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1. Introduction and Summary.

M. P. Singh [2] has considered the generalised inflated binomial distribution inflated at r-th cell and has obtained the estimates of the parameters and also the asymptotic variances and covariances of the estimates by the method of maximum likelihood. I. D. Patel [1] has obtained the estimates of the parameters of an inflated power series distribution inflated at zero by considering the maximum likelihood method. This very distribution has been generalised here, called as the generalised inflated power series distribution and we define it as follows:

\[ P[X = R] = \begin{cases} 
1 - \alpha + \alpha \frac{\theta^r}{f(\theta)} & \text{for } R = r \\
\alpha \frac{a^R \theta^r}{f(\theta)} & \text{for } R \in S
\end{cases} \tag{1.1} \]

where \( S \) is a subset of non-negative integers, not containing \( r \), \( \alpha \) is the inflation parameter \( (0 < \alpha \leq 1) \), \( f(\theta) = \sum_{R \in S} a^R \theta^R + a^r \theta^r \) and \( f(\theta) \) is convergent and differentiable also.

Note that Singh's note becomes a particular case of this generalised inflated power series distribution. The purpose of this paper is to obtain the maximum likelihood estimates of the parameters \( \alpha \) and \( \theta \) and also the asymptotic variances and covariance of these estimates. As an application of this method, the generalised inflated Poisson distribution is considered here.


Let \( N_R \) be the observed frequency at \( R \in S \) and \( N_r \) at \( R = r \), and let

\[ N = \sum_{R \in S} N_R \]

Then the logarithm of the likelihood function will be given by
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\[(2 -1) \log L = N_r \cdot \log \left[ 1 - \alpha + \alpha \cdot \frac{a \theta^r}{f(\theta)} \right] + \sum_{R \in S} N_R \log \left[ a \cdot \frac{a_R \theta^R}{f(\theta)} \right] \]

Hence the maximum likelihood estimates \( \hat{\alpha} \) and \( \hat{\theta} \) of the parameters \( \alpha \) and \( \theta \) from (2.1) can be given by

\[(2.2) \quad \hat{\alpha} = \frac{\left( N - N_r \right) / N}{1 - \frac{\alpha}{f(\theta)}} ; \quad \text{and} \quad \hat{\theta} = \frac{T}{N - N_r} \cdot \frac{G(\hat{\theta})}{G'(\hat{\theta})} \]

where \( G(\theta) = f(\theta) - a \theta^r \) and \( G'(\theta) \) is its derivative (w.r. to \( \theta \))

and \( T = \sum_{R \in S} R N_R \)

The asymptotic distribution of these estimates can easily be obtained from the covariance matrix given by (2.3)

\[
\begin{bmatrix}
N \left( 1 - \frac{a \theta^r}{f(\theta)} \right) & -N \cdot \frac{a \theta^r}{f(\theta)} \left( \frac{f'(\theta)}{f(\theta)} - \frac{r}{\theta} \right) \\
\alpha \left( 1 - \alpha + \alpha \cdot \frac{a \theta^r}{f(\theta)} \right) & \left( 1 - \alpha + \alpha \cdot \frac{a \theta^r}{f(\theta)} \right)
\end{bmatrix}^{-1} =
\begin{bmatrix}
-N \cdot \frac{a \theta^r}{f(\theta)} \left( \frac{f'(\theta)}{f(\theta)} - \frac{r}{\theta} \right) & \left( 1 - \alpha + \alpha \cdot \frac{a \theta^r}{f(\theta)} \right)
\end{bmatrix}
\]

Here \( f'(\theta) \) and \( f''(\theta) \) are the first and the second derivatives of the generating function \( f(\theta) \) with respect to \( \theta \).
3. Approximate Estimate of $a$ and $\theta$.

Since the equations in (2.2) are complicated to solve, we suggest the following approximate estimates. These approximate estimates are not comparable with the M.L. estimates, but, they can be used as an initial estimate with which M.L. iteration can be started.

