PART - I

BUDGET DECISION
2.1 Introduction:

In Chapter 1, the recognition of utility of advertising by different business and non-business organisations in India has been discussed. Many companies in India spend huge amount of money on advertising and their expenditure runs into millions and still advertising expenditure does not appear to have become noticeably more rational and less intuitive.

Often advertising expenditure decisions are arbitrary and, at times, merely a guess work. As a result, a gross inadequacy of funds at one extreme and extravagance at another extreme are noticed. In other words, what advertising expenditure specially requires is good planning. Budgeting is a form of the plan which protects the advertiser from an unbalanced situation, wherein he may find that he has either spent more or less than he should have.
The preparation of an advertising budget generally determines the size of advertising expenditure. However, for this purpose in India there are no simple and scientific methods but only arbitrary conventional practices which are generally followed but do not quite make the process of budgeting\(^{(19)}\). The motive behind this study is to provide Budget Decision Models which include almost all factors which a Good Budget Decision Model* should consist of and at the same time should be simple and easy to understand.

2.2 Literature Survey:

The main purpose of this section is to highlight the reasons for our having to develop the new Budget Decision Models and hence we do not present a very detailed review of all published literature on this subject.

From the study of published literature on this subject two approaches are discernible.

* Refer Dhalla's\(^{(24)}\) suggestions for a Good Budget Decision Model discussed in Chapter-1.
2.2.1 **Operational Budgeting Approach:**

The detailed review of these methods are given by Kotler \(^{63,65}\), Aaker and Myer \(^{2}\), C. H. Sandage \(^{107}\), Colin Gilligan \(^{40}\), Chunawalla \(^{19}\), etc. These methods however, cannot be blindly employed, for there is no single method which is applicable to all the situations.

2.2.2 **Analytical Budgeting Approach:**

The determination of the total advertising budget by using only the operational methods has little basis in logic. Each of these methods has serious flaws which cannot easily be obviated. The alternative to these methods, which in practice is usually a complementary approach to the operational methods, is the budgeting of advertising according to an analytic advertising approach. Advertising researchers have proposed several models for setting the advertising budget which are discussed here briefly.
2.2.2.1 Static Budget Decision Models:

Those models which do not consider scheduling of budget over time are described here.

1. Simple Model:

In this model, current advertising is the only variable considered for affecting current period of sales. The carry-over effect of advertising, other affecting factors of marketing mix and competitive situations are not considered here. Hence the method has very limited application in practice.

2. Competitive Model:

In simple competitive model it is assumed that advertising does not affect total industry demand but affects firm's market share. Also, relative advertising expenditures are assumed to determine the share of market gained by each of the firms in the industry. This simple model may be extended to account for differential sales effectiveness of advertising between the firms by multiplying each term by a constant or by raising each advertising term to a constant exponent.
The first game theory applications to advertising were developed by Friedman. He was concerned with promotional competition between two competitors in geographic areas. The focus was the final demand in each geographic region which would be allocated to the competitors on the basis of their respective advertising shares. Sales in each area that are not influenced by advertising were ignored. The formulation of the problem assumed that an amount spent on advertising is equally effective for both advertisers. Each competitor is also assumed to have a budget constraint. Since the total sales influenced by advertising are constant, in this case the problem becomes a two person constant sum game. Friedman showed that the optimum allocation of funds to each area by each competitor should be proportional to the sales potential in each area. He also studied one firm's optimal strategy when another firm does not use its optimal strategy.

Mills extended Friedman's model by constant sum games with more than two competitors. He also used a more realistic market share term. A variable sum approach was considered by Baligh. Shakun extended Friedman's model by constant sum games with more than two competitors. He also used a more realistic market share term. A variable sum approach was considered by Baligh. Shakun extended Friedman's model by constant sum games with more than two competitors. He also used a more realistic market share term. A variable sum approach was considered by Baligh.
has been concerned with developing game theory models for market in which there are product line interdependencies. Recently, Erickson (30) has developed a dynamic model of advertising rivalry between competitors in a duopoly. He obtains analytical results for the case of pure market share rivalry in a mature market. A more general model, allowing for market expansion as well as market share rivalry, is analysed here numerically.

Mills (10) models are developed in a competitive situation when each competitor has only one control variable. The models which consider more than one control variable do not consider the effect of advertising efforts separately but merge with selling efforts. Though the models discussed here are easily applicable, they are based on restrictive assumptions, which are far from reality.

3. Adaptive Models:

John D. C. Little (79) has developed a model for the adaptive control of advertising expenditures. The application of the model begins with a determination of the profit maximizing level of expenditure.
for advertising, given a model of sales response to advertising. Since it is not possible to known the parameters of the sales response function with much precision and the response may change over time, an experiment is designed to measure the sales response to advertising during the next period. The model assumes a simple quadratic sales response to advertising that exhibits diminishing returns and the budget obtained here is the profit maximising level, given current estimates of the parameters.

Thus the model is based on a decision procedure that is continuously updated with new information. This information is generated by observing the results of the model's last decision and the outcomes of carefully designed experiments. It is then fed back to the model, where the decision model parameters themselves are updated.

The procedure adopted here is lengthy. Also this model ignores the possibility of carry over effects, and competitive effects are only partially considered. In addition, it ignores the inter-dependencies in the marketing mix of the firm.
4. Determinants of the Advertising to Sales Ratio

Paul Fanis and Mark S. Albion\(^{(32)}\) have tried to establish here the condition for optimal advertising outlays by a profit maximising monopoly which requires the value of marginal product of advertising and price elasticity of demand.

