regime $\eta > \eta_c$ the effect of rapidity dominates over that of anisotropy and as the anisotropy parameter rises, the screening length continually drops and the bound states become progressively loosely bound. Thus, the bottom line of our analysis is that for quarkonia with high momentum, the generic effect of anisotropy is to make the bound state more susceptible to melting which leads to suppression in the yield of various quarkonia that is measured in the collider experiments.
4.4.2 Screening length in a special case

In this section we find out an analytical expression for the screening length $\ell_{\text{max}}$ in a restricted regime of the parameter space of the rapidity $\eta$ and the non-commutativity parameter $\theta$. The expression for the regularized $Q-\bar{Q}$ separation length $\ell(q)$ as a function of the constant of motion $q$ is given in Eq. 4.35. However, as we have mentioned, it is not possible to perform the integration, in general, and give an exact analytic expression for $\ell(q)$ which compelled us to resort to numerical means to solve Eq. 4.35 and plot $\ell(q)$ against $q$ in the previous subsection. It must be realized that this has nothing to do with the non-commutativity of the underlying gauge theory and this happens also for the case of commutative theory. For the case of commutative theory it is possible to give an exact analytic expression of $\ell(q)$ only in the large velocity or large rapidity limit. Non-commutativity, on
the other hand, makes the analysis a little bit more involved and in this case it is possible to obtain the analytic expression only when the rapidity is large and the non-commutativity is small with the product remaining small. For large \( \eta \) or large \( y_c \), the expression for \( \ell(q) \) in Eq. 4.35 can be expanded as follows,

\[
\ell(q) = \left[ 2q \int_{y_c}^{\Lambda} \frac{1 + a^4 r_0^4 y^4}{y^2 \sqrt{y^4 - y_c^4}} dy + \frac{q}{\sqrt{1 - a^4 r_0^4 q^2}} \int_{y_c}^{\Lambda} \frac{1 + a^4 r_0^4 y^4}{y^6 \sqrt{y^4 - y_c^4}} dy 
\right.
\]

\[
+ \frac{3q}{4 \sqrt{1 - a^4 r_0^4 q^2}} \int_{y_c}^{\Lambda} \frac{1 + a^4 r_0^4 y^4}{y^{10} \sqrt{y^4 - y_c^4}} dy + \cdots - \frac{2qa^4 r_0^4}{\sqrt{1 - a^4 r_0^4 q^2}} \Lambda \left|_{\Lambda \to \infty} \right. \] (4.40)

When \( \Lambda \to \infty \), the above integrals can be evaluated and \( \ell(q) \) can be written as a series expansion in inverse powers of \( y_c \) as,
4.4. \( Q-\bar{Q} \) POTENTIAL IN THERMAL NCYM PLASMA

\[
\ell(q) = \frac{q\sqrt{\pi}y_c}{\sqrt{1 - a^4r_0^4q^2}} \left[ -2a^4r_0^4\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + (2 + a^4r_0^4)\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \frac{1}{y_c^4} \right. \\
+ \left. \left(1 + \frac{3}{4}a^4r_0^4\right)\frac{\Gamma\left(\frac{5}{4}\right)}{4\Gamma\left(\frac{3}{4}\right)} \frac{1}{y_c^8} + \cdots \right] \tag{4.41}
\]

By construction the divergent last term in Eq. 4.40 gets canceled with the divergent term in the first integral when \( \Lambda \to \infty \). The other integrals are convergent and makes the expression for \( \ell(q) \) finite. By taking the first three terms in the series we can obtain the values of \( q \) and \( y_c \) which maximize \( \ell(q) \) as,

\[
q^2 = 2\cosh^2\eta(1 - 15a^4r_0^4\cosh^2\eta) \\
y_c^4 = \frac{\cosh^2\eta + q^2}{(1 - a^4r_0^4q^2)} = 3\cosh^2\eta(1 - 8a^4r_0^4\cosh^2\eta). \tag{4.42}
\]

![Figure 4.11: Normalized Q-\( \bar{Q} \) potential \( E(\ell) \) as a function of \( \ell \) for the same set of \( \eta \) (as in Figure 4.10) at \( ar_0 = 0.1 \)]
In obtaining the above expressions we have assumed $a^4 r_0^4 \ll 1$ and $a^4 r_0^4 \cosh^2 \eta \ll 1$.