Let us consider a particular problem with the set $S\{a, a+1, a+2, \ldots, b\}$ not containing $r$ in the expression (1.1). The approximate estimates are given by

\[
(3.1) \quad \hat{a}_t = \frac{(N - N_r)/N}{[1 - \frac{a_r}{f(\hat{\theta}_t)}]}, \quad \text{and} \quad \hat{\theta}_t = \frac{\sum_{x \in S_1} a x \cdot N x + 1}{\sum_{x \in S_1} a x + 1 \cdot N x}
\]

where $S_1 = \{a, a+1, \ldots, b-1\}$, if $r < a$ and $r > b$; and

$S_1 = \{a, a+1, \ldots, r-2, r+1, \ldots, b-1\}$, if $a < r < b$.

The amount of bias to order $1/N$ for $\hat{a}_t$ and $\hat{\theta}_t$ are

\[
(3.2) \quad b(\hat{a}_t) = \frac{\hat{a}_t}{2N} \left[ \left( \frac{\sum_{x \in S_1} a_{x}^2 \cdot P}{\sum_{x \in S_1} a \cdot P} \right)^{-1} \left( \mu'_2 - \mu'_1(r) - (1 + 2\mu'_1(r)) \right) \right]
\]
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(3.3) \( b(\theta_1) = \frac{\theta_1}{N} \left[ \frac{\Sigma_{x \in S_1} a x^2 + P x}{\left( \frac{\Sigma_{x \in S_1} a x + P x}{x} \right)^2} \right] - \frac{\Sigma_{x \in S_1} a x + P x + 1}{\left( \frac{\Sigma_{x \in S_1} a x + P x}{x} + 1 \right)^2} \)


Let \( f(\lambda) = e^\lambda \), and \( a_x = 1/x! \), then from (1.1) we get the generalised inflated Poisson distribution as follows

(4.1) \( P(X = k) = \begin{cases} 
1 - \alpha + \alpha \cdot \frac{e^{-\hat{\lambda}} \cdot \lambda^r}{r!} & \text{for } k = r \\
\alpha \cdot \frac{e^{-\hat{\lambda}} \cdot \lambda^k}{k!} & \text{for } k \neq S 
\end{cases} \)

The maximum likelihood estimates \( \hat{\alpha} \) and \( \hat{\lambda} \) of the parameters \( \alpha \) and \( \lambda \) are given by

(4.2) \( \hat{\alpha} = \frac{(N - N_r)/N}{1 - \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!}} \); and \( \hat{\lambda} = \frac{T}{(N - N_r)} \cdot \left[ \frac{1 - \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!}}{1 - \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^{r-1}}{(r-1)!}} \right] \)

The asymptotic distribution of these estimates can be obtained from the following covariance matrix (4.3)

\[
\begin{bmatrix}
\frac{N}{\alpha} \left[ 1 - \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!} \right] & N \cdot \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!} \cdot \left( 1 - \frac{r}{\hat{\lambda}} \right) \\
N \cdot \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!} \cdot \left( 1 - \frac{r}{\hat{\lambda}} \right) & \frac{Na}{\lambda} \cdot \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!} \cdot \left[ \frac{1 - \frac{r}{\hat{\lambda}}}{1 - \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!}} \right] \cdot (1 - \alpha)
\end{bmatrix}^{-1}
\]

The approximate estimates of \( \alpha \) and \( \lambda \) are given by

(4.4) \( \hat{\alpha} = \frac{(N - N_r)/N}{1 - \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^r}{r!}} \); and \( \hat{\lambda} = \frac{\Sigma_{x \in S_1} a_x^2 + P x}{\left( \frac{\Sigma_{x \in S_1} a_x + P x}{x} + 1 \right)^2} \)
The amount of bias to order $1/N$ for these approximate estimates are easily obtained from (3-2) and (3-3) simply by putting $a_x = 1/x$.

