CHAPTER 2

Problem of Efficiency
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This chapter reviews literature in the areas of productivity and performance measurement using mathematical programming and multi-level planning for resource allocation and efficiency measurement. The review is organized as follows. Firstly, definitions and concepts of efficiency are presented. Then work in performance and efficiency measurement with DEA is summarized. It then delves into the literature on weight restriction DEA models. A part of the section on weight restriction models is dedicated to describing the applications of those models to real-life examples. Finally, this chapter contains a discussion on fuzzy set theory, fuzzy numbers, fuzzy linear programming and applications of fuzzy set theory in DEA.

2.1 DEFINITIONS AND CONCEPTS

In the management science literature, productivity and performance measurement have traditionally been concerned with some factors (inputs and outputs), processes, or machines rather than the organizational whole. For example, one measurement technique is to calculate the ratio of total output to a particular input i.e., partial factor productivity. The most common measure is that of labor productivity (e.g., output per man-hours) while another common measure is capital productivity (e.g., rate of return on capital utilized) (Stainer (1997)). According to Stainer (1997), such ratios face a fundamental problem wherein external factors may affect their computation and have no relationship to efficient resource usage.

Productivity research led to the development of other measures that incorporated all the important factors in aggregated form. These measures offered more insight about technical and financial performance of an organization. The concept of technical efficiency introduced by Farrell (1957) is a result of these concerns. Charnes et al.(1978) further extended Farrell’s (1957) work and developed a mathematical programming
approach to measure relative efficiency of decision making units. Basic concepts and their definitions are summarized below.

2.1.1 Production Technology

A production technology is defined as the set \((X, Y)\) such that inputs \(X = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l_+\) are transformed into outputs \(Y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k_+\). Fare et al. (1994) describe production technology with the following notation.

\(L(y)\) is the input set such that:

\[
L(y) = \{x: (y, x) \text{ is feasible}\},
\]

\(\forall y \in \mathbb{R}^k_+ \exists \text{ an isoquant } Isol(y)\) such that:

\[
Isol(y) = \{x: x \in L(y), \lambda x \notin L(y), \lambda \in [0, 1]\},
\]

and an efficient subset \(EffL(y)\)

\[
\text{subject to } x'_i \leq x_i, \forall i = 1, \ldots, m \text{ and } x'_i < x_i \text{ for at least one component}\]

\[
EffL(y) = \{x: x \in L(y), x' \notin L(y), x' \leq x\}
\]

2.1.2 Production Function

A production function is defined as the relationship between the outputs and inputs of a production technology. Mathematically, a production function relates the amount of output \((Y)\) as a function of the amount of input \((X)\) used to generate that output. Technical efficiency (Section 2.2) is assumed for a production function \(i.e.,\) every feasible combination of inputs generates the maximum possible output or all outputs are produced using the minimum feasible combination of inputs. For example, the production function for input \(X\) and output \(Y\) is:

\[
Y = f(X)
\]
2.1.3 Isoquant

An *isoquant* is defined as the locus of points that represent all possible input-output combinations that defines the production function for a constant level of output (or a constant level of input). Each point on the isoquant represents a unique production technology. For example, the isoquant (input orientation) for output level $Y^0$ (a specific realization of output $Y$ in section 2.1.2 above) from input $X$ is:

$$Y^0 = g(X) \quad (2.5)$$

This isoquant is shown in Figure 2.1. In this case, further the isoquant is from the origin in the positive quadrant, the greater is the output level.

Figure 2.1 Isoquant for Output Level $Y^0$
2.1.4 Radial and Non-Radial Measures of Efficiency

Consider the isoquant in Figure 2.1 for output level \( Y^o \). Suppose there are firms A, B, C, D, E, F, and G, each of which produces the same output \( Y^o \) consuming inputs \( X \) and \( X_2 \). Firms A, B, D, E, and F are the firms consuming the least amount of each input to produce \( Y^o \). Therefore, these firms define the isoquant and also lie on it. Firms C and G consume more inputs to generate \( Y^o \), are enveloped by the isoquant and are therefore inefficient. The isoquant thus serves as the standard of comparison for the firms. This is the essence of the concept of relative efficiency and is explained in detail in Section 2.2. There are two ways to measure the efficiency for a given firm, (i) radially and (ii) non-radially.

Consider the inefficient firm C. Let \( C' \) be a virtual firm that is the convex combination of firms B and D. C and \( C' \) lie on the same ray through the origin and \( C' \) lies on the isoquant. Therefore, the radial measure of technical efficiency for C is:

\[
TE_{Radial}(C) = \frac{CO'}{OC}
\]  

(2.6)

This means that for C to become efficient it must operate at \( C' \)'s input levels i.e., C must radially or equi-proportionately reduce its inputs to \( C' \)'s levels. However, an equi-proportional reduction in inputs may not always be feasible. In this case, the non-radial measure of efficiency is more appropriate. The non-radial measure of technical efficiency for C for inputs \( X \) and \( X_2 \) is:

\[
TE_{Non-Radial}(C) = \frac{C_1C'_1}{C_1C}
\]  

(2.7)

\[
TE_{Non-Radial}(C) = \frac{C_2C'_2}{C_2C}
\]  

(2.8)

Here, the inputs are reduced individually by different proportions (non-radially) to reach the efficient subset ABDE while maintaining the same level of output and not altering
the input levels of the remaining inputs. Therefore, separate efficiency scores are obtained for each input.

2.1.5 Returns to Scale

In production theory the change in output levels due to changes in input levels is termed as returns to scale. Returns to scale can be constant or variable. Constant Returns to Scale (CRS) implies that an increase in input levels by a certain proportion results in an increase in output levels by the same proportion. Figure 2.2 shows this linear relationship between the inputs and outputs. Variable Returns to Scale (VRS) implies that an increase in the input levels need not necessarily result in a proportional increase in output levels i.e., the output levels can increase (increasing returns to scale) or the output levels can decrease (decreasing returns to scale) by a different proportion than the input increment.

Figure 2.2 Constant and Variable Returns to Scale
Geometrically, this means that the linear relationship between inputs and outputs in the case of CRS is replaced by a curve with a changing slope. Figure 2.2 shows the piece wise linear curve with varying slopes. As the slope of the curve increases the production technology displays increasing returns to scale (e.g., from B to D). Where the slope of the curve decreases the production technology displays decreasing returns to scale, (e.g., from D to E). And where the curve has a zero slope (from point E to ∞) the production technology experiences no increase in output for any further increase in input. Where the curve has a zero slope (from X₁ to A) the output jumps from 0 to Y₁ for an input usage of X₁.