Using Eq. 4.42 we obtain the maximum value of $\ell$ up to next to leading order as,

$$
\ell_{\text{max}} = 2\sqrt{2} \pi \Gamma \left( \frac{3}{4} \right) \left( \frac{\eta}{\cosh^{1/2} \eta} \right) \left[ 1 - \frac{7}{2} a^4 r_0^4 \cosh^2 \eta + \cdots \right].
$$

By using Eqs. 4.20, 4.21 and 4.23 we can rewrite $\ell_{\text{max}}$ in Eq. 4.43 in terms of the gauge theory parameters as,

$$
\ell_{\text{max}} = 0.74333 (1 - v^2)^{1/4} \left[ 1 - \frac{7}{2} \frac{\hat{T}^4 \theta^2}{2} \frac{1}{1 - v^2} + \cdots \right].
$$

where we have used $\cosh \eta = \gamma = 1/\sqrt{1 - v^2}$, with $v$ being the velocity of the dipole.

In Eq. 4.44 the term outside the square bracket is the commutative result (when we put
4.4. Q-\bar{Q} POTENTIAL IN THERMAL NCYM PLASMA

\( \theta = 0 \) and represents the usual \( J/\Psi \) suppression of the high velocity \( Q-\bar{Q} \) pair produced in the QGP in the heavy ion collision observed in RHIC [70, 101]. However, we note that non-commutativity reduces this result due to the second term in the square bracket in Eq. 4.44. The quantity \( L_{\text{max}} = \ell_{\text{max}}/(\pi T) \) can be thought of as the screening length of the dipole since this is the maximum value of \( L \) beyond which we have two dissociated quark and antiquark or two disjoint world-sheet for which \( E(\ell) = 0 \). As the screening length gets smaller less and less dipoles will be created and there will be more suppression of quark-antiquark bound states like \( J/\Psi \). Non-commutativity makes the interaction between the quark and the antiquark weaker due to non-locality and that is the reason it makes the screening length shorter. Note that the velocity of the dipole has an opposite effect in the correction term due to non-commutativity, i.e., higher the velocity, lower would be the correction term due to non-commutativity. Also the correction term is more pronounced.

![Figure 4.13](image-url)  

**Figure 4.13:** Normalized \( Q-\bar{Q} \) potential \( E(\ell) \) as a function of \( \ell \) for the same set of \( \eta \) (as in Figure 4.12) at \( ar_0 = 10 \).
at higher temperature. We would also like to remark that non-commutativity gives a range for the temperature. Eq. 4.44 is valid when $a^4 r_0^4 \cosh^2 \eta \ll 1$ which, in turn, gives a range for the temperature as,

$$T \ll \left( \frac{1}{\pi^4 \lambda^2 (1 - v^2) \theta^2} \right)^{\frac{1}{4}}. \quad (4.45)$$

When the temperature is above this value the expansion in Eq. 4.43 will break down and the screening length will no longer be given by Eq. 4.44. In that case the screening length has to be computed in the opposite limit where $a^4 r_0^4 \cosh^2 \eta \gg 1$. However, in this limit we have not been able to write a closed form analytic expression for the screening length.

### 4.4.3 Jet quenching parameter - another look

In the previous section we alluded to the two different regimes in which we can compute the Wilson loops. In the regime discussed just now, the rapidity $\eta$ remains finite and $\sqrt{\cosh \eta} < \Lambda$. So, the velocity of the background is in the range $0 < v < 1$ and the Wilson loop is time-like. Here we discuss the other regime where $\sqrt{\cosh \eta} > \Lambda$ whence, the Wilson loop becomes light-like. Now we can recover the expression for the jet quenching parameter $\hat{q}_{\text{NCYM}}$ which we had already obtained in §4.3. Here we rederive the result just for the sake of completeness. This also provides us the excuse to be brief in our discussion here.

Note that as $\cosh^2 \eta$ is now greater than $\Lambda^4$, where $\Lambda$ is the upper limit of $y$, the factor $(y^4 - \cosh^2 \eta)$ appearing in the action (Eqs. 4.30 and 4.31) is negative and the action becomes imaginary. So, we rewrite the action in Eq. 4.30 as,

$$S = \frac{i T r_0}{\pi \alpha'} \int_0^{t/2} d\sigma L = i TT \sqrt{\lambda} \int_0^{t/2} d\sigma L \quad (4.46)$$
where
\[ L = \sqrt{(\cosh^2 \eta - y^4) \left( \frac{1}{1 + a^4 r_0^4 y^4} + \frac{y^2}{y^4 - 1} \right)}. \] (4.47)

