Chapter 3

SIMPLE AND SEMISIMPLE

$L$-MODULES

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3.1 Introduction.

The concepts of simple and semisimple modules form an important area of study in the theory of $R$-modules. Recall that a left module $M$ over a ring $R$ is said to be simple if it does not contain any submodule other than 0 and $M$, and if $M \neq 0$. A left module $M$ is said to be semisimple if each of its proper submodules is a direct summand of $M$ and there are several other equivalent definitions in the literature. In this chapter we extend these notions to the fuzzy setting and investigate some properties.

3.2 Simple $L$-Modules.

In this section we introduce the concept of simple $L$-modules and prove that if $L$ is regular, then $M$ is simple if and only if $1_M$ is a simple left $L$-module.

3.2.1 Definition:

Let $\mu \in L(M)$ be a left $L$-module. Then $\lambda \in L^M$ is said to be an $L$-submodule of $\mu$ if $\lambda$ itself is a left $L$-module such that $\lambda \subseteq \mu$. That is if

(i) $\lambda(0) = 1$

(ii) $\lambda(x + y) \geq \lambda(x) \land \lambda(y) \quad \forall x, y \in M$

(iii) $\lambda(rx) \geq \lambda(x) \quad \forall r \in R, \forall x \in M$

(iv) $\lambda(x) \leq \mu(x) \quad \forall x \in M$

3.2.2 Definition:

Let $\mu : M \rightarrow L$ be a left $L$-module. Then a left $L$-module $\eta : M \rightarrow L$ is said
to be a strictly proper $L$-submodule of $\mu$ if $\eta \subseteq \mu$, $\eta \neq 1_{\{0\}}$, $\eta(x) = \mu(x)$ $\forall x$
for which $\eta(x) > 0$ and $\eta^\ast \subseteq \mu^\ast$; and $\eta : M \to L$ is said to be a proper $L$
submodule of $\mu$ if $\eta \subseteq \mu$, $\eta \neq 1_{\{0\}}$, $\eta^\ast \subseteq \mu^\ast$.

3.2.3 **Definition:**

$\mu \in L(M)$ is said to be a simple left $L$-module if $\mu$ has no proper $L$
submodules.

3.2.4 **Example:**

Let $D$ be a division ring. Let $R = M_n(D)$ be the set of all $n \times n$ matrices
with entries in $D$. Let $R_i = \{A \in R : j^{\text{th}}$ column of $A$ is $0$, for $j \neq i\}$. Then $R_i$ is a
left $R$-module.

For $i = 1, 2, 3..., n$, define $\mu_i : R \to [0, 1]$ as

$$\mu_i(A) = \begin{cases} 
1 & \text{if } A = 0 \\
\frac{1}{2^i} & \text{if } A \in R_i - \{0\} \\
0 & \text{if } A \not\in R_i 
\end{cases}$$

Then $\mu_i; i = 1, 2, 3..., n$ are simple left $L$-modules.

3.2.5 **Theorem:**

Suppose $L$ is regular. Then $M$ is simple if and only if $1_M$ is a simple left
$L$-module.

**Proof:**

Suppose $M$ is simple. Then $M$ has no proper submodules. If possible let
$1_M$ be not a simple left $L$-module. Then $1_M$ has a proper left $L$-submodule say $\mu$
such that $\lambda \neq 1_{\{0\}}$, $\lambda^* \subset 1_M^* = M$. Since $\lambda \in L(M)$, and since $L$ is regular, $\lambda^*$ is a submodule of $M$ and $\lambda^* \neq \{0\}$, $\lambda^* \neq M$. That is $\lambda^*$ is a proper submodule of $M$. This contradicts the fact that $M$ is simple.

Conversely suppose that $1_M$ is a simple left $L$-module. If possible assume that $M$ is not simple. Let $N$ be a proper submodule of $M$. Then $N \neq \{0\}, N \neq M$. Define $\lambda : M \to L$ by

$$\lambda(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}$$

Then $\lambda \in L(M)$; $\lambda \subset 1_M$, $\lambda \neq 1_{\{0\}}$ or $1_M$ and $\lambda^* \subset M = 1_M^*$. Hence $\lambda$ is a proper $L$-submodule of $1_M$ which is a contradiction.