2.1.6 Definitions of Technical Efficiency

The concepts presented above enable the discussion of the two definitions of technical efficiency that are reported in the literature (Fare and Lovell (1978)). The first one is the radial definition presented by Debreu (1951) and Farrell (1957). The input-reducing radial measure of technical efficiency for a unit is defined as the difference between unity (100% efficiency) and the maximum equi-proportional reduction in inputs (while maintaining the production of originally specified output levels). If this difference is zero then the unit is efficient else it is inefficient. The output-increasing radial measure of technical efficiency is defined as the difference between unity (100% efficiency) and the maximum augmentation of outputs (while still utilizing the originally specified input levels). Again, the unit is efficient if this difference is zero else it is inefficient.

The second definition is Koopmans’ (1951) definition of technical efficiency. The firm is technically efficient if and only if an increase in one output results in a decrease in another output so as not to compromise the input level or else results in the increase of at least one input. Stated otherwise, the definition implies that a decrease in one input must result in an increase in another input so as not to compromise the output, or else must result in the decrease of at least one output.
The difference between the two definitions is explained through Figure 2.1. The radial definition provided by Debreu (1951) and Farrell (1957) terms all firms on the isoquant with output level $Y^*$ as efficient. However, Koopman's (1951) non radial-definition deems firm F as inefficient. This is due to the fact that though firm F lies on the isoquant it does not lie on the efficient subset of the isoquant defined by ABDE. In other words, E produces the same output with fewer inputs (lesser amount of $x_1$) than F, and therefore F is inefficient. This case highlights the situation where a unit may lie on the isoquant but still consume excess inputs compared to other units on the isoquant. The next section provides a detailed discussion of technical efficiency.

Since DEA is a technical efficiency measurement technique, we start this chapter with a review of the traditional techniques used for efficiency measurement. The objective of this and the subsequent section of this chapter are to trace the evolution of the DEA approach.

### 2.1.6.1 Average Productivity of Labor

For a long time, efficiency was assessed by measuring the average productivity of labor (Farrell (1957)). Though this was a very popular measure, it had a drawback. The drawback was that it ignored all inputs except labor and was found to be unsatisfactory when the process or organization being evaluated had multiple inputs and outputs.

### 2.1.6.2 Indices of Efficiency

Because of the unsatisfactory nature of the labor productivity measure, attempts were made to develop measures of efficiency, which combined all the factors by aggregating a firm's inputs. One set of measures developed as a result of those efforts is called indices of efficiency. Here, the input vectors are first stripped of their dimensions. The dimensionless quantities are weighted and then added up. Thus, indices of efficiency involve a comparison of weighted-average of inputs with the output. The weighted average is equivalent to a valuation of the inputs at prices proportional to the weights in the index. Thus, an attempt to compare efficiency by this measure can be regarded as making a cost comparison. The choice of a set of prices introduces an arbitrary element.
2.2 TECHNICAL EFFICIENCY USING THE PRODUCTION FUNCTION

To eliminate the above mentioned drawbacks associated with traditional efficiency measures, Farrell (1957) introduced a new measure of (technical) efficiency, which employs the concept of the efficient production function. This method of measuring technical efficiency of a firm consists in comparing it with a hypothetical perfectly efficient firm represented by the production function. The efficient production function is some postulated standard of perfect efficiency and is defined as the output that a perfectly efficient firm could obtain from any given combination of inputs.

The first step in calculating the technical efficiency by this method is determining the efficient production function. There are two ways in which the production function can be determined. It could either be a theoretical function or an empirical one. The problem with using a theoretical function is that it is very difficult to define a realistic theoretical function for a complex process. The empirical efficient production function, on the other hand, is estimated from observations of inputs and outputs of a number of firms. Therefore, it is far easier to compare performances with the best actually achieved (the empirical production function) than to compare with some unattainable ideal (the theoretical function).

To understand the concept of an efficient production function, we take the example of a set of firms employing two factors of production (inputs) to produce a single product (output) under conditions of Constant Returns to Scale. Constant Returns to Scale mean that increase in the inputs by a certain proportion results in a proportional increase in the output. An isoquant diagram is the one in which all firms producing the same output lie in the same plane. Each firm in an isoquant diagram is represented by a point so that a set
of firms yields a scatter of points. An efficient production function is a curve, which joins all the firms in an isoquant diagram utilizing the inputs most efficiently.

While drawing the isoquant from the scatter plot, two more assumptions, in addition to constant returns to scale are made:

1. The isoquant is convex to the origin. This means that if two points are attainable in practice then so is their convex combination.
2. The slope of the isoquant is nowhere positive which ensures that an increase in both inputs does not result in a decrease in the output.

![Figure 2.3 Representation of the Production Function (Isoquant) SS'](image)

Figure 2.3 Representation of the Production Function (Isoquant) SS'

In Figure 2.3, isoquant SS' represents a production function. Point P represents an inefficient firm, which uses the two inputs per unit of output in a certain proportion. Point Q represents an efficient firm which produces the same output as P, uses the two inputs in the same proportion as P but uses only a fraction OQ/OP as much of each input. Point Q could also be thought of as producing OP/OQ times as much output from the same inputs. Therefore, the ratio OQ/OP is defined as the technical efficiency of firm P. This measure of efficiency ignores the information about the prices of the factors. To incorporate the price information, another type of efficiency measure called price (or allocative) efficiency is used. Price efficiency is a measure of the extent to which a firm uses the various factors of production in the best proportions, in view of their prices.
In Figure 2.3, if has a slope equal to the ratio of the prices of the two input factors, then $Q'$ and not $Q$ is an optimal method of production. Although both $Q$ and $Q'$ represent 100 percent technical efficiency, the costs of production at $Q'$ will only be a fraction of $OR/OQ$ of those at $Q$. Therefore, the ratio $OR/OQ$ is called the **price efficiency** of both firms $P$ and $Q$. The product of technical efficiency and price efficiency is called overall efficiency. In Figure 2.3, the ratio $OR/OP$ represents the overall efficiency of firm $P$. We see that an important feature of Farrell's (1957) method outlined above is the distinction between price and technical efficiency. While the price efficiency measures a firm’s success in choosing an optimal set of inputs which minimize the cost of production, the technical efficiency measures its success in producing maximum output from a given set of inputs.

Traditionally, labor productivity was considered as an overall measure of efficiency (Farrell (1957)). According to Farrell (1957) this ratio (e.g., units produced divided by labor hours) was inappropriate as a measure of technical efficiency (TE) as it incorporated only labor and ignored other important factors such as materials, energy, and capital. Thus Farrell (1957) proposed a measure of **technical efficiency** that incorporated all inputs in an aggregated scalar form and also overcame the difficulty of converting multi-component input vectors into scalars. Thus the technical efficiency formulation for multiple input-output configurations is:

$$TE = \frac{\text{Aggregate output measure}}{\text{Aggregate input measure}}$$

The inputs are all resources that are consumed to generate the outputs. From equation (2.9) it can be seen that technical efficiency for a firm relates to its ability to:

(i) produce maximum outputs for a constant input usage (output-increasing efficiency), or

(ii) use minimum inputs to generate a constant output production (input-reducing efficiency).
Technical efficiency measurement generally involves comparing a Decision Making Unit's (DMUs) production plan that lies on the efficient production frontier or isoquant (Fried et al. (1993), Fare et al. (1994), Charnes et al. (1994)). As presented in the next chapter, a production plan for a DMU represents its input usage and output production. The concept of a production plan motivates two types of technical efficiency measurement, input-reducing and output-increasing. Input-reducing efficiency refers to the production of a constant output set while reducing the level of inputs used to the least possible. Output-increasing efficiency refers to maintaining a fixed level of inputs while producing the maximum possible set of outputs.