which supplies the equation of motion
\[ y' = \frac{\sqrt{1 + a^4 r_0^4 q_0^2} \sqrt{(y^4 - 1)(y_m^4 - y^4)}}{q_0(1 + a^4 r_0^4 y^4)} \] (4.48)

where
\[ y_m^4 = \frac{\cosh^2 \eta - q_0^2}{1 + a^4 r_0^4 q_0^2} \] (4.49)

and \( q_0 \) is the constant of motion. On integration, Eq. 4.4.3 gives us,
\[ \ell = 2 \int_0^{\ell/2} d\sigma = \frac{2q_0}{\sqrt{1 + a^4 r_0^4 q_0^2}} \int_1^{\Lambda} \frac{1 + a^4 r_0^4 y^4}{\sqrt{(y^4 - 1)(y_m^4 - y^4)}} dy. \] (4.50)

Substituting the value of \( y' \) from Eq. 4.4.3 into the action (Eq. 4.47), we simplify it as,
\[ S(\ell) = \frac{iTT \sqrt{\lambda}}{\sqrt{1 + a^4 r_0^4 q_0^2}} \int_1^{\Lambda} \frac{\cosh^2 \eta - y^4}{\sqrt{(y^4 - 1)(y_m^4 - y^4)}} dy. \] (4.51)

Now since \( \ell \) is very small compared to other length scales in the theory from Eq. 4.50 it is evident that \( q_0 \) is also a small parameter whence one has
\[ q_0 = \frac{\ell \cosh \eta}{2} \left[ \int_1^{\Lambda} \frac{1 + a^4 r_0^4 y^4}{\sqrt{y^4 - 1}} dy \right]^{-1}. \] (4.52)

In this limit \( S(\ell) \) in Eq. 4.51 can be expanded as,
\[ S(\ell) = S^{(0)} + q_0^2 S^{(1)} + \mathcal{O}(q_0^4) \] (4.53)
where

\[ S^{(0)} = iTT\sqrt{\lambda} \int_1^\Lambda \frac{\sqrt{\cosh^2 \eta - y^4}}{\sqrt{y^4 - 1}} dy \]

\[ q_0^2 S^{(1)} = \frac{iTT\sqrt{\lambda}}{2} q_0^2 \int_1^\Lambda \frac{1 + a^4 r_0 y^4}{\sqrt{\cosh^2 \eta - y^4 \sqrt{y^4 - 1}}} dy. \]  

(4.54)

It can be shown [74,75] that as \( q_0 \to 0 \), \( S^{(0)} \) above is equal to \( S_0 \), the self-energy of the dissociated quark and antiquark or area of the two disjoint world-sheets. So, subtracting the self-energy we obtain the action as

\[ S - S_0 = q_0^2 S^{(1)} = iTT\sqrt{\lambda} \frac{\ell^2}{4} c\cosh \eta \left[ \int_1^\Lambda \frac{1 + a^4 r_0 y^4}{\sqrt{y^4 - 1}} dy \right]^{-1} \]  

(4.55)

where we have used Eq. 4.52 and have taken \( \eta \to \infty \). Now \( T \cosh \eta \) in Eq. 4.55 can be identified as \( L^-/\sqrt{2} \), where \( L^- \) is the length of the Wilson loop in the light-like direction.

Using the relation in Eq. 3.46, we get

\[ \hat{q}_{\text{NCYM}} = \pi^2 \sqrt{\lambda} T^3 \left[ \int_1^\Lambda \frac{1 + a^4 r_0 y^4}{\sqrt{y^4 - 1}} dy \right]^{-1}. \]  

(4.56)

As expected the above integral diverges as \( \Lambda \to \infty \) and hence, needs to be regularized. Here we just furnish the expression for the regularized integral,

\[ \int_1^\infty \frac{1 + a^4 r_0 y^4}{\sqrt{y^4 - 1}} dy = \left( 1 + \frac{a^4 r_0^4}{3} \right) a_3, \quad \text{with} \quad a_3 = \frac{\sqrt{\pi} \Gamma \left( \frac{5}{4} \right)}{\Gamma \left( \frac{3}{4} \right)}. \]  

(4.57)

Substituting Eq. 4.57 in Eq. 4.56 and expressing \( a^4 r_0^4 \) in terms of the gauge theory parameters from Eqs. 4.20, 4.21 and 4.23 we obtain,
\[ \hat{q}_{\text{NCYM}} = \frac{\pi \frac{3}{2} \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \sqrt{\lambda T^3} \left( 1 + \frac{\pi^4 T^4 \hat{\lambda} \theta^2}{3} \right)^{-1}. \]  