### 3.3 Semisimple $L$-Modules.

Now we introduce the notion of semisimple $L$-modules and prove the fuzzy analogues of the theorems 'every submodule of a semisimple module is semisimple' and 'every semisimple module contains a simple submodule' in the crisp case. We also prove some other theorems which are relevant in the fuzzy setting.

#### 3.3.1 Definition:

Let $\mu \in L(M)$. Then $\mu$ is said to be a semisimple left $L$-module if whenever $\lambda$ is a strictly proper $L$-submodule of $\mu$, there exists a strictly proper $L$-submodule $\eta$ of $\mu$ such that $\mu = \lambda \oplus \eta$. 
That is if \( \lambda \) is a proper \( L \)-submodule of \( \mu \) such that \( \lambda(x) = \mu(x) \ \forall x \) for which \( \lambda(x) > 0 \); then there exists a proper \( L \)-submodule \( \eta \) of \( \mu \) satisfying \( \eta(x) = \mu(x) \ \forall x \) for which \( \eta(x) > 0 \), such that \( \mu = \lambda \oplus \eta \).

### 3.3.2 Example:

Let \( D \) be a division ring. Consider \( R = M_3(D) = \{3 \times 3 \text{ matrices over } D\} \), which is a ring with unity with respect to the addition and multiplication of matrices. Let \( R_i = \{A \in R : j^{th} \text{ column of } A \text{ is } 0, \text{ for } j \neq i\} \). Then \( R_i \) is a simple left module over \( R \) for \( i = 1, 2, 3 \) and \( _RR \) is a semisimple left module.

Define \( \mu : R \rightarrow [0, 1] \) by

\[
\mu(A) = \begin{cases} 
1 & \text{if } A \neq 0 \\
\frac{1}{2} & \text{if } A \in R_i - \{0\} \\
\frac{1}{3} & \text{if } A \in R_i + R_2 - \{R_i\} \\
\frac{1}{4} & \text{if } A \in R_i + R_2 + R_3 - \{R_i + R_2\}
\end{cases}
\]

Then \( \mu \) is a semisimple left \( L \)-module.

### 3.3.3 Theorem:

Let \( M \) be a left module over a ring \( R \). Then \( M \) is semisimple if and only if \( 1_M \) is a semisimple left \( L \)-module.

**Proof:**

Suppose \( M \) is semisimple. To prove that \( 1_M \) is a semisimple left \( L \)-module.

Let \( \mu \) be a strictly proper \( L \)-submodule of \( 1_M \). To show that there exists a strictly proper \( L \)-module \( \eta \in \mathcal{L}(M) \) such that \( 1_M = \mu \oplus \eta \).
For this let \( S = \{ x \in M : \mu(x) = 1 \} \). Then obviously \( S \) is a submodule of \( M \); \( S \neq 0, S \neq M \). Therefore since \( M \) is semisimple, \( S \) is a direct summand of \( M \). Hence we can write \( M = S \oplus T \) for some submodule \( T \) of \( M \). Now define \( \eta : M \to L \) by,

\[
\eta(x) = \begin{cases} 
1 & \text{if } x \in T \\
0 & \text{if } x \notin T 
\end{cases}
\]

Then \( \eta \in L(M) \). Further \( \eta(x) = 1_M(x) \quad \forall \ x \) for which \( \eta(x) > 0 \). Now \((\mu + \eta)(x) = \vee \{ \mu(y) \land \eta(z) : y, z \in M, y + z = x \} \). Since \( M = S \oplus T \), \( x \in M \) can be uniquely expressed as \( x = s + t \), where \( s \in S \) and \( t \in T \). Thus \( x = s + t \), where \( \mu(s) = 1 \), \( \eta(t) = 1 \). Therefore \( (\mu + \eta)(x) = 1 \quad \forall \ x \in M \). Thus we get \( \mu + \eta = 1_M \). Also, since \( S \cap T = \{0\} \), we get \( \mu \cap \eta = 1_{\{0\}} \) and hence \( 1_M = \mu \oplus \eta \). This proves the first part.