The notion of comparisons of production plans leads to the need for deriving a "standard of excellence" to serve as a benchmark. This standard must represent that level of technical efficiency that is achieved with (i) the least amount of inputs and constant outputs (for input-reducing efficiency) and (ii) the maximum production of outputs with constant inputs (for output-increasing efficiency). The literature reports three approaches to measure technical efficiency: (i) the index numbers approach (ii) the econometric approach, and (ii) the mathematical programming approach. The index numbers approach includes multi-factor productivity models, financial and operational ratios (Parkan (1997)). The econometric approach presupposes a theoretical production function to serve as the standard of technical efficiency. The Cobb-Douglas, Translog, and Leontief type functions are most commonly used to approximate the production function as they are easily transformed into linear forms. Econometric models are further divided into deterministic and stochastic models. For a detailed discussion of the econometric approach the reader is referred to Girod (1996) and Lovell (1993). The mathematical programming approach does not require the use of a specified functional form for the production data. This approach was pioneered by Charnes, Cooper and Rhodes (1978) and is called Data Envelopment Analysis (DEA). DEA is defined by Giokas (1997) as follows:

"DEA measures relative efficiency [of DMUs] by estimating an empirical production function which represents the highest values of outputs/benefits that could be
generated by inputs/resources as given by a range of observed input/output measures during a common time period."

While econometric methods (e.g., regression analysis) employ "average observations", mathematical programming methods (e.g., DEA) uses "production frontiers" or "best practice observations" for efficiency analysis. A detailed discussion of input-reducing and output-increasing orientations of technical efficiency and DEA is provided in the subsequent sections.

2.2.1 Input-Reducing and Output-Increasing Orientations of Technical Efficiency

The concepts of input-reducing and output-increasing orientations of Farrell's (1957) technical efficiency measure are presented through the following example. Consider a Decision-Making Unit (DMU) that uses \( i = 1, 2, \ldots, I \) inputs to produce \( j = 1, 2, \ldots, J \) outputs. Let the matter of interest be DMU productivity performance over time (say, one year or twelve months) i.e., let each month represent a DMU. Denote the input vector for the \( n^{th} \) month as \( X_{\text{in}} = [x_{ni}] \) and the output vector for the \( n^{th} \) month as \( Y_{\text{in}} = [y_{nj}] \) (Figure 2.4).

![Figure 2.4 Input and Output Vectors for the \( n^{th} \) Month](image)

It is assumed that each component of the input and output vectors is uniquely identifiable and quantifiable (Hoopes and Triantis (1999)). Thus, the objective for the DMU (month) would be to minimize usage of each input resource and maximize the
production of each output type. Farrell (1957) defined the "efficient" transformation of inputs into outputs as the efficient production frontier or isoquant. An isoquant can be oriented for input-reduction or output-augmentation.

An input-reducing isoquant is defined by the observations that are efficient relative to the other observations in the data set. This isoquant represents the minimum input usage that is required to produce a constant set of outputs. Points on the isoquant are observations with different input mixes producing the same level of outputs. Such an isoquant is assumed to be convex to the origin and to have a negative slope. The convexity assumption allows for virtual production plans that are obtained as a weighted combination of actual production plans.

**Strong disposability of inputs** i.e., an increase in any input keeping other inputs constant must result in an increase in the level of outputs. In other words, given the negative slope assumption, different input mixes can be obtained without compromising the output level. With strong disposability of inputs an increase in any input without change in other inputs must result in the observation moving to a higher isoquant, if the output level remains constant than it would imply weak disposability of inputs.

To illustrate an input-reducing isoquant consider a production plan with a constant output set \( Y \) and two inputs \( X = [X_0, X_1] \). Let the production technology exhibit constant returns to scale where an increase in inputs results in an equi-proportional increase in the output. The resultant input-reducing isoquant then represents the output level. For every output level, there exists an input-reducing isoquant. Therefore, the input-reducing isoquant is a function of the output level. In Figure 2.4, SS' represents the input-reducing isoquant for output level \( Y' \) attained with inputs \( X = [X_0, X_1] \). The case of multiple inputs (greater than two) is extended similarly.

An output-reducing isoquant or production possibility frontier is constructed similarly by observations deemed relatively efficient in the data set. This frontier
represents the maximum output production possible with consumption of constant inputs. Efficient observations are points on the frontier with different output mixes produced with the same level of inputs. Once again, let the production technology exhibit concavity with respect to the origin, negative slope, and constant returns to scale. The concavity property permits observations obtained as weighted combinations of actual observations. The negative slope permits strong disposability of outputs i.e., a decrease in any one output keeping all other outputs constant must result in the decrease in the level of inputs. Therefore different output mixes can be obtained without compromising the level of inputs. The constant returns to scale assumption imply that a decrease in outputs results in an equi-proportional decrease in inputs. The strong disposability of outputs ensures that a decrease in any output without change in other outputs must result in the observation moving to a lower frontier. If the input level remains the same then it implies weak disposability of outputs.

To illustrate an output-increasing frontier consider a production plan with two outputs $Y = [Y_0, Y_1]$ and a constant input set $X'$. The resultant output-increasing frontier then represents the input level. For every input level there exists an output-increasing frontier. Therefore, the output-increasing frontier is a function of the input level. In Figure 2.5 RR' represents the output-increasing frontier for input level $X'$ attained with outputs $Y = [Y_0, Y_1]$. The case of multiple outputs (greater than two) is extended similarly.
Figure 2.6: Output – Increasing Isoquant Orientation

Figure 2.5: Input – Reducing Isoquant Orientation
It should be noted that while the isoquant constructs are obtained from input/output levels, the production function is unique for a data set. In other words, isoquants are particular realizations of a production function for a given input/output orientation. Farrell (1957) uses the isoquant construct(s) presented above as the efficient frontier to compare performance of different observations. Since the isoquant is constructed from observed data, the relative comparison of observations is based on the Pareto-Koopmans condition or Pareto optimality condition. For the input-reducing case this condition is stated by Charnes et al. (1978, p.13) as:

"If, for a given observation's input-output mix, it is not possible to find an observation or combination of observations that produce the same amount of output with less of some input and no more of other inputs, then the given observation is efficient. Otherwise, the given observation is inefficient."