So, for small non-commutativity, \( \theta \ll 1 \), the jet quenching parameter is given as,

\[ \hat{q}_{\text{NCYM}} = \frac{\pi \frac{3}{2} \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{3}{2} \right)} \sqrt{\lambda T^3} \left( 1 - \frac{\pi^4 T^4 \hat{\lambda} \theta^2}{3} + O(\theta^4) \right) \]  

whereas, for large non-commutativity, \( \theta \gg 1 \), the jet quenching parameter takes the form,

\[ \hat{q}_{\text{NCYM}} = \frac{3 \Gamma \left( \frac{3}{2} \right)}{\pi \frac{3}{2} \Gamma \left( \frac{3}{2} \right)} \frac{1}{\sqrt{\lambda T \theta^2}} \left( 1 - \frac{3}{\pi^4 T^4 \hat{\lambda} \theta^2} + O\left( \frac{1}{\theta^4} \right) \right). \]  

This is in perfect agreement with our findings in §4.3. For small non-commutativity we have \( a^4 r_0^4 \ll 1 \), which yields a range for the temperature due to non-commutativity,

\[ T \ll \left( \frac{1}{\pi^4 \hat{\lambda} \theta^2} \right)^{\frac{1}{4}}. \]  

When the temperature is above this value, the jet quenching expression will no longer be given by Eq. 4.59. In that case we have to use the expression Eq. 4.60 which is valid when the temperature is given by the limit

\[ T \gg \left( \frac{1}{\pi^4 \hat{\lambda} \theta^2} \right)^{\frac{1}{4}}. \]  

### 4.5 Conclusion

In this concluding section let us recapitulate the results of this chapter. In this chapter we considered the first of our anisotropic models - the non-commutative Yang-Mills theory at
finite temperature. We took the non-commutativity parameter $\theta$ as a measure of anisotropy that breaks the $SO(3)$ symmetry of the gauge theory living on the Minkowski space $\mathbb{R}^{1,3}$ at the boundary to a $SO(2)$ symmetry in the non-commutative Moyal plane $\mathbb{R}^2_\theta$. The four-dimensional space-time now decomposes as $\mathbb{R}^{1,1} \times \mathbb{R}^2_\theta$. The presence of anisotropy thus singles out a particular direction, in this case $x^1$, in space which should have qualitative effects on the experimental observables in the heavy ion colliders. Translated into the language of the heavy ion colliders we thus have a special direction in the thermal medium, which we can reasonably take as the direction along which the collisions take place. We studied the propagation of heavy $Q\bar{Q}$ bound states in this anisotropic thermal medium.

First, we computed the jet quenching parameter and found out how it picks up corrections arising from a non-zero value of $\theta$. Then we considered two limits when $\theta$ is very small or very high and found analytical expression for the jet quenching parameter. We also explored any possibility of whether any signature of non-commutativity can be detected in the present collider experiments. Using some benchmark values of $\theta$ available in the literature we computed the correction to the jet quenching parameter coming from the presence of non-commutativity and concluded that even if non-commutativity is present, its existence is too weak to have any appreciable effect on the heavy quark observables in the accessible range of energies. Its presence can only be felt in higher energy domain that may be reached in some future collider. Next we evaluated the expectation values of time-like Wilson loops using the standard recipe and from there extracted information about the bound state interaction potential. To keep the discussion simple, we considered only the case when the dipole is aligned along a non-commutative direction and moves along the commutative direction. We found out how the potential varies with the separation between the quark and the antiquark with the dipole velocity and the non-commutativity as parameters. We explored various regimes of this parameter space and plotted the results.
4.5. CONCLUSION

While the details vary, we observed that generically, introduction of non-commutativity aka anisotropy makes the bound state prone to dissociation. We were also successful in obtaining an analytical expression of the screening length in a restricted domain of the parameter space. Finally, by considering the limit $v \to 1$ we considered light-like Wilson loop and from there recovered the form of the jet quenching parameter calculated in an earlier section. In the case of jet quenching parameter also, we found that turning on a small non-commutativity leads to a decrease in the value of $\hat{q}_{\text{NCYM}}$. A plausible explanation of this decrease can be attributed to the intrinsic non-locality of the underlying non-commutative theory that rules out point-like interactions among the partons. The underlying fuzziness of the theory is thus responsible for an increase in suppression of the yield of quarkonia like $J/\Psi$. 