Conversely suppose that \( 1_M \) is a semisimple left \( L \)-module. To prove that \( M \) is semisimple. For this let \( S \) be any proper submodule of \( M \). To prove that \( S \) is a direct summand of \( M \). Define \( \mu \in L^M \) by,

\[
\mu(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \notin S 
\end{cases}
\]

Then clearly \( \mu \subseteq L(M) \) and \( \mu \) is a strictly proper \( L \)-submodule of \( 1_M \). Since \( 1_M \) is semisimple, \( 1_M = \mu \oplus \eta \) for some strictly proper \( L \)-submodule \( \eta \) of \( 1_M \). Take \( T = \{ x \in M : \eta(x) = 1 \} \). Then \( T \) is a submodule of \( M \). We show that \( M = S \oplus T \). For all \( x \in M \), we have,

\[
1 = 1_M(x) = (\mu + \eta)(x) = \vee \{ \mu(y) \land \eta(z) : y + z = x, \ y, z \in M \}
\]
which implies \( \mu(y) = \eta(z) = 1 \) for some \( y, z \in M \), where \( y + z = x \) (since \( \mu(y) = 1 \) or \( 0 \) and \( \eta(z) = 1 \) or \( 0 \)). Thus if \( x \in M \) then \( x = y + z \) for some \( y \in S, \ z \in T \).

So \( M = S + T \). Also, since \( \mu \cap \eta = 1_{(0)} \), we get \( S \cap T = \{0\} \). Therefore \( M = S \oplus T \). This completes the proof.

### 3.3.4 Theorem:

Let \( \mu \in L(M) \) be a semisimple left \( L \)-module. Then \( \mu_a^> \) is a semisimple submodule of \( M \) \( \forall \ a \neq 0 \in L \).

**Proof:**

Given \( \mu \in L(M) \) is semisimple. To prove that \( \mu_a^> \) is a semisimple submodule of \( M \) \( \forall \ a \neq 0 \in L \). Assume \( a \neq 0 \). Let \( A \) be a submodule of \( \mu_a^> \). To show that \( A \) is a direct summand of \( \mu_a^> = \{x \in M : \mu(x) > a\} \). Define \( \eta \in L^M \) by

\[
\eta(x) = \begin{cases} 
\mu(x) & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
\]

Then clearly \( \eta \in L(M) \) and \( \eta \) is a strictly proper \( L \)-submodule of \( \mu \) such that \( \eta_a^> = A \). Since \( \mu \) is semisimple and \( \eta \) is a strictly proper \( L \)-submodule of \( \mu \), there exists a strictly proper \( L \)-submodule \( \nu \) of \( \mu \) such that \( \mu = \eta \oplus \nu \). Then \( \nu(x) = \mu(x) \ \forall \ x \) for which \( \nu(x) > 0 \). Take \( B = \nu_a^> = \{x \in M : \nu(x) > a\} \). We prove that \( \mu_a^> = A \oplus B \). That is we prove that \( \mu_a^> = \eta_a^> \oplus \nu_a^> \).

For: \( x \in \mu_a^> \Rightarrow \mu(x) > a \)

\[
\Rightarrow (\eta \oplus \nu)(x) > a
\]

\[
\Rightarrow \nu(\eta(y) \wedge \nu(z) : y, z \in M; \ y + z = x) > a
\]
Thus $x \in \mu^> \Rightarrow \exists y, z \in M$ with $y + z = x$ such that $\eta(y) \wedge \nu(z) > a$

$\Rightarrow \exists y, z \in M$ with $y + z = x$ such that $\eta(y) > a$ and $\nu(z) > a$

Hence $\mu^> = \eta^> + \nu^>$.