The Pareto optimality concept is illustrated graphically as follows. Figure 2.6 presents the input-reducing case. For graphical simplicity, suppose that all DMUs utilize two inputs $X_1$ and $X_2$ to produce a constant output set $Y$. Here, DMUs A, C, D, and E are efficient as they lie on the efficient frontier (or isoquant for output level $Y$) while DMU B is inefficient as it lies above the isoquant. Let $B'$ be a convex combination of DMUs C and D. Then, both DMU B and virtual DMU $B'$ produce the same output set $Y$, but DMU B consumes more resources than virtual DMU $B'$. Therefore, the input-(in) efficiency score of DMU B is given as:

$$\text{TE}_{\text{input}}(B) = \frac{OB}{OB'}$$  \hspace{1cm} (2.10)

Similarly, Figure 2.7 presents the output-increasing case. Again, for graphical simplicity, suppose that all DMUs produce two outputs $Y_1$ and $Y_2$ from a constant input set $X$. Here, DMUs F, G, H, and I are efficient as they lie on the frontier for input level $X$ while DMU J is inefficient as it lies under the frontier. Let $J'$ be a convex combination of DMUs G and H. Then, both DMU J and virtual DMU $J'$
consume the same amount of inputs, but virtual DMU $J'$ produces more output than DMU $J$. Therefore, the output-(in) efficiency score of DMU $J$ is given as:

$$D(x) = \bigcap_{i=1}^{m+1} \mu_i = \min \{ \mu_i(x) \}$$

**Figure 2.8: Input-Reducing Technical Efficiency**
The above review of Farrell’s (1957) technical efficiency measure clearly outlines the importance of the assumptions of convexity, negative slope and Constant Returns to Scale which define the production function. Econometric methods theoretically fit a function to the production technology. Aigner and Chu (1968) first applied the Cobb-Douglas function to estimate the efficient production frontier. However, it has been observed that theoretically derived functions provide inaccurate approximations of the production technology as the complexity of the technology increases. This prompted researchers to look closely at the mathematical programming methods that empirically derive the efficient production frontier from observed data. Charnes, Cooper, and Rhodes (1978) extended Farrell’s (1957) work in the measurement of technical efficiency and developed Data Envelopment Analysis (DEA). The DEA methodology allows the relaxation and the enhancement of some of Farrell’s (1957) assumptions for the production function and the production technology. The Literature on DEA is reviewed and presented in next chapter.

Figure 2.9: Output-Increasing Technical Efficiency
2.3 FUZZY DECISION MAKING

The realm of fuzzy decision making is discussed in this section. The literature review is divided into the following subsections. First, the mathematical concepts and notions of fuzzy set theory are introduced. Second, the linkage of fuzzy set theory and decision-making is established. In this subsection the definitions of fuzzy goal, fuzzy constraint, fuzzy decision, and optimal fuzzy decision are outlined. Third, the connection between fuzzy decision making and linear programming is discussed. In particular the model formulation proposed by Zimmermann (1976) and adapted to DEA by Sengupta (1992) is illustrated. Sengupta's (1992) formulation is adapted to the GoDEA model (Athanassopoulos (1995)) and developed in this research to provide a fuzzy decision-making environment incorporating goal programming and data envelopment analysis.

2.3.1 Fuzzy Sets

The concept of fuzzy sets was first introduced by Lofti Zadeh (1965) to deal with the issue of uncertainty in systems modeling. Zadeh (1965, p. 338) defined a fuzzy set as "a class of objects with a continuum of grades of membership [...] and characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one." The concept of fuzzy set theory challenged conventional two-valued logic as follows:

When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false, as required by two-valued logic, but it may be true only to some degree - the degree to which x is actually a member of A.

The degrees of membership in fuzzy sets are most commonly expressed by numbers in the closed unit interval [0, 1]. Thus fuzzy sets express gradual transitions from membership (membership value of 1) to non-membership (membership value of 0) and vice versa.
Suppose \( X \) is a space of positive real values associated with a variable and \( x \) is a generic element of \( X \). Mathematically, a fuzzy set \( A \) in \( X \) is defined as the set of ordered pairs:

\[
A = \{(x, \mu_A(x)) | x \in X\}
\]  

(2.12)

The fuzzy set \( A \) in \( X \) is characterized by a membership function \( \mu_A(x) \) such that each point in \( X \) is associated with a real number in the interval \([0,1]\). The value of \( \mu_A(x) \) denotes the degree of membership of \( x \) in \( A \) and, therefore, the closer the value of \( \mu_A(x) \) to unity the higher is degree of membership of \( x \) in \( A \). When \( \mu_A(x) \) takes on only two values 1 and 0 corresponding to whether \( x \) does or does not belong to \( A \) then \( \mu_A(x) \) reduces to the ordinary characteristic function of \( A \) (i.e., \( A \) is a non-fuzzy set).

Zadeh (1965) extended the definitions for ordinary sets to derive definitions for fuzzy sets. These definitions are consistent with topological concepts such as equality, complementation, containment, union, intersection, algebraic product, algebraic sum, normality, support, relation, composition, mapping, convexity, and concavity. These definitions are outlined below.

**Empty Set:** A fuzzy set \( A \) is empty if and only if its membership function is identically zero on \( X \).

\[
i.e., A = \emptyset \iff \mu_A(x) = 0 \quad \forall \ x \in X
\]

(2.13)

**Equality:** Two fuzzy sets \( A \) and \( B \) are equal if and only if their membership functions are equal for all \( x \in X \).

\[
i.e., A = B \iff \mu_A(x) = \mu_B(x) \quad \forall \ x \in X
\]

(2.14)

**Complementation:** The complement of a fuzzy set \( A \) is a fuzzy set \( A' \) with a membership function \( \mu_A(x) \) and is defined as

\[
\mu_{A'}(x) = 1 - \mu_A(x) \quad \forall \ x \in X
\]

(2.15)

**Containment:** Fuzzy set \( A \) is contained in fuzzy set \( B \) (or \( A \) is a subset of \( B \)) if and only if \( \mu_A(x) \leq \mu_B(x) \) for all \( x \in X \).

\[
i.e., A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \quad \forall \ x \in X
\]

(2.16)
**Union:** The union of two fuzzy sets \( A \) and \( B \) with membership functions \( \mu_A(x) \) and \( \mu_B(x) \) respectively is defined as a fuzzy set \( C \) with a membership function \( \mu_C(x) \) such that \( C \) is the smallest fuzzy set containing both \( A \) and \( B \).

\[
\therefore \mu_C(x) = \mu_A(x) \lor \mu_B(x) \quad (2.17)
\]

\[i.e., \quad \mu_C(x) = \text{Max} [\mu_A(x), \mu_B(x)] = \mu_A(x) \text{ if } \mu_A(x) \geq \mu_B(x) \quad (2.18)\]

\[
\mu_C(x) = \text{Max} [\mu_A(x), \mu_B(x)] = \mu_B(x) \text{ if } \mu_A(x) \leq \mu_B(x) \quad (2.19)
\]