5. Example.

Let us consider for the sake of computation the following example which is given for the inflated Poisson distribution inflated at $x = 1$.

The theoretical parameters are $\lambda = 1.32$ and $\alpha = 0.8$ inflation being taken at $x = 1$.

<table>
<thead>
<tr>
<th>Table</th>
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</thead>
<tbody>
<tr>
<td>$x$</td>
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<tr>
<td>------</td>
</tr>
<tr>
<td>0</td>
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<td>1</td>
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<td>3</td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7 and above</td>
</tr>
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</table>

Here $N = 402$, $T = 317$, $N - N_1 = 209$ and hence $T/(N - N_1) = 1.5167$.

The approximate estimates of $\alpha$ and $\lambda$ using the formula (3-1) comes out as follows:

$$\hat{\alpha}_1 = 0.8008 \text{ and } \hat{\lambda}_1 = 1.3406$$

With these approximate estimates the expected frequencies are shown in the above table.

The maximum likelihood equation in $\hat{\lambda}$ is

$$\hat{\lambda} \left[ 1 - \exp \left( -\hat{\lambda} \right) \right] / \left[ 1 - \exp \left( -\hat{\lambda} \right) \right] = 1.5167$$

which is solved by iteration procedure given by

$$\lambda_{f+1} = 1.5167 \left[ 1 - \lambda_f \exp \left( -\lambda_f \right) \right] / \left[ 1 - \exp \left( -\lambda_f \right) \right]$$

where $\lambda_f$ is the $f$-th iterative solution. With the initial solution as $\hat{\lambda}_0 = 1.3406$, we obtain $\hat{\lambda} = 1.335$ as our maximum likelihood estimate of the parameter $\lambda$.

Hence the estimate of the inflation parameter $\alpha$ is found out as

$$\hat{\alpha} = 209 / \left\{ 402 \left[ 1 - \exp \left( -\hat{\lambda} \right) \right] \right\} = 0.8015.$$  

The expected frequencies obtained by the maximum likelihood estimates are also given in the table.
The variances and covariance of \( \hat{a} \) and \( \hat{X} \) are given by
\[
\text{Var}(\hat{a}) = 0.001527 \\
\text{Var}(\hat{X}) = 0.004341 \\
\text{Cov}(\hat{a}, \hat{X}) = 0.0004728
\]

6. Acknowledgement

We are thankful to Dr. C. G. Khatri for his helpful suggestions.

References


TEST PROCEDURES FOR HOMOGENEITY OF GENERALIZED POWER SERIES DISTRIBUTION AND A TEST OF INFLATION FOR AN INFLATED POWER SERIES DISTRIBUTION

Y. K. Shah and I. D. Patel
Mathematics Department
Gujarat University, Ahmedabad-9

1. Summary and Introduction

The generalized power series distribution as defined by Patil [4] is as follows:

Let $S$ be a non-empty countable subset of non-negative integers and define the generating function

$$f(\theta) = \sum_{x \in S} \alpha_x \cdot \theta^x$$

with $\alpha_x > 0$, $\theta > 0$ so that $f(\theta)$ is positive, finite and differentiable. Then the generalized power series distribution (gpsd) of a random variable $X$ is given by

$$(1) \quad \Pr(X=x) = \frac{\alpha_x \theta^x}{f(\theta)}, \quad x \in S$$

Note that proper choice of $S$ and $f(\theta)$ reduces the gpsd in particular to binomial, negative binomial, poisson and logarithmic series distributions and their truncated forms.