He finds that (i) firms with high profit margins on sales should generally have high ratio, (ii) firms with high cost of capital should have generally lower ratio, (iii) firms that depend on current advertising expenditure generate large percentage of total sales and should also have high ratios.

In this study certain variables were found to be consistently related to industry ratio. But it is difficult and controversial to measure the influence of advertising on sales. Also, the parameters of the model are difficult to estimate.

2.2.2.2 Dynamic Budget Decision Models:

Some important models which consider scheduling of budget over time are discussed here.
1. **Simple Model**

A simple model for determining an advertising budget in the presence of carry-over effects is given by Julian Simon (113). He assumes that revenues realized in future periods due to advertising expenditure in the present period will decrease by a constant rate per period into the indefinite future.

Nerlove and Arrow (92) considered a more general formulation of the effect of advertising decisions in each of a number of future periods and determine the optimum on the basis of the dynamic effects of the series of decisions. They allowed dynamic advertising effects to accumulate in a storage variable called 'goodwill'. To determine the optimum level of goodwill, the classical calculus model is applied with enough assumptions to assure that the necessary conditions for the optimum are also sufficient.

An interesting finding which results from the study is that in the special case of (i) constant marginal costs of production (ii) a demand function that is linear in its logarithm (iii) a stationary environment (iv) exogenous competition, the optimal advertising budget is a constant percentage of sales
and it proves to be optimal in this case. But these models show a limited view of the functions of advertising and hence they have very limited utility.

2. Sales Response and Decay Models

Vidale and Wolfe (123) developed a model in which they attempted to explain the rate of change of sales which occurs as the result of advertising. They considered dynamic advertising effects in a model that is based upon experimental evidence. In their studies for several major industrial concerns, they found that the rate of change in sales may be related to advertising expenditures.

Although there is some empirical evidence that this model for budgeting advertising is a reasonable one and there is considerable logical justification for the three prominent aspects of the model—sales decay, sales response and a sales saturation level, the basic drawback of the model is its concentration on potential customers and consequent neglect of current customers. To certain products which have large bodies of loyal customers, the model may be directly applicable. For others, where purchase quantity affects usage or the problem of brand-switching arises, the model may require modification.
3. Regression Models

By far the most extensive application of regression analysis has used time-series data. In this context, the task is to predict or explain the dependent variable, which is either sales or market share. Among the independent or explanatory variables measures of advertising expenditures are considered. The regression coefficient corresponding to the advertising variable is then a measure of the short-term response to advertising. In an early classic study, Paldai\(^{(94)}\) applied this model to annual advertising and sales data of 'Lydia Pinkham Company'.

The preceding approaches have suggested that a change in advertising effort will cause a change in a sales response. It may be that a sales change will actually cause in some later period a change in advertising effort. Under such circumstances, the assumption that advertising effort creates sales, and not the reverse, is suspected.

There is a mechanism to explore the direction of causation that is operating between advertising and sales. This involves introducing two or more simultaneous equations into a regression model. Bass\(^{(12)}\)
has done pioneering model building and model testing in this area. A key assumption in such a model is that the time period used reflects the cause effect delay.

There are a series of issues, considerations and difficulties that arise when regression analysis is used to determine the effect of advertising on sales. They include data problems, measuring the carry-over effect of advertising, the direction of causation issues and the inclusion of other marketing variables.

4. Transfer-Function Modeling:

According to Adams (5), sales forecasts are essential in planning market strategy and the management should consider it important to know the pattern of advertisement's influence on sales. In particular it is important to know if the effect of changes in advertising expenditure from one period to another will even be measurable on current or subsequent period of sales. One product's advertising-sales relation was studied by comparing the best available regression model to Transfer-Function Model.
The Transfer Function results are demonstrably superior to the best of regression models. It is seen to provide a richer description of the underlying relation between advertising and sales. In particular it permits inferences about the duration of advertising effects as well as offers suggestions on the timing of advertising expenditure. Though the results of this model are superior to the regression models, like regression models it does not consider adverse effect of competition and at the same time the question of data problem stands.

5. Some Advanced Models based on empirical Approach :

M. M. Metwally (84) has developed a Model of sales Response to Advertising based on eight Australian products. He proved that the more attached the customers are to a particular brand, the lower the sales elasticity of advertising. Moreover advertising sales ratio are higher for brands selling in markets with greater brand consciousness.

According to Dhall (24) the whole problem of advertising budget setting can be best examined through the three-fold approach (i) preliminary model building from historical data (ii) model validation
through controlled experiment (iii) model updating.

According to him a correct approach requires three preliminary steps. (i) Adjusting the sales figures for the delayed effect of advertising. (ii) Taking into account negative effect of competitive economy. (iii) Formulating a nonlinear relationship between brand advertising and sales.

These models are based on empirical results which may not be fitted for all situations. Furthermore, the reliability of the results depends to some extent on the number of test areas used. If only a few are used, it may be difficult to place sufficient confidence in the results. Also, for application of empirical approach, obtaining data to develop Budget Decision Models, is difficult.

2.3 Summary of the Present Study:

Most of the models indicated here make budget determination recommendations based on the assumption that advertising is the only variable. This is an assumption that is usually not true. Price, channels of distribution and personal selling efforts are a few of the other variables important
in the over-all marketing programme. Advertising is part of marketing mix and should be determined simultaneously with other elements of marketing mix.