**Note:** \( \lor \) has the associative property, i.e., \( A \lor (B \lor C) = (A \lor B) \lor C \)

**Intersection:** The intersection of two fuzzy sets \( A \) and \( B \) with membership functions \( \mu_A(x) \) and \( \mu_B(x) \) respectively is defined as a fuzzy set \( C \) with a membership function \( \mu_C(x) \) such that \( C \) is the largest fuzzy set contained in both \( A \) and \( B \).

\[
\therefore \mu_C(x) = \mu_A(x) \land \mu_B(x) \quad (2.20)
\]

\[i.e., \quad \mu_C(x) = \text{Min} [\mu_A(x), \mu_B(x)] = \mu_A(x) \text{ if } \mu_A(x) \leq \mu_B(x) \quad (2.21)\]

\[
\mu_C(x) = \text{Min} [\mu_A(x), \mu_B(x)] = \mu_B(x) \text{ if } \mu_A(x) \geq \mu_B(x) \quad (2.22)
\]

**Algebraic Product:** The algebraic product of fuzzy sets \( A \) and \( B \) is denoted as \( AB \) and defined such that for all \( x \in X \):

\[
\mu_{AB}(x) = \mu_A(x) \mu_B(x) \quad (2.23)
\]

**Algebraic Sum:** The algebraic sum of fuzzy sets \( A \) and \( B \) is denoted as \( A \oplus B \) and defined such that for all \( x \in X \):

\[
\mu_{A \oplus B}(x) = \mu_A(x) + \mu_B(x) - (\mu_A(x) \mu_B(x)) \quad (2.24)
\]

**Relation:** A fuzzy relation \( R \) in the product space \( X_1 \times X_2 \) is a fuzzy set with a membership function \( \mu_R(x_1, x_2) : X_1 \times X_2 \rightarrow R \) which associates a degree of membership \( \mu_R(x_1, x_2) \) in \( R \) with each ordered pair \( (x_1, x_2) \).
**Decomposition:** Consider fuzzy set \( C \) in \( X \times Y = \{x, y\} \) with a membership function \( \mu_C(x, y) \) and fuzzy sets \( A \) and \( B \) with membership functions \( \mu_A(x) \) and \( \mu_B(y) \) respectively. Then \( C \) is decomposable along \( X \) and \( Y \) if and only if:

\[
\mu_C(x) = \min (\mu_A(x), \mu_B(y))
\]  

(2.25)

**Mapping:** Consider \( T: X \rightarrow Y \) a mapping from \( X \) to \( Y \). Let \( B \) be a fuzzy set in \( Y \) with a membership function \( \mu_B(y) \). The inverse mapping \( T' \) induces a fuzzy set \( A \) in \( X \) with a membership function

\[
\mu_A(x) = \mu_B(y), \quad y \in Y
\]  

(2.26)

for all \( x \in X \) which are mapped by \( T \) into \( Y \).

Now, consider conversely that \( A \) is a fuzzy set in \( X \). Then \( T \) induces a fuzzy set \( B \) in \( Y \) such that:

\[
\mu_B(y) = \max_{x \in T^{-1}(y)} (\mu_A(x)), \quad y \in Y
\]  

(2.27)

where \( T^{-1}(y) \) is the set of points in \( X \) which are mapped into \( Y \) by \( T \).

**Concavity and Convexity:** A fuzzy set \( A' \) is concave if its complement \( A \) is convex. A fuzzy set \( A \) is convex if and only if for every \( x_1, x_2 \in X \) and all \( \beta \in [0, 1] \),

\[
\mu_A(\beta x_1 + (1-\beta)x_2) \geq \min (\mu_A(x_1), \mu_A(x_2))
\]  

(2.28)

**Normality:** A fuzzy set \( A \) is normal if and only if the supremum of \( \mu_A(x) \) over \( X \), \( \sup_{x \in X} \mu_A(x) \), is equal to 1. Otherwise \( A \) is subnormal.

**Support:** The support of a fuzzy set \( A \) is a subset of \( X \), \( S(A) \), such that:

\[
x \in S(A) \iff \mu_A(x) > 0.
\]  

(2.29)

**2.3.2 Membership Functions**

A membership function is a function which assigns to each element \( x \) of \( X \) a number, \( \mu_A(x) \), in the closed unit interval \([0, 1]\) that characterizes the degree of membership of \( x \) in \( A \). The closer the value of \( \mu_A(x) \) is to one, the greater the membership of \( x \) in \( A \). Thus, a fuzzy set \( A \) can be defined precisely by associating with each element \( x \), a number
between 0 and 1, which represents its grade of membership in $A$. The membership function of a fuzzy set $A$ can also be represented as $A(x)$.

### 2.3.3 α-cut and Strong α-cut (Klir and Yuan (1995))

Given a fuzzy set $A$ defined on $X$ and any number $\alpha \in [0, 1]$, the $\alpha$-cut of the fuzzy set $A$ is the crisp set $^\alpha A$ that contains all the elements of the universal set $X$ whose membership grades in $A$ are greater than or equal to the specified value of $\alpha$.

Mathematically: $^\alpha A = \{ x | A(x) \geq \alpha \}$

On the other hand, the strong $\alpha$-cut of a fuzzy set $A$ is the crisp set $^*\alpha A$ that contains all the elements of the universal set $X$ whose membership grades in $A$ are greater than the specified value of $\alpha$.

Mathematically: $^*\alpha A = \{ x | A(x) > \alpha \}$

**2.3.3.1 Special Cases of α-cuts:**

1-cut: The 1-cut of a fuzzy set $A$ is the crisp set which contains all elements of $X$ whose membership grades in $A$ are equal to 1. The 1-cut is often called the core of $A$.

Mathematically: $^1 A = \{ x | A(x) = 1 \}$.

**Support:** The support of a fuzzy set $A$ within a universal set $X$ is the crisp set that contains all the elements of $X$ that have nonzero membership grades in $A$. Clearly the support of $A$ is exactly the same as the strong $\alpha$-cut of $A$ for $\alpha = 0$.

**2.3.4 Intersection of Fuzzy Sets**

The intersection of fuzzy set $A$ and fuzzy set $B$ is the largest fuzzy set contained in both $A$ and $B$. Such a set is denoted $A \cap B$. The membership function of $A \cap B$, for all $x \in X$, can be given by:

$$A \cap B(x) = \min (A(x), B(x)) = A(x); \quad \text{if } A(x) \leq B(x)$$

$$= \min (A(x), B(x)) = B(x); \quad \text{if } A(x) \geq B(x)$$
2.3.5 Fuzzy Numbers

Fuzzy sets that are defined on the set $\mathbb{R}$ of real numbers are called fuzzy numbers (Klir and Yuan (1995)). Membership functions of these sets have a quantitative meaning and are represented as:

$$A: \mathbb{R} \rightarrow [0, 1]$$

The membership functions of fuzzy numbers tend to capture the intuitive conception of approximate numbers i.e. "numbers close to a given real number." Therefore, they are useful for characterizing states of fuzzy variables.