In the above if $S$ is the entire set of non-negative integers then a gpsd reduces to a psd as defined by Noack [3]. Hence an inflated power series distribution (inflated at zero) is given by

$$\Pr(X=0) = 1 - a + \frac{a \cdot a_0}{f(\theta)}$$

$$(2) \quad \Pr(X=x) = a \frac{a_x \theta^x}{f(\theta)}, \quad x = 1, 2, \ldots,$$
where \( a \) \((0 < a \leq 1)\) is the inflation parameter. Note that when \( f(\theta) = \exp(\theta) \), we get the inflated poisson distribution as defined by S. N. Singh [5]. Also (2) is similar to the extension of a truncated power-series distribution in the sense of A.C. Cohen [1] provided \( a \) is replaced by \( \frac{a \cdot f(\theta)}{f(\theta) - a_0} \). Large sample test of homogeneity for gpsd is given by Patil [4] and a test of inflation in Poisson distribution is given by S. N. Singh [5].

The purpose of this note is to develop UMP or UMPU test procedures for the following four hypothesis for two gpsd with parameters \( \theta_1 \) and \( \theta_2 \):

(3) \( H_1 : \theta_2 \leq \theta_0 \cdot \theta_1 \) vs. \( K_1 : \theta_2 > \theta_0 \cdot \theta_1 \)

(4) \( H_2 : \theta_0 \cdot \theta_1 \leq \theta_2 < \theta_1 \) vs. \( K_2 : \theta_0 \cdot \theta_1 < \theta_2 < \theta_1 \cdot \theta_1 \)

(5) \( H_3 : \theta_2 = \theta_0 \cdot \theta_1 \) vs. \( K_3 : \theta_2 \neq \theta_0 \cdot \theta_1 \)

where \( \theta_0 \) and \( \theta_1 \) are given constants.

Also we shall develop a UMP test procedure for testing the hypothesis of no inflaction for an inflated psd, which is equivalent to testing

(4) \( H_5 : \alpha = 1 \) vs. \( K_5 : \alpha < 1 \)

2. Derivation of the test procedure for homogeneity of two gpsd.

Let \( X \) and \( Y \) be independently distributed as gpsd with generating functions \( f(\theta_1) \) and \( f(\theta_2) \) respectively and probabilities given by

\[
Pr(X = x) = \frac{a_x \cdot \theta_1^x}{f(\theta_1)}, \quad x \in S_1
\]

and

\[
Pr(Y = y) = \frac{b_y \cdot \theta_2^y}{f(\theta_2)}, \quad y \in S_2
\]

where \( S_1 \) and \( S_2 \) are non-empty countable subsets of non-negative integers. The joint distribution of \( X \) and \( Y \) is given by

(5) \( Pr(X = x, Y = y) = \frac{a_x \cdot b_y}{f(\theta_1) \cdot f(\theta_2)} \exp[y \log \frac{\theta_2}{\theta_1} + (x+y)\log \theta_1] \)
which belongs to the exponential family of distributions and hence by Lehmann [2; p. 134–6] there exists UMP tests for $H_1$ and $H_2$, and UMPU tests for $H_3$ and $H_4$ concerning the parameter $\theta = \log \frac{\theta_2}{\theta_1}$ or equivalently concerning the ratio $\rho := \frac{\theta_2}{\theta_1}$.

From (5) the conditional distribution of $Y$ given $X+Y=t$ can be written as

$$ P_r(Y = y|X+Y = t) = c_t(\rho) \cdot a_{t-y} \cdot b_y \cdot \rho^y, \quad y \in S_t, $$

where $S_t$ is the set of values that $Y$ can take in the conditional distribution and

$$ \{ c_t(\rho) \}^{-1} = \sum_{y \in S_t} a_{t-y} \cdot b_y \cdot \rho^y. $$

Then the original four hypothesis $H_1, H_2, H_3$ and $H_4$ reduce to the corresponding ones about the parameter $\rho$ of the conditional distribution given by (6), e.g., the hypothesis $H_1$ reduce to $H'_1: \rho \leq \rho_0$ vs. $H''_1: \rho > \rho_0$ and the test function is given by

$$ \phi(y|t) = \begin{cases} 1 & \text{if } y > k_t, \\ \gamma_t & \text{if } y = k_t, \\ 0 & \text{if } y < k_t. \end{cases} $$