There appears to be a need for greater communication and discussion of advertising budgeting methods between theories and the practitioner. Practical methods used for determining budget decision are too simple. Although quantitative models are potentially of value, research undertaken in order to make them simpler to apply and easier to understand is still necessary. In developing new budget decision models, we have endeavoured to keep a balance between theory and practice.

Here, in budget determination, we have considered competitive situation where it is assumed that sales of goods depend upon advertising outlay, selling effort and price. To increase sales volume, one can either decrease price or increase advertising and selling efforts. We need to decide whether such a strategy is worth pursuing under competitive strategies, in which the competitor manipulates his controllable variables so as to maximize his profit function.
Section (2.4) of this chapter presents Model-1 which determines the advertising budget in a fixed market with two different cases. In Case-1, a model for a marketing situation with two competitors is discussed, whereas Case-2 is generalised for n competitors. The model derives conditions under which equilibrium solutions exist and the sensitivity of the model for small deviations from their equilibrium values in the decision variables is also tested. The equilibrium solutions have been found by simultaneously maximizing the profit functions of all competitors.

In Section (2.5), Model-2 is discussed which deals with multi-competitors and considers the model in which the market is independent of all control variables. Here, we have determined the budget under three different situations as stated by Tull Donald (121) where the total market potential is considered as a function of total advertising efforts of all competitors. Case-1 discusses budget determination under Diminishing Returns Market Response, Case-2 presents budget determination under Quadratic Market Response and Case-3 shows determination of budget under Saturation Market
Response. The general structure of both the models are identical. Moreover, their analysis proceeds in the same manner. The only difference between them is that in Model-1, the market size is fixed whereas Model-2 deals with the changing total amount of effective advertising expenditure for the product.

Finally, in section (2.6), general conclusion is presented in which the attention is directed to the current limitations of the models and the suggestions are advanced for extending their applications.

2.4 Model-1: Budget determination in a fixed market.

In this model, under the stated assumptions, equilibrium values are obtained using game theoretic analysis. Budget determination based on two competitors is discussed in Case-1 whereas Case-2 deals with more than two competitors.

2.4.1 Case-1: Competition between two brands.

2.4.1.1 Assumptions:

1. Only two brands are competing in the market.
2. The total sales volume of the product is fixed.
3. Each competitor knows manufacturing cost of all other competitors* and tries to maximize his profit.
4. Each competitor's share of market depends on his relative effectiveness of advertising effort, selling effort**, and the relative proportion of the deviation of his price from that of his opponent.

2.4.1.2 Problem Formulation:

Here we consider a fixed market in which only two brands compete and total market potential represents total sales of all competitors under a given set of strategies.

The contribution of advertising and selling effort to the marketshare of i^th brand is proportional to

\[ \frac{\alpha_1 a_1}{\alpha_1 a_1 + \alpha_2 a_2} \quad \text{and} \quad \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} \]

for i=1,2 respectively\(^{(3)}\). Where \(a_i\) is the expenditure on advertising, \(s_i\) is the expenditure on selling for \(i^\text{th}\) brand. \(a_i, \beta_i\) are relative

* There are certain techniques available for estimating cost of production of competitor brands\(^{(63)}\).

** Selling efforts include here tools for consumer promotion, trade promotion, sales force promotion, personal selling etc.\(^{(65)}\)
effectiveness of advertising expenditure and selling expenditure per rupee of ith brand.

The contribution of price to the share of the market of ith brand is assumed to be

\[ w_i \left( \frac{p^i}{p_i} - 2p_i \right), \]

where \( p_i \) is the price charged by ith brand and \( w_i \) is a price constant and \( w_i > 0, \ i = 1,2 \).

Thus the share of market \( M_i \) for ith brand is given by

\[
M_i = u \frac{\alpha_i a_i}{\alpha_1 a_1 + \alpha_2 a_2} + v \frac{\beta_i s_i}{\beta_1 s_1 + \beta_2 s_2} + \sum_{j=1}^{2} \frac{p_j}{p_i} - 2p_i, \quad i = 1,2, \quad (2.1)
\]

where \( \sum_{j=1}^{2} M_j = 1 \) and \( 0 < M_j < 1. \quad (2.2) \)

Here advertising constant \((u)\), selling constant \((v)\) and price constants \((w_i, i = 1,2)\) are assumed
to be known since their values can be determined from sample data using least squares Technique (99).

Also it is assumed that $u, v, w_i, i = 1, 2$ are all positive constants such that

$$2 \sum_{i=1}^{2} w_i - u - v > 0.$$ 

We simplify our notation in profit function as:

Profit = (total sales volume) \cdot (margin of profit)

- (advertising expenditure)
- (selling expenditure)
- (fixed expenditure).

Using (2.1) in profit function, we obtain,

$$P_i = V \left[ u \frac{a_i a_i}{a_1 a_1 + a_2 a_2} + v \frac{\beta_i s_i}{\beta_1 a_1 + \beta_2 s_2} \right.

\left. + w_i \frac{\sum_{j=1}^{2} p_j - 2p_i}{p_i} \right] h_i - a_i

- s_i - F_i, \quad i = 1, 2,$$

\text{ Where for } i^{th} \text{ brand, }

$P_i$ is profit.