To qualify as a fuzzy number, a fuzzy set $A$ on $\mathbb{R}$ must possess at least the following three properties:

1. $A$ must be a normal fuzzy set as defined in section 2.3.1;
2. $A$ must be a closed interval for every $\alpha \in (0,1]$;
3. The support of $A$, $0+ A$, must be bounded.

The most commonly used shapes for fuzzy numbers are the triangular and trapezoidal. The triangular functions express the proposition "close to a real number $r." The trapezoidal membership function represents a fuzzy interval. Graphically the triangular and trapezoidal membership functions are represented as follows:

![Diagram showing triangular fuzzy number](image)

**Figure 2.9:** Triangular Fuzzy Number $R$ close to the crisp number $r$.
Figure 2.10: crisp number $r$

Figure 2.11: Fuzzy interval $r-s$

Figure 2.12: Crisp interval $r-s$
2.3.5.1 α-Cuts of Fuzzy Numbers

The α-cut of a fuzzy number is a closed interval and is defined completely by specifying its ends. The ends of an α-cut are points of intersection of the line "membership degree = α" and the rightmost and leftmost lines in the graphical representation of the membership function of the fuzzy number. Refer to figure 2.5 where the α-cut of the fuzzy number $R$ approximating the real number $r$ is the crisp set (marked by sloping lines) containing $x$ values between $LaR$ and $RaR$. $LaR$ is the left end of the α-cut given by the intersection of the line $R(x) = α$ and the line $x = r - q + qR(x)$, representing the change in the membership function between $r - q$ and $r$. Similarly the right-end of the α-cut, $RaR$, is the intersection of the line $R(x) = α$ and the line $x = r + p - pR(x)$. Therefore:

$$αR = [LaR, RaR] = [r - q + αq, r + p - αp]$$

The 1-cut of $R$ will be:

$$1R = [r, r]$$

The support $R$ will be:

$$0R = [r - q, r + p]$$

Note that the 1-cut of the fuzzy number contains only its most desirable element ($r$), while the support contains all elements belonging to the fuzzy number. Obviously, the ends of the support are the least desirable elements of the fuzzy number.

2.3.5.2 Arithmetic Operations on Fuzzy Numbers

Fuzzy arithmetic is based on two properties of fuzzy numbers (Klir and Yuan (1995)):

Each fuzzy number can fully and uniquely be represented by its α- cuts and; All α-cuts ($α ∈ [0,1]$) of each fuzzy number are closed intervals of real numbers.

Arithmetic operations on fuzzy numbers are therefore defined in terms of arithmetic operations on their α-cuts i.e. arithmetic operations on closed intervals.

Let fuzzy numbers $A$ and $B$ be represented in terms of their α-cuts as:

$$αA = [a, b]$$

$$αB = [c, d]$$

In general, if * represents an arithmetic operation between two fuzzy numbers, then we define a fuzzy set $A*B$ on $R$ by defining its α-cut $α(A*B)$ as
\( a(A*B) = {}^aA \ast {}^aB \) for any \( a \in (0,1] \).

Since \( a(A*B) \) is a closed interval for each \( a \in (0,1] \) and \( A, B \) are fuzzy numbers, \( A*B \) is also a fuzzy number. In terms of \( \alpha \)-cuts, the four arithmetic operations on the fuzzy numbers \( A \& B \) would then be defined as follows:

**Addition:**
\[ a(A + B) = {}^aA + {}^aB = [a, b] + [c, d] = [a + c, b + d] \]

**Subtraction:**
\[ a(A - B) = {}^aA - {}^aB \]
\[ = [a, b] - [c, d] = [a - c, b - d] \]

**Multiplication:**
\[ a(A*B) = {}^aA \ast {}^aB = [a, b]*[c, d] = [\min(ac, bd, be), \max(ad, ac, bd, be)] \]

**Division:**
\[ a(A/B) = {}^aA / {}^aB \]
\[ = [a, b]/[c, d] = [\min(a/d, a/c, b/d, b/c), \max(a/d, a/c, b/d, b/c)] \]
\[ c, d \neq 0 \]

### 2.3.5.3 Lattice of Fuzzy Numbers
Klir and Yuan (1995) define the MIN and MAX operations on fuzzy numbers \( A \& B \) as follows:

\[ \text{MIN}(A, B)(Z) = \sup_{z = \min(x, y)} \min[A(x), B(y)] \]
\[
\text{MAX}(A, B)(Z) = \sup_{z=\max(x,y)} \min[A(x), B(y)]
\]

for all \( z \in \mathbb{R} \)

2.3.5.4 Partial Ordering of Fuzzy Numbers

Klir and Yuan (1995) define the partial ordering between those two numbers as

\( A \leq B \) iff \( \alpha A \leq \alpha B \) for all \( \alpha \in (0, 1] \).

\( A \geq B \) iff \( \alpha A \geq \alpha B \) for all \( \alpha \in (0, 1] \).

Where the partial ordering of closed intervals is defined in the following way:

\([a_1, a_2] \leq [b_1, b_2]\) iff \( a_1 \leq b_1 \) and \( a_2 \leq b_2 \),

\([a_1, a_2] \geq [b_1, b_2]\) iff \( a_1 \geq b_1 \) and \( a_2 \geq b_2 \),

2.3.5.5 Fuzzy Relations between Real Numbers and Fuzzy Numbers

While comparing a real number with a fuzzy number, we cannot say, like in the crisp case, that one is strictly greater than the other. A real number can be greater (or smaller) than a fuzzy number to only a certain degree. In this research, we define this degree in the following way:

The fuzzy relation \( ax \leq B \) (where \( B \) is a fuzzy number and \( ax \) is a real number) will be defined in the following way:

\( ax \) satisfies \( ax \leq B \) to a degree equal to \( \lambda \) iff \( \lambda = \sup \{ \alpha : ax \leq \alpha B \} \).

Similarly, the fuzzy relation \( ax \geq C \) (\( C \) is a fuzzy number) will be defined in the following way:

\( ax \) satisfies \( ax \geq C \) to a degree equal to \( \lambda \) iff \( \lambda = \sup \{ \alpha : ax \geq \alpha C \} \).

The following statements follow from the above definitions:

1. \( ax \leq B \) is satisfied to a degree greater than or equal to \( \lambda \) if \( ax \leq ^{R_B}B \).

2. \( ax \geq C \) is satisfied to a degree greater than or equal to \( \lambda \) if \( ax \geq ^{L_A}C \).
2.3.6 Fuzzy Decision-making

According to Zimmerman (1996) a decision is characterized by:

- A set of decision alternatives (the decision space). The decision space can be described by enumeration or be defined by a number of constraints.
- A set of states of nature (the state space);
- A relation assigning to each pair of a decision and state a result;
- A utility function or objective function that orders the decision space via the one-to-one relationship of results to decision alternatives.