Also, $x \in \eta^> \cap \nu^> \Rightarrow x \in \eta^> \cap \nu^>$

$\Rightarrow \eta(x) > a$, $\nu(x) > a$

$\Rightarrow (\eta \cap \nu)(x) = \eta(x) \wedge \nu(x) \geq a > 0$

$\Rightarrow x = 0$ (since $\eta \oplus \nu$ is a direct sum)

Thus $\eta^> \cap \nu^> = \{0\}$. Hence $\mu^> = \eta^> \oplus \nu^> = A \oplus B$. That is $A$ is a direct summand of $\mu^>$. So $\mu^>$ is a semisimple submodule of $M \forall a \neq 0 \in L$.

Note: In the above theorem, if $L$ is regular, $\mu^>$ is semisimple even if $a = 0$.

That is $\mu^>$ is a semisimple submodule of $M$, if $L$ is regular.

3.3.5 Theorem:

Suppose $L$ satisfies the complete distributive property. Then every strictly proper $L$-submodule of a semisimple left $L$-module is semisimple.

Proof:

Let $\mu$ be a given semisimple left $L$-module and $\lambda$ be a strictly proper $L$-submodule of $\mu$. To show that $\lambda$ is a semisimple left $L$-module. For this let $\eta$ be a strictly proper $L$-submodule of $\lambda$. Since $\lambda$ is a strictly proper $L$-submodule of $\mu$ we see that $\eta$ is a strictly proper $L$-submodule of $\mu$. Since $\mu$ is semisimple there exists a strictly proper $L$-submodule $\delta$ of $\mu$ such that $\mu = \eta \oplus \delta$. 
Now we prove that $\lambda \cap (\eta \oplus \delta) = (\lambda \cap \eta) \oplus (\lambda \cap \delta)$. We have

$$[\lambda \cap (\eta + \delta)](x)$$

$$= \lambda(x) \land (\eta + \delta)(x)$$

$$= \lambda(x) \land (\lor \{\eta(y) \land \delta(z) : y, z \in M; y + z = x\})$$

$$= (\lor \\{\lambda(y) \land \lambda(z) : y, z \in M; y + z = x\}) \land (\lor \{\eta(y) \land \delta(z) : y, z \in M; y + z = x\})$$

(Since $\lambda$ being an $L$-module, for $x = y + z$, $\lambda(x) = \lambda(y + z)$)

$$\lambda(y) \land \lambda(z); \text{ and equality is attained for } x = x + 0 \text{ or } x = 0 + x$$

$$= \lor \\{(\lambda(y) \land \lambda(z)) \land (\eta(y) \land \delta(z)) : y, z \in M; y + z = x\}$$

$$= \lor \\{(\lambda(y) \land \eta(y)) \land (\lambda(z) \land \delta(z)) : y, z \in M; y + z = x\}$$

$$= \lor \\{(\lambda \cap \eta)(y) \land (\lambda \cap \delta)(z) : y, z \in M; y + z = x\}$$

$$= [(\lambda \cap \eta) + (\lambda \cap \delta)](x)$$

Thus $\lambda \cap (\eta + \delta) = (\lambda \cap \eta) + (\lambda \cap \delta) = \eta + (\lambda \cap \delta)$ (since $\eta \subseteq \lambda$)

Now $(\eta \cap (\lambda \cap \delta))(x) = (\eta \cap (\delta \cap \lambda))(x)$

$$= ((\eta \cap \delta) \cap \lambda)(x)$$

$$= 1_{\{0\}}(x) \land \lambda(x)$$

$$= 1_{\{0\}}(x)$$

Thus $\eta \cap (\lambda \cap \delta) = 1_{\{0\}}$. Hence $\eta + (\lambda \cap \delta) = \eta \oplus (\lambda \cap \delta)$.

Therefore $\lambda \cap (\eta \oplus \delta) = \eta \oplus (\lambda \cap \delta)$. So we get $\lambda = \lambda \cap \mu = \lambda \cap (\eta \oplus \delta) = \eta \oplus (\lambda \cap \delta)$. Obviously $\lambda \cap \delta$ is a strictly proper $L$-submodule of $\lambda$. Therefore $\lambda$ is a semisimple $L$-module. This completes the proof of theorem.
3.3.6 Theorem:

Suppose $L$ is regular. Let $\mu \in L(M)$ be a semisimple left $L$-module. Then $\mu$ contains a simple left $L$-module.