$V$ is fixed unit sales volume.
\( c_i \) is unit variable cost of production, 
\( h_i \) is logistic margin where \( h_i = p_i - c_i \), 
and 
\( F_i \) is fixed costs.

2.4.1.3 Problem Solution:

Advertising Budget Determination:

Since both \( a_1 \) and \( a_2 \) are positive, the necessary and sufficient conditions for the maximum profit of \( i^{th} \) competitor (\( i = 1,2 \)) are given by (64)

\[
(1) \quad \frac{\delta P_i}{\delta p_i} = \frac{\delta P_i}{\delta a_i} = 0 \quad (2.4)
\]

\[
(ii) \quad (1) \quad \frac{\delta^2 P_i}{\delta p_i^2} < 0, \quad \frac{\delta^2 P_i}{\delta a_i^2} < 0 \quad (2.5)
\]

\[
(2) \quad \begin{vmatrix}
\frac{\delta^2 P_i}{\delta p_i^2} & \frac{\delta^2 P_i}{\delta p_i \delta a_i} \\
\frac{\delta^2 P_i}{\delta a_i \delta p_i} & \frac{\delta^2 P_i}{\delta a_i^2}
\end{vmatrix} > 0 \quad (2.6)
\]
Conditions for Branc-1:

From (2.3), considering \( i = 1 \), it can be shown that

\[
\frac{\delta P_i}{\delta a_i} = \frac{Vu(\alpha_1^2 a_2^a_2)h_1}{(\alpha_1a_1 + \alpha_2a_2)^2} - 1 \quad (2.7)
\]

\[
\frac{\delta P_i}{\delta p_i} = VM_1 + Vh_1 [\omega_i(-\frac{p_2}{p_1})] \quad (2.8)
\]

From (2.7), we get,

\[
\frac{\delta^2 P_i}{\delta a_1^2} = -\frac{2uV(\alpha_1^2 a_2^a_2)h_1}{3(\alpha_1a_1 + \alpha_2a_2)} \quad (2.9)
\]

From (2.8), we get,

\[
\frac{\delta^2 P_i}{\delta h_1^2} = V\left[-\frac{2\omega_1 p_2}{p_1} + \frac{2h_1\omega_1 p_2}{p_1^3}\right]
\]

\[
= -\frac{2V\omega_1 p_2 c_1}{p_1^3} \quad (\text{since } h_1 = p_1 - c_1). \quad (2.10)
\]

From (2.8), we get,

\[
\frac{\delta^2 P_i}{\delta a_1 \delta p_1} = \frac{Vu(\alpha_1^2 a_2^a_2)}{(\alpha_1a_1 + \alpha_2a_2)^2} \quad (2.11)
\]
Similarly from (2.7),

\[ \frac{\partial^2 p_1}{\partial p_1 \partial a_1} = \frac{Vu(\alpha_1 \alpha_2 a_2)}{(\alpha_1 a_1 + \alpha_2 a_2)^2} \] ....(2.12)

Using necessary condition (2.4), results (2.7) and (2.8) can be rewritten as

\[ \frac{Vu(\alpha_1 \alpha_2 a_2) h_1}{(\alpha_1 a_1 + \alpha_2 a_2)^2} - 1 = 0, \]

which yields, \[ Vu(\alpha_1 \alpha_2 a_2) h_1 = (\alpha_1 a_1 + \alpha_2 a_2)^2 \] ....(2.13)

and

\[ VM_1 + Vh_1 \left[ w_1 \left( - \frac{p_2}{p_1} \right) \right] = 0, \]

shows that

\[ VM_1 = Vw_1 h_1 \left( \frac{p_2}{p_1} \right). \]

Dividing both the sides by \( V \), we get,

\[ M_1 = w_1 h_1 \left( \frac{p_2}{p_1} \right). \] ....(2.14)
From results (2.9) and (2.10), it can be observed that sufficient condition (2.5) is satisfied since $\alpha_i, a_i, p_i, c_i, h_i, V$ are all positive quantities for $i = 1, 2$.

From (2.6), we have,

$$\begin{vmatrix}
-2uV(a_1 a_2 a_2) & Vu(a_1 a_2)
\end{vmatrix}
\begin{vmatrix}
\alpha_1 a_1 + \alpha_2 a_2
\end{vmatrix}^2
(\alpha_1 a_1 + \alpha_2 a_2)^2
\begin{vmatrix}
u(a_1 a_2)
\end{vmatrix}
\begin{vmatrix}
\alpha_1 a_1 + \alpha_2 a_2
\end{vmatrix}^2
= 0. \quad (2.15)

[ Using the results (2.9), (2.10), (2.11), (2.12) in (2.6), ]

This will yield the condition

$$\frac{4h_1 w_1 p_2 c_1}{p_1^3} > u \frac{a_2 a_2}{\alpha_1 a_1 + \alpha_2 a_2}. \quad (2.16)$$

Conditions for Brand-2:

Similar results can be obtained for Brand-2, proceeding in the same manner.