Bellman and Zadeh (1970) suggest a model for decision making in a fuzzy environment. They consider a situation of decision making in which the objective functions as well as the constraint(s) are fuzzy. The fuzzy objective function and the fuzzy constraints are both characterized by their membership functions. Since we want to satisfy the objective function as well as the constraints, a decision in a fuzzy environment is defined as the selection of activities that simultaneously satisfy the objective function "and" the constraints. In other words, decision making in a fuzzy environment seeks a compromise between satisfying the objective function and satisfying the constraints.

Assuming that the constraints are non-interactive (independent), the logical "and" corresponds to intersection. Thus a fuzzy decision can be viewed as the intersection of fuzzy constraints and fuzzy objective function. We see that the relationship between objective functions and constraints is fully symmetric because both can be represented using membership functions. The relationship would have been unsymmetrical if one of them was not expressed as a membership function.

A formal definition of a decision in a fuzzy environment stated by Bellman and Zadeh (1970) is as follows:

Assume that we are given a fuzzy goal $\tilde{G}$ and a fuzzy constraint $\tilde{C}$ in a space of alternatives $X$. Then $\tilde{G}$ and $\tilde{C}$ combine to form a decision $\tilde{D}$, which is a fuzzy set resulting from intersection of $\tilde{G}$ and $\tilde{C}$. In symbols, $\tilde{D} = \tilde{G} \cap \tilde{C}$ and correspondingly,
\[ D(x) = \min \{G(x), C(x)\}. \]

More generally, suppose that we have \( n \) goals \( G_1, \ldots, G_n \) and \( m \) constraints \( C_1, \ldots, C_m \). Then the resultant decision is the intersection of the given goals \( G_1, \ldots, G_n \) and the given constraints \( C_1, \ldots, C_m \). That is, \( \tilde{D} = G_1 \cap G_2 \cap \ldots \cap G_n \cap C_1 \cap C_2 \cap \ldots \cap C_m \) and correspondingly,

\[ D(x) = \min(G_1(x), G_2(x), \ldots, G_n(x), C_1(x), C_2(x), \ldots, C_m(x)) \]

\[ = \min \{G_i(x), C_j(x)\} \]

\[ = \min \{A_{ij}(x)\} \]

where \( A_{ij}(x) \) is a generalized representation for the membership functions of goals and constraints.

According to Zimmerman (1996), the above definition implies essentially three assumptions:

- The "and" connecting the goals and the constraints in the model corresponds to the "logical and"

- The logical "and" corresponds to the set-theoretic intersection.

- The intersection of fuzzy sets is defined by the "min"-operator.

Bellman and Zadeh (1970) indicated that the min-interpretation of the intersection might have to be modified depending upon the context. Therefore, they stated the following broad definition of the concept of decision: "Decision = Confluence of Goals and Constraints."

### 2.3.7 Fuzzy Linear Programming

Linear programming models are special kinds of decision models where the decision space is defined by linear constraints and the "goal" is defined by a linear objective function. A typical linear programming model (Bazaraa et al. (1990)) is expressed as follows: Find \( x \) which:
Max $f(x) = C^T x$

subject to $A x \leq b$

$x \geq 0$

With $c, x \in R^n, b \in R^m, A \in R^{m \times n}$

(2.21)

where $f(x) = c^T x$ defines the objective function, $A x \leq b$ the constraints, and $x \geq 0$, the decision variables. $c = (c_1, c_2, \ldots, c_n)$ is known as the revenue coefficient vector, $x = (x_1, x_2, \ldots, x_n)$ as the vector of decision variables, $b = (b_1, b_2, \ldots, b_m)$ as the right-hand-side (resource) vector, and $A = [a_{ij}]$ as the $n \times m$ constraint matrix. The $a_{ij}$ elements of $A$ are called technological coefficients.

The above classical model makes the following assumptions:

- All the coefficients $A$, $b$, and $c$ are crisp numbers,
- $\leq$ is meant in a crisp sense,
- "Maximize" is a strict imperative.

If the classical linear program in (2.21) is used to model decisions in a fuzzy environment, Zimmerman (1996) suggests quite a number of possible modifications to it. Firstly, the decision-maker might not want to actually maximize or minimize the objective function. He/she might just be interested in "improving the present cost situation." Therefore, he/she might end up specifying some aspiration levels for the objective function that may not be definable crisply.

Secondly, the constraints might be vague in one of the following ways:

- The constraints may represent aspiration levels or sensory requirements that cannot adequately be approximated by a crisp constraint. The $\leq$ sign may not be meant in the strictly mathematical sense and smaller violations might well be acceptable.
- The coefficients of the vectors $b$ or $c$ or of the matrix $A$ can have fuzzy character either because they are fuzzy in nature or because the perception of them is fuzzy.
Finally, the decision-maker might attach different degrees of importance to violations of different constraints. As a result, the role of the constraints in fuzzy linear programming can be different from that in classical linear programming, where the violation of any single constraint by any amount renders the solution infeasible.

2.3.7.1 Types of Fuzzy Linear Programming Models

In contrast to classical linear programming, "fuzzy linear programming" is not a uniquely defined type of model and many variations are possible, depending on the assumptions or features of the real situation being modeled. In this thesis, we use two types of fuzzy LP models:

1. Zimmerman's (1996) basic fuzzy LP models which can be either symmetric or unsymmetrical and
2. Fuzzy models with fuzzy coefficients of the matrix A.

Depending upon whether the objective function is crisp or fuzzy, Zimmerman (1996) classifies his basic fuzzy LPs into the following two types:

- Symmetric Fuzzy LP where both the objective function and the constraints are fuzzy.
- Unsymmetrical Fuzzy LP where the constraints are fuzzy but the objective function is crisp.

2.3.7.1.1 Symmetric Fuzzy LP (Zimmerman (1996))

In this model, it is assumed that the decision maker can establish an aspiration level, z, for the value of the objective function and that each of the constraints is modeled as a fuzzy set. The fuzzy LP then becomes:

Find \( x \) such that

\[
C^T x \geq Z \\
A x \leq b \\
x \geq 0
\]  

(2.22)

Here the relation \( \leq \) denotes the fuzzified version of \( \leq \) and has the linguistic interpretation "the real number on the LHS is essentially smaller than or equal to the real
number on the RHS. The relation \( \preceq \) denotes the fuzzified version of \( \geq \) and has the linguistic interpretation "the real number on the LHS is essentially greater than or equal to the real number on the RHS". Model (2.22) is fully symmetric with respect to objective function and constraints. This can be made more obvious by substituting 

\[
\begin{pmatrix}
-c \\
A
\end{pmatrix} = B \quad \text{and} \quad \begin{pmatrix}
-z \\
b
\end{pmatrix} = d.
\]

After making these substitutions, model (2.22) becomes:

Find \( x \) such that

\[
Bx \preceq d \\
x \geq 0
\]

(2.23)

Each of the \((m+1)\) rows of model (2.23) shall now be represented by the fuzzy set \( \mu_i(x) \). \( \mu_i(x) \) can be interpreted as the degree to which \( x \) fulfills (satisfies) the fuzzy inequality

\[ B_i x \preceq d_i \]

(where \( B_i \) denotes the \( i \)th row of \( B \)).