Proof:

Given that $\mu \in L(M)$ is semisimple. Then for $a \in L$, $\mu_a$ is a semisimple submodule of $M$. Therefore $\mu_a$ contains a simple submodule say $A$. That is $A$ has no proper submodule.

Define $\eta : M \to L$ by

$$
\eta(x) = \begin{cases} 
\mu(x) & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases}
$$

We claim that $\eta$ is a simple left $L$-module. If not $\eta$ has a proper $L$-submodule $\nu$; $\nu \neq 1_{\{0\}}$, $\nu \subset \eta \subseteq A$. Thus $\{0\} \subset \nu \subset A$. But $\nu = \{x \in M : \nu(x) > 0\}$ is clearly a submodule of $A$. (since $L$ is regular and $\nu \in L(M)$). Thus $\nu$ is a proper submodule of $A$ which is a contradiction. Hence $\eta$ is a simple left $L$-module. 

For a left $R$-module $\_\_M$ the equivalence of the following three properties is well known in crisp theory.

(1) $M$ is semisimple.

(2) $M$ is the sum of a family of simple submodules.

(3) $M$ is the direct sum of a family of simple submodules.

Similar to this result we have the following theorem in the fuzzy case.
3.3.7 Theorem:

Let $L$ be a complete distributive lattice and let $\mu \in L(M)$ be a left $L$-module. Then the following are equivalent.

(1) $\mu$ is semisimple.

(2) $\mu$ is the sum of a family of strictly proper simple $L$-submodules $\mu_i$, $(i \in I)$ of $\mu$.

(3) $\mu$ is the direct sum of a family of strictly proper simple $L$-submodules $\mu_j$, $(j \in J)$ of $\mu$.

Proof:

(1) $\Rightarrow$ (2). Suppose $\mu \in L(M)$ is semisimple. Let $\lambda$ be the sum of all strictly proper simple $L$-submodules $\mu_i$, $(i \in I)$ of $\mu$, where $\mu_i(x) = \mu(x) \forall x$ for which $\mu_i(x) > 0$, $(i \in I)$. Then clearly $\lambda$ is a strictly proper $L$-submodule of $\mu$ such that $\lambda(x) = \mu(x) \forall x$ for which $\lambda(x) > 0$. Therefore there exists a strictly proper $L$-submodule $\eta$ of $\mu$ such that $\lambda(x) = \mu(x) \forall x$ for which $\lambda(x) > 0$. We claim that $\eta = 1_{\{0\}}$ so that $\mu = \lambda$. If not, being an $L$-submodule of $\mu$ which is strictly proper, $\eta$ is semisimple and so $\eta$ contains a simple $L$-submodule say $\delta$. Moreover we can choose $\delta$ such that $\delta(x) = \eta(x) \forall x$ for which $\delta(x) > 0$, and so $\delta(x) = \mu(x) \forall x$ for which $\delta(x) > 0$ (since $\eta$ is a strictly proper submodule of $\mu$). Then $\delta \neq 1_{\{0\}}$, $\delta \subseteq \eta$ and $\delta^* \subseteq \eta^*$.

Also being a strictly proper simple $L$-submodule of $\mu$ such that $\delta(x) = \mu(x) \forall x$ for which $\delta(x) > 0$, we get $\delta \subseteq \lambda$. Thus we get $\delta \subseteq \lambda \cap \eta$ which in turn implies that $\delta = 1_{\{0\}}$. This is a contradiction. Hence $\eta = 1_{\{0\}}$ and so $\mu = \lambda$. 
Conversely let $\mu$ be the sum of a family of strictly proper simple $L$-submodules $\mu_i$ ($i \in I$) of $\mu$ say $\mu = \sum_{i \in I} \mu_i$, where for $i \in I$, $\mu_i(x) = \mu(x)$ $\forall x$ for which $\mu_i(x) > 0$. To show that $\mu$ is a semisimple left $L$-module. That is to show that corresponding to any strictly proper $L$-submodule $\lambda$ of $\mu$ there exists a strictly proper $L$-submodule $\eta$ of $\mu$ such that $\mu = \lambda \oplus \eta$.