Here we obtain,
\[
\frac{\delta p_2}{\delta a_2} = \frac{V u (a_1 a_2 a_1) h_2}{(a_1 a_1 + a_2 a_2)^2} - 1 \quad (2.17)
\]

\[
\frac{\delta p_2}{\delta p_2} = V M_2 + V h_1 \left[ w_1 \left( - \frac{p_2}{p_1^2} \right) \right] \quad (2.18)
\]

\[
\frac{\delta^2 p_2}{\delta a_2^2} = - \frac{2 u V (a_1 a_2 a_1) h_2}{(a_1 a_1 + a_2 a_2)^3} \quad (2.18)
\]

\[
\frac{\delta^2 p_2}{\delta p_2 \delta a_2} = \frac{V u (a_1 a_2 a_1)}{(a_1 a_1 + a_2 a_2)^2} = \frac{\delta^2 p_2}{\delta p_2 \delta a_2}
\]

Using the necessary condition (2.4) for results (2.17) and (2.18), we obtain,

\[
V u (a_1 a_2 a_1) h_2 = (a_1 a_1 + a_2 a_2)^2 \quad (2.19)
\]

\[
M_2 = w_2 h_2 \left( \frac{p_1}{p_2} \right)^2 \quad (2.20)
\]

As in the previous case, it can be shown here that sufficient condition (2.5) is satisfied but condition (2.6) is satisfied only if

\[
\frac{4 h_2 w_2 p_1 c_2}{p_2^3} > u \left( \frac{a_1 a_1}{a_1 a_1 + a_2 a_2} \right) \quad (2.21)
\]
Hence from (2.4) and (2.6), the optimum $p_i$ and $a_i$ of Brand-$i$ for $i = 1,2$ should satisfy the relations stated in (2.13), (2.14), (2.16) and (2.19), (2.20), (2.21) respectively.

Simultaneous Optimization Conditions:

From (2.13),
\[ Vu(a_1 a_2 a_1)h_1 = (a_1 a_1 + a_2 a_2)^2 \]
from (2.19),
\[ Vu(a_1 a_2 a_1)h_2 = (a_1 a_1 + a_2 a_2)^2 \]
Equating results (2.13) and (2.19), we get
\[ Vu(a_1 a_2 a_1)h_1 = Vu(a_1 a_2 a_1)h_2 \]
Dividing both sides of the above equation by $Vu a_1 a_2$, we obtain,
\[ a_2 h_1 = a_1 h_2 \]
i.e.
\[ \frac{a_1}{a_2} = \frac{h_1}{h_2} \quad (2.22) \]
Now from (2.14),
\[ M_1 = w_1 h_1 (p_2/p_1^2) \]
and from (2.20),

$$\hat{m}_2 = w_2 h_2 \left( \frac{p_1}{p_2} \right)^2.$$  

Adding results (2.14) and (2.20), we get,

$$\sum_{i=1}^{2} m_i = w_1 h_1 \left( \frac{p_2}{p_1} \right) + w_2 h_2 \left( \frac{p_1}{p_2} \right). \tag{2.23}$$

But from (2.2), sum of the market share of all brands is always 1 and hence the result (2.23) can be rewritten as

$$1 = p_1 p_2 \sum_{i=1}^{2} \frac{w_i h_i}{p_i^2} \tag{2.24}.$$  

But considering \( i = 1 \) in (2.1), we have,

$$m_1 = u \frac{a_1 a_2}{a_1 a_1 + a_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + w_1 \frac{(p_2 - p_1)}{p_1} \tag{2.24}.$$  

Considering value of \( m_1 \) in (2.14), we get

$$u \frac{a_1 a_1}{a_1 a_1 + a_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + w_1 \frac{(p_2 - p_1)}{p_1} = \frac{w_1 h_1}{p_2} \frac{p_2}{p_1}.$$  

i.e. \( u \frac{a_1 \alpha_1}{\alpha_1 a_1 + \alpha_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} \)

\[= w_1 h_1 \left( \frac{p_2}{p_1} \right) \cdot w_1 \left( \frac{p_2 - p_1}{p_1} \right) \]

\[= - \frac{w_1 p_2 c_1}{p_1} + w_1 \text{ (Considering } h_1 = p_1 - c_1, \text{ )} \]

\[= w_1 \left( 1 - \frac{p_2 c_1}{p_1} \right) \quad \ldots(2.25) \]

Similarly considering value of \( \bar{w}_2 \) in (2.20), and on further simplification, it can be easily shown that

\[ u \frac{a_2 \alpha_2}{\alpha_1 a_1 + \alpha_2 a_2} + v \frac{\beta_2 s_2}{\beta_1 s_1 + \beta_2 s_2} = w_2 \left( 1 - \frac{p_1 c_2}{p_2} \right) \quad \ldots(2.26) \]

Adding results (2.25) and (2.26), we get,

\[ u + v = - p_1 p_2 2 \sum_{i=1}^{w_i \frac{1}{p_1}} + 2 \frac{w_i}{p_1} \sum_{i=1} \]

i.e. \( p_1 p_2 \sum_{i=1}^{w_i \frac{1}{p_1}} = \sum_{i=1}^{w_i} - u - v \)
\[ \begin{align*}
\text{i.e. } l &= \frac{1}{\sum_{i=1}^{2} w_i - u - v} \left[ \frac{2}{p_1 p_2} \sum_{i=1}^{2} \frac{w_i c_i}{p_i^3} \right]. \\
\text{Comparing results (2.24) and (2.27), we get,} \\
\frac{2}{p_1 p_2} \sum_{i=1}^{2} \frac{w_i}{p_i^3} (h_i) &= \frac{2}{\sum_{i=1}^{2} w_i - u - v} \sum_{i=1}^{2} \frac{w_i c_i}{p_i^3} \\
\text{i.e. } \frac{2}{\sum_{i=1}^{2} w_i - u - v} \left[ h_i - \frac{c_i}{p_i} \right] &= 0 \quad \text{.....(2.28)} \\
\text{But here } w_i > 0, \ p_i > 0, \text{ for } i = 1, 2 \text{ and hence the condition (2.28) is possible only if} \\
\frac{c_i}{p_i} &= \frac{1}{\sum_{i=1}^{2} w_i - u - v} \quad \text{.....(2.29)} \\
\text{Let } Q = \frac{1}{\sum_{i=1}^{2} w_i - u - v} \text{ and hence equation (2.29) yields,} \\
h_i &= \frac{c_i}{Q} \quad \text{.....(2.30)} \\
\text{which gives,} \\
p_i &= \frac{c_i (1 + Q)}{Q} \quad \text{.....(2.31)} \\
&\quad \text{(considering } h_i = p_i - c_i).}
\end{align*} \]
Therefore, from (2.33),