where the constants $k_t$ and $\gamma_t$ are to be determined from the size condition

$$ E[\phi(y|t)] = \alpha', $$

i.e., $k_t$ and $\gamma_t$ are to be determined from the equation

$$ \sum_{y \in S'_t} c_t(\rho_0) \cdot a_{t-y} \cdot b_y \cdot \rho_0^y + \gamma_t \cdot c_t(\rho_0) \cdot a_{t-k_t} \cdot b_{k_t} \cdot \rho_0^{k_t} = \alpha' $$

where $\alpha'$ is the size of the test and

$$ S'_t = \{ y \in S_t; \quad y > k_t \}. $$

The above tests are also applicable when we have two samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ from two gpsi's with
parameters $\theta_1$ and $\theta_2$. Since in that case the random variables

$$X_m = \sum_{i=1}^{m} X_i$$

and

$$Y_n = \sum_{i=1}^{n} Y_i$$

have g.p.d with generating functions

$$\{f(\theta_1)\}^m$$

and

$$\{f(\theta_2)\}^n$$

and taking values in some sets $S_{1m}$ and $S_{2n}$ respectively. The sets $S_{1m}$ and $S_{2n}$ may be given by

$$S_{1m} = \left\{ x_m = \sum_{i=1}^{m} x_i : x_i \in S_i \text{ for all } i \right\}$$

and

$$S_{2n} = \left\{ y_n = \sum_{i=1}^{n} y_i : y_i \in S_i \text{ for all } i \right\}$$

2.1. Application to truncated binomial distribution

Let $S_1 = \{ c_1, c_1 + 1, \ldots, m \}$ and $S_2 = \{ c_2, c_2 + 1, \ldots, n \}$

and define

$$f_{\theta_1}(x) = \sum_{x=c_1}^{m} \binom{m}{x} \theta_1^x \quad \text{and} \quad f_{\theta_2}(y) = \sum_{y=c_2}^{n} \binom{n}{y} \theta_2^y$$

Then the probability density functions of the random variables $X$ and $Y$ are given by

$$P_r(X=x) = \frac{\binom{m}{x} \theta_1^x}{\sum_{x=c_1}^{m} \binom{m}{x} \theta_1^x} \quad x = c_1, c_1 + 1, \ldots, m$$

and

$$P_r(Y=y) = \frac{\binom{n}{y} \theta_2^y}{\sum_{y=c_2}^{n} \binom{n}{y} \theta_2^y} \quad y = c_2, c_2 + 1, \ldots, n$$

where $\theta_i = \frac{p_i}{1-p_i} (i=1, 2)$ so that the above probabilities reduce to those of truncated binomial variates with probability of success $p_i (i=1, 2)$

The conditional distribution of $Y$ given $X+Y=t$ is

$$P_r(Y=y|X+Y=t) = c_1(p) \cdot \left( \frac{m}{y} \right) \left( \begin{array}{c} n \\\ y \end{array} \right) \theta_2^y$$

$$y = c_2, c_2 + 1, \ldots, t - c_1$$
where
\[ \left\{ c_t \left( \frac{1}{p^t} \right) \right\}^{-1} = \sum_{y=c_2}^{t-c_1} \binom{m}{t-y} \binom{n}{y} p^y \]
and
\[ \rho = \frac{\theta_2}{\theta_1} = \frac{\theta_2 (1 - \theta_1)}{\theta_1 (1 - \theta_2)} \]

The test of the hypothesis \( H_1 : \theta_2 < \theta_1 \) vs. \( K_1 : \theta_2 > \theta_1 \) is equivalent to the test of the hypothesis \( H_1' : \theta < 1 \) vs. \( K_1' : \theta > 1 \) in the conditional distribution.

2.2. An example

While sampling for studies in albinism, sampling has necessarily to be restricted to families having at least one child.

Suppose that two families were examined. In the first family out of nine, four were albino child and in the second family out of ten, six were albino child.