Zimmerman assumes \( \mu_i(x) \) to take a value 0 if the constraints (or the objective function) are strongly violated and a value 1 if they are very well satisfied i.e. satisfied in the crisp sense. The values between 0 and 1 represent the "in between" satisfaction.

\[
\mu_i(x) = \begin{cases} 
1 & \text{if } B_i x \leq d_i, \\
\epsilon [0,1] & \text{if } d_i < B_i x \leq d_i+P_i, \\
0 & \text{if } B_i x > d_i+P_i. 
\end{cases}
\]

(2.24)

where \( P_i \) are subjectively chosen constants of admissible violations of the constraints and the objective function. Zimmerman (1996) assumes that the membership function of the fuzzy set corresponding to constraint (or objective function) \( i \) increase linearly over the "tolerance interval" \([d_i, d_i+P_i]\) and is given by:

\[
\mu_i(x) = \begin{cases} 
1 & \text{if } B_i x \leq d_i, \\
1-\frac{B_i x-d_i}{P_i} & \text{if } d_i < B_i x \leq d_i+P_i, \\
0 & \text{if } B_i x > d_i+P_i.
\end{cases}
\]

(2.25)

The membership function of the fuzzy set "decision" of model (2.23) is equal to the intersection of the fuzzy sets \( \mu_i \) and is given by.
Since the decision-maker is interested not in a fuzzy set but in a crisp "optimal" solution, Zimmerman (1996) suggests finding the "maximizing solution" to equation (2.26). The maximizing solution to (2.26) would be the solution to the following problem:

Max \( D(x) \) = Max min \{\mu_i(x)\}  \tag{2.27} 

Replacing \( D(x) \) by a new variable \( \lambda \), we arrive at the following aggregate model:

Maximize \( \lambda \) 
Such that \( \lambda p_i + Bx \leq d_i + p_i \)  \tag{2.28} 

\( \lambda \leq 1 \) 
\( x \geq 0 \) 

The aggregate model (2.28) is a problem of finding a point (say \( x_0 \)), which satisfies all the constraints and the goal (objective function) with the maximum degree. The point \( x_0 \) is the maximizing solution of model (2.22).

2.3.7.1.2 Unsymmetrical Fuzzy LP (Fuzzy LP with Crisp Objective Function)

If the objective function has to be either maximized or minimized, it is considered as a crisp. A model, in which the constraints are fuzzy and the objective function is crisp, is no longer symmetric because the constraints and the objective function play different roles (Zimmerman (1996)). The former define the decision space and the latter induces an order of decision alternatives just like in classical LP models. Therefore, the approach used for arriving at the solution in the symmetric case is not applicable here. To arrive at a solution in unsymmetrical models, we need to somehow aggregate the crisp objective function with fuzzy constraints. For that purpose, Zimmerman (1996) proposes determining an extremum of the crisp function over a fuzzy domain. To determine the extremum of the objective function, we use the notion of "maximizing set" introduced by Zadeh (1972). After the maximizing set for the objective function is determined, the model becomes symmetric and can be solved like the symmetric case by determining a
"maximizing solution." Let us now digress a little and understand the concepts of extremum of fuzzy functions and maximizing sets.

Traditionally, the extremum (maximum or minimum) of a crisp function \( f \) over a domain \( D \) is attained at the same point \( x_0 \) at which the function achieves an optimal value when it is the objective function of a decision model. The point \( x_0 \) in the latter case is called the "optimal decision." Thus, in classical theory, there is an almost unique relationship between the extremum of a function and the notion of optimal decision of the decision model. However, in case of fuzzy models, this unique relationship does not exist (Zimmerman (1996)). According to Bellman and Zadeh (1970, p.150), "In decision models, the optimal decision is often considered to be the crisp set, \( D_m \), that contains those elements of the fuzzy set decision attaining the maximum degree of membership."

When considering functions in general (not as part of a decision model), the concept of a "maximizing set" is equivalent to the notion of an optimal decision defined above.

Zadeh (1972) provides the following definition for the maximizing set:

Let \( f \) be a real-valued function in \( X \). Also, let \( f \) be bounded from below by \( \inf(f) \) and from above by \( \sup(f) \). The fuzzy set \( M(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)} \) where

\[
M(x) = \frac{f(x) - \inf(f)}{\sup(f) - \inf(f)}
\]

is called the maximizing set.

Where

- \( \sup \) stands for supremum (upper bound or maximum);
- \( \inf \) stands for infrenum (lower bound or minimum)

### 2.3.8 Decision Making In a Fuzzy Environment

Bellman and Zadeh (1970) extended fuzzy set theory and developed a framework for decision-making in a fuzzy environment. A fuzzy environment is defined as an environment where the goals and/or constraints are fuzzy. Conventional decision-making environments consist of three parts, namely, objectives, constraints and alternatives. The
alternatives define the decision space (from which a solution may be chosen) and is restricted by the constraints. The objective(s) or goal provides the selection criteria for the solution and assigns a utility value to all possible choices.

Bellman and Zadeh (1970) forwarded the premise that objectives and constraints can be treated as fuzzy sets in the decision space and a fuzzy decision then would be obtained as the intersection of these fuzzy sets. Their conceptual definition is stated as:

"Decision = Confluence of Goals and Constraints"

(2.30)

Accordingly, the formal definition (Bellman and Zadeh (1970) is stated as:

"Assume that we are given a fuzzy goal \( G \) and a fuzzy constraint \( C \) in a space of alternatives \( X \). Then \( G \) and \( C \) combine to form a decision \( D \), which is a fuzzy set resulting from intersection of \( G \) and \( C \). In symbols, \( D = G \cup C \) and correspondingly, \( \mu_D(x) = \mu_{G \cup C}(x) \)."

In the general case, suppose there are \( n \) goals \( G_i (i = 1, 2, \ldots, n) \) and \( m \) constraints \( C_j (j = 1, 2, \ldots, m) \). The resultant fuzzy decision is given as the intersection of the \( n \) goals and \( m \) constraints as:

\[
D = G_1 \cup G_2 \ldots \cup G_n \cup C_1 \cup C_2 \cup C_m
\]

(2.31)

i.e., \( \mu_D(x) = \text{Min} (\mu_{G_1}(x), \ldots, \mu_{G_n}(x), \mu_{C_1}(x), \ldots, \mu_{C_m}(x)) \)

(2.32)

Given a set of fuzzy decisions Bellman and Zadeh (1970) addressed the optimal fuzzy decision. They proposed that the optimal fuzzy decision in the decision space \( X \) was a maximizing decision i.e., a decision that maximized \( \mu_D(x) \).