\[ h_1 = \frac{c_1}{Q} \quad \text{and} \quad h_2 = \frac{c_2}{Q} \]

and hence ratio of logistic margins is given by

\[ \frac{h_1}{h_2} = \frac{c_1}{c_2} \quad \text{....(2.32)} \]

Similarly from result (2.31), it can be shown that

\[ \frac{p_1}{p_2} = \frac{c_1}{c_2} \quad \text{....(2.33)} \]

But from (2.22), we have

\[ \frac{a_1}{a_2} = \frac{h_1}{h_2} \quad \text{....} \]

and hence comparing results (2.22), (2.32), and (2.33), we get,

\[ \frac{a_1}{a_2} = \frac{p_1}{p_2} = \frac{c_1}{c_2} \quad \text{....(2.34)} \]

Sufficient condition under optimization:

From (2.16), the sufficient condition for Brand -1 is given by

\[ \frac{4h_1^2w_1p_2^c_1}{p_1^3} > u \frac{a_2 \alpha_2}{\alpha_1 \alpha_1 + \alpha_2 \alpha_2} \]
But under optimization, from (2.30), (2.31) and (2.34), we get,

\[ h_1 = \frac{c_i}{Q}, \quad p_1 = c_i \frac{(1+Q)}{Q} \quad \text{and} \]

\[ \frac{a_1}{a_2} = \frac{c_1}{c_2}. \]

Using these values for \( i = 1 \), in (2.16), we get,

\[ \frac{4Q}{(1+Q)^2} \left( \frac{\omega_1 c_2}{c_1} \right) > u \frac{c_2 a_2}{\alpha_1 c_1 + \alpha_2 c_2} \quad \ldots (2.35) \]

Similarly for Branch-2, repeating the same procedure we can show that

\[ \frac{4Q}{(1+Q)^2} \left( \frac{\omega_2 c_1}{c_2} \right) > u \frac{\alpha_1 c_1}{\alpha_1 c_1 + \alpha_2 c_2} \quad \ldots (2.36) \]

Adding results (2.35), and (2.36), we get,

\[ \frac{4Q}{(1+Q)^2} \left[ \frac{\omega_1 c_2 + \omega_2 c_1}{c_1 c_2} \right] > u \quad \ldots (2.37) \]

Optimization of advertising expenditure:

From (2.22), we have

\[ \frac{a_1}{a_2} = \frac{h_1}{h_2}, \quad \text{i.e.,} \quad a_2 = \frac{h_2}{h_1} a_1. \quad \ldots (2.38) \]
Now from (2.13)

\[
\frac{Vu(h_2) (a_1 a_2) a_1}{(a_1 a_1 + a_2 a_2)^2} = 1
\]

i.e. \[ a_1 = \frac{(a_1 a_1 + a_2 a_2)^2}{V u(h_2) (a_1 a_2)} \] \[ \ldots (2.39) \]

Using result (2.38), in (2.39), we get,

\[
a_1 = \frac{[a_1 a_1 + a_2] h_2^2 a_1^2}{V u(h_2) (a_1 a_2)}
\]

and on further simplification we obtain,

\[
a_1 = \frac{a_1^2 [a_1(h_1) + a_2(h_2)]^2}{V u(h_2) (a_1 a_2)}
\]

i.e. \[ Vu(h_1)(h_2)(a_1 a_2) = a_1[a_1(h_1) + a_2(h_2)]^2 \]

i.e. \[ a_1 = h_1 Vu \left[ \frac{[a_1(h_1)] [a_2(h_2)]}{[a_1(h_1) + a_2(h_2)]^2} \right] \] \[ \ldots (2.40) \]

But from (2.32),

\[
\frac{h_1}{h_2} = \frac{c_1}{c_2}
\]
i.e. \[ h_1 = \frac{c_1}{c_2} (h_2) \] \[ \ldots (2.41) \]

Considering (2.41) in (2.40) and on further simplification, we get,

\[ a_1 = h_1 \frac{V_i a_1 (c_1) (a_2 c_2)}{(a_1 c_1 + a_2 c_2)^2} \] \[ \ldots (2.42) \]

Assuming,

\[ A = \frac{(a_1 c_1) (a_2 c_2)}{[a_1 c_1 + a_2 c_2]^2} \] and considering result (2.30) for \( i = 1 \), in (2.41) we get optimum advertising expenditure for Brand - 1 as

\[ a_1^o = \frac{c_1}{Q} (V_i A) \]