Let $\lambda$ be a strictly proper $L$-submodule of $\mu$. Consider subsets $J \subseteq I$ with the properties

(i) \[ \sum_{j \in J} \mu_j \text{ is a direct sum } \bigoplus_{j \in J} \mu_j \]

(ii) \[ \lambda \cap \sum_{j \in J} \mu_j = 1_{\{0\}} \]

Consider the family $F$ of all such $J$'s with respect to ordinary inclusion. $F \neq \emptyset$ as it contains the empty set. By Zorn's lemma there exists a maximal element in $F$.

Take such a maximal $J$. For this $J$, let $\mu' = \lambda + \sum_{j \in J} \mu_j = \lambda \oplus (\bigoplus_{j \in J} \mu_j)$. Then $\mu'$ is such that $\mu'(x) = \mu(x)$ $\forall x$ for which $\mu'(x) > 0$. Now we show that $\mu' = \mu$. For this we prove that $\mu_i \subseteq \mu'$ $\forall i \in I$. Suppose not. Then $\mu_i \not\subset \mu'$ for some $i$. Consider $\mu' \cap \mu_i$ for this $i$. It is an $L$-submodule of $\mu_i$. Since $\mu_i$ is simple we have $\mu' \cap \mu_i = 1_{\{0\}}$ or $(\mu' \cap \mu_i)^* = \mu_i^*$. Therefore $\mu' \cap \mu_i = 1_{\{0\}}$ or $\mu_i$ (since $L$ is regular, if $(\mu' \cap \mu_i)(x) > 0$ then both $\mu'(x)$, $\mu_i(x) > 0$; and then $\mu_i(x) = \mu(x) = \mu'(x)$). Since $\mu_i \not\subset \mu'$ we get $\mu' \cap \mu_i = 1_{\{0\}}$. Therefore $\mu' + \mu_i$ is a direct sum $\mu' \oplus \mu_i = \lambda \oplus (\bigoplus_{j \in J} \mu_j) \oplus \mu_i$. This contradicts the maximality of $J$. Therefore $\mu_i \subseteq \mu'$.
\( \forall i \in I. \) This implies \( \mu = \sum_{i \in I} \mu_i \subseteq \mu' \). That is \( \mu \subseteq \mu' \). Clearly \( \mu' \subseteq \mu \). Hence \( \mu = \mu' = \lambda \oplus \sum_{j \in J} \mu_j \). Thus there exists a strictly proper \( L \)-submodule \( \eta = \sum_{j \in J} \mu_j \) of \( \mu \), where \( \eta(x) = \mu(x) \ \forall x \) for which \( \eta(x) > 0 \), such that \( \mu = \lambda \oplus \eta \). Therefore \( \mu \) is semisimple.

(2) \( \Rightarrow \) (3). Suppose \( \mu \in L(M) \) is the sum of a family of strictly proper simple \( L \)-submodules \( \mu_i, (i \in I) \) of \( \mu \) where \( \mu_i(x) = \mu(x) \ \forall x \) for which \( \mu_i(x) > 0 \). To show that \( \mu \) is the direct sum of a family of such simple \( L \)-submodules.

Consider \( \mu = \sum_{i \in I} \mu_i \) where \( \mu_i \)'s are strictly proper simple \( L \)-submodules of \( \mu \) such that \( \mu_i(x) = \mu(x) \ \forall x \) for which \( \mu_i(x) > 0 \). Consider the family \( F = \{J \subseteq I : \sum_{j \in J} \mu_j \text{ is a direct sum} \} \) with respect to the ordinary inclusion. Then \( F \) contains a maximal element \( J \). Then as in the proof of (2) \( \Rightarrow \) (1) it is easy to see that \( \mu = \bigoplus_{j \in J} \mu_j \).

(3) \( \Rightarrow \) (2). This is obvious.