Let \( \theta_1 = \text{Prob for an albino child in the first family} \)
\( \theta_2 = \text{Prob for an albino child in the second family} \).

We want to test
\[ H_{11} : \theta_2 < \theta_1 \quad \text{vs.} \quad K_{11} : \theta_2 > \theta_1 \]
i.e.,
\[ H_{11}' : \theta < 1 \quad \text{vs.} \quad K_{11}' : \theta > 1. \]

In the terminology of \((2.1)\)
\( m = 9, \quad n = 10, \quad x = 4, \quad y = 6, \quad c_1 = c_2 = 1, \quad t = 10 \)
and from \((7)\) with \( \rho_0 = 1 \) & \( t = 10 \) we have
\[ \left\{ c_1 (1) \right\}^{-1} = \frac{9}{1} \binom{9}{10} \binom{10}{y} = 92379 \]
and the constants \( k \) and \( \gamma \) of \((8)\) are to be determined from the equation, for a test at 5% level of significance
\[ \sum_{y=3+1}^{9} \frac{1}{92379} \binom{9}{10-y} \binom{10}{y} + \gamma \frac{1}{92379} \binom{9}{10-K} \binom{10}{K} = 0.05 \]
which gives \( k = 7, \gamma = 0.2886. \)

Hence with observed \( y = 6 (< k = 7) \), we accept \( H_{11}' \).
3. A test of inflation for an inflated power series distribution:

Let \((x_1, x_2, \ldots, x_n)\) denote a random sample of size \(n\) from a population whose distribution is given by (2) and let \(n_i\) observations have values equal to \(i\) (\(i = 0, 1, 2, \ldots\)) \((\sum n_i = n)\). Then the joint distribution of \(n_0, n_1, n_2, \ldots\) is given by

\[
\phi(n_0, n_1, n_2, \ldots) = \frac{n!}{n_0!} \left[ 1 - \alpha + \frac{\alpha a_0}{f(\theta)} \right]^{n_0} \prod_{k=1}^{\infty} \left[ \frac{\alpha a_k \cdot g_k}{f(\theta)} \right]^{n_k}
\]

So that the joint distribution of \(n_0\) and \(t = \sum kn_k = \sum x_i\) is given by

\[
P_r(n_0, t) = \left( \begin{array}{c} n \\ n_0 \end{array} \right) \phi(n_0, t) \cdot A(t, n_0)
\]

where

\[
\xi = \frac{(1 - \alpha) f(\theta) + \alpha a_0}{\alpha a_1}
\]

and

\[
A(t, n_0) = \sum (n-n_0)! \prod_{k=2}^{\infty} \left( \frac{a_k}{a_1} \right)^{n_k}
\]

Hence the conditional distribution of \(n_0\) given \(t\) is

\[
P_r(n_0|t) = c_t(\xi) \left( \begin{array}{c} n \\ n_0 \end{array} \right) \phi(n_0) \cdot A(t, n_0), \quad n_0 = 0, 1, 2, \ldots
\]

where \(\{c_t(\xi)\}^{-1} = \sum_n \left( \begin{array}{c} n \\ n_0 \end{array} \right) \phi(n_0) \cdot A(t, n_0)\).

In this conditional distribution, the test of \(H_0\) is equivalent to testing

\[H_0' : \xi = \xi_0 \quad \text{vs.} \quad H_0' : \xi > \xi_0\]

where \(\xi_0 = \frac{a_0}{a_1}\) and there exists an UMP test procedure.
which is given by:

\[ \phi(n_0/t) = \begin{cases} 1 & \text{if } n_0 > N_0 \\ \gamma & \text{if } n_0 = N_0 \\ 0 & \text{if } n_0 < N_0 \end{cases} \]

where the constants \( N_0 \) and \( \gamma \) are to be determined from the equation

\[ \sum_{n_0=N_0}^{N_0+1} P_r(n_0/t) + \gamma P_r(N_0/t) = a'. \]

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**References**