Similarly optimum advertising expenditure for Brand-2 can be obtained as

\[ a_2^o = \frac{c_2}{Q} (V_i A) \]

i.e. \[ a_i = \frac{c_i}{Q} (V_i A), \quad i = 1, 2, \ldots (2.43) \]
Optimization of Selling expenditure:

When both \( s_1 \) and \( s_2 \) are positive, the necessary and sufficient conditions for the maximum profit of \( i \)th competitor \((i = 1, 2)\) are given by

\[
\begin{align*}
(i) \quad & \frac{\delta P_i}{\delta p_i} = \frac{\delta P_i}{\delta s_i} = 0 \\
(ii) \quad & \frac{\delta^2 P_i}{\delta p_i^2} < 0, \quad \frac{\delta^2 P_i}{\delta s_i^2} < 0
\end{align*}
\]

\[
\begin{align*}
(2) \quad & \frac{\delta^2 P_i}{\delta p_i^2} - \frac{\delta^2 P_i}{\delta p_i \delta s_i} > 0, \\
& \frac{\delta^2 P_i}{\delta s_i^2} > 0.
\end{align*}
\]

Proceeding in the same way, we can easily show that simultaneous optimization of selling expense will yield

\[
\frac{s_1}{s_2} = \frac{a_1}{a_2} = \frac{c_1}{c_2}
\]

which implies that

\[
\frac{a_1}{a_2} = \frac{s_1}{s_2}
\]
The equilibrium sales expenditure can be obtained as

$$s_{i}^{o} = \frac{c_{i}}{Q} (\text{VuB}), \quad i = 1, 2 \quad \ldots \ldots (2.45)$$

where

$$B = \frac{(\beta_{1}c_{1}) (\beta_{2}c_{2})}{(\beta_{1}c_{1} + \beta_{2}c_{2})^{2}}.$$

Optimization of Profits:

From (2.3) for $i = 1$, Profit function of Brand-1 can be given by

$$P_{1} = V\left[u \frac{\alpha_{1}a_{1}}{\alpha_{1}a_{1} + \alpha_{2}a_{2}} + v \frac{\beta_{1}s_{1}}{\beta_{1}s_{1} + \beta_{2}s_{2}} \right.$$

$$+ \omega_{1} \frac{(p_{2}-p_{1})}{p_{1}} h_{1} - a_{1} - s_{1} - F_{1} \left. \right].$$

Therefore optimum profit can be given by

$$P_{1} = \delta_{1}^{o} + \delta_{2}^{o} + \delta_{3}^{o} - F_{1} \quad \ldots \ldots (2.46)$$

Where optimum value of $\delta_{1}$ is given by

$$\delta_{1}^{o} = VU_{1} \frac{\alpha_{1}a_{1}}{\alpha_{1}a_{1} + \alpha_{2}a_{2}} - a_{1}.$$
But from (2.34),

\[
\frac{a_1}{a_2} = \frac{c_1}{c_2} \quad \text{i.e.} \quad a_1 = \frac{c_1}{c_2} \qquad a_2
\]

Considering the above expression in \(d_1^o\),
we get,

\[
d_1^o = V_u h_1 \left[ \frac{\alpha_1 c_1}{\alpha_1 c_1 + \alpha_2 c_2} \right] - a_1^o.
\]

Using results (2.35) for \(i = 1\), and (2.43) in above expression, we get,

\[
d_1^o = V_u \left[ \frac{c_1}{Q} \frac{\alpha_1 c_1}{\alpha_1 c_1 + \alpha_2 c_2} \right] - V_u \frac{c_1}{Q} \left[ \frac{(\alpha_1 c_1)(\alpha_2 c_2)}{(\alpha_1 c_1 + \alpha_2 c_2)^2} \right]
\]

\[
= V_u \frac{c_1}{Q} \left( \frac{\alpha_1 c_1}{\alpha_1 c_1 + \alpha_2 c_2} \right)^2 \quad \text{......}(2.47)
\]

Similarly optimum value of \(d_2^o\) is given by

\[
d_2^o = V_v h_1 \left[ \frac{s_1}{\beta_1 s_1 + \beta_2 s_2} \right] - s_1^o.
\]

Proceeding as earlier, it can be easily shown that

\[
d_2^o = V_v \frac{c_1}{Q} \left( \frac{\beta_1 c_1}{\beta_1 c_1 + \beta_2 c_2} \right)^2 \quad \text{......}(2.48)
\]
Optimum Value of $d_3$ is given by

$$d_3^o = \sqrt{\omega_1 \left( \frac{p_2 - p_1}{p_1} \right) h_1^o}.$$  

Using (2.30), and (2.31) for $i = 1$, in the above expression, we get,

$$d_3^o = \sqrt{\omega_1 \frac{c_1}{Q} \left( \frac{c_2 - c_1}{c_1} \right)} \quad \ldots (2.49)$$

Using (2.47), (2.48) and (2.49) in (2.46), the optimum profit can be obtained as

$$P_1^o = \sqrt{\frac{c_1}{Q} \left[ u \left( \frac{\alpha_1 c_1}{\alpha_1 c_1 + \alpha_2 c_2} \right)^2 + v \left( \frac{\beta_1 c_1}{\beta_1 c_1 + \beta_2 c_2} \right)^2 \right]}$$

$$+ \omega_1 \left( \frac{c_2 - c_1}{c_1} \right) - F_1 \quad \ldots (2.50)$$

Similarly it can be shown that optimum profit for Brand-2 is given by

$$P_2^o = \sqrt{\frac{c_2}{Q} \left[ u \left( \frac{\alpha_2 c_2}{\alpha_1 c_1 + \alpha_2 c_2} \right)^2 + v \left( \frac{\beta_2 c_2}{\beta_1 c_1 + \beta_2 c_2} \right)^2 \right]}$$

$$+ \omega_2 \left( \frac{c_1 - c_2}{c_2} \right) - F_2 \quad \ldots (2.51)$$
From results (2.50), and (2.51) the following particular cases can be observed:

1. If the cost of production for both the brands is the same, the profit will not depend on the price effectiveness.

2. If $c_1 = c_2$ and if no selling effort has been put up,

$$P_1^o + F_1 = \sqrt{\frac{c_1}{Q}} \left[ u \left( \frac{\alpha_1}{\alpha_1 + \alpha_2} \right)^2 \right]$$

and

$$P_2^o + F_2 = \sqrt{\frac{c_2}{Q}} \left[ u \left( \frac{\alpha_2}{\alpha_1 + \alpha_2} \right)^2 \right]$$

and hence,

$$\frac{P_1^o + F_1}{P_2^o + F_2} = \frac{\alpha_1^2}{\alpha_2^2}.$$  

This is very similar to Mills' result which shows that the ratio of the net contribution margins of the two brands is equal to the ratio of the square of their advertising effectiveness.

Similarly if $c_1 = c_2$ and if no advertising efforts are made,
3. If no advertising effort or selling effort is put up, the net contribution margins of the two brands are given by

\[
\frac{P_1 + F_1}{P_2 + F_2} = \frac{\beta_1^2}{\beta_2^2}
\]

If no advertising effort or selling effort is put up, the net contribution margins of the two brands are given by

\[
P_1 + F_1 = \nu_1 \frac{c_1}{Q} \left( \frac{c_2 - c_1}{c_1} \right)
\]

\[
P_2 + F_2 = \nu_2 \frac{c_2}{Q} \left( \frac{c_1 - c_2}{c_2} \right)
\]

i.e. \[
\frac{P_1 + F_1}{P_2 + F_2} = \frac{\omega_1}{\omega_2} \left( \frac{c_2 - c_1}{c_1 - c_2} \right)
\]

i.e. \[
\frac{P_1 + F_1}{P_2 + F_2} = -\frac{\omega_1}{\omega_2}
\]

i.e. \[(P_1 + F_1)\omega_2 + (P_2 + F_2)\omega_1 = 0.\]

Since \(\omega_i > 0, i = 1, 2\), the above expression is possible only if \(P_1 + F_1 = 0\) and \(P_2 + F_2 = 0\).

This means that if no promotional effort is made, the net contribution margins of two brands are equal to zero.
4. If Brand-1 is not spending on advertising as well as on selling, the profit functions of the two brands are given by

\[ P_1^0 + F_1 = V \frac{c_1}{Q} \left[ \omega_1 \frac{(c_2 - c_1)}{c_1} \right] \]

\[ P_2^0 + F_2 = V \frac{c_2}{Q} \left[ u + v + \omega_2 \frac{(c_1 - c_2)}{c_2} \right] \]

i.e. \((P_2^0 + F_2) - (P_1^0 + F_1) = \frac{V}{Q} \left[ c_2(u+v)+(\omega_2+\omega_1) \right] \frac{(c_1-c_2)}{c_1} \]

In this case, the net profit contribution of Brand-2 is greater than Brand-1 if cost of production of Brand-2 is equal to or less than that of Brand-1. But if cost of production of Brand-2 is greater, no general statement can be made.

5. If both the Brands are spending on advertising and selling their maximum net profit contribution can be shown as:

\[ (P_1^0 + F_1) + (P_2^0 + F_2) = \frac{V}{Q} \left[ u \frac{a_1c_1^2 + a_2c_2^3}{(a_1c_1 + a_2c_2)^2} \right. \]

\[ \left. + \frac{v(\beta_1c_1^3 + \beta_2c_2^3)}{(\beta_1c_1 + \beta_2c_2)^2} + (c_2-c_1)(\omega_1-\omega_2) \right] \]
2.4.1.4 Sensitivity Analysis:

Let Brand-1 be the brand under consideration and Brand-2 be the opponent brand. We measure the sensitivity of net profit contribution for Brand-1 with respect to its price and advertising expenditure as well as that of its opponent.

1. Analysis of Price Sensitivity:

(i) Change in the price level of brand under consideration.

Let us assume that the new price of the given brand is $P^0 = P_1 + \delta$

where $\delta$ is a small nonzero constant.

Using (2.3), the new net profit contribution function is given by

$$P_{1} + F_1 = V(h_1 + \delta) \left[ u \frac{\alpha_1 a_1}{\alpha_1 a_1 + \alpha_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} \right]$$

$$+ \omega_1 \left( \frac{p_2 - (p_1 + \delta)}{p_1 + \delta} \right)$$

$$= V(h_1 + \delta) \left[ u \frac{\alpha_1 a_1}{\alpha_1 a_1 + \alpha_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} \right]$$

$$+ \omega_1 \left( \frac{p_2}{p_1(1 + \delta)} - 1 \right)$$
Ignoring the high power of \( \delta \), we get,

\[
P_1^o + F_1 = V(h_1 + \delta)[u \frac{a_1 a_1}{a_1 a_1 + a_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega_1 \left( \frac{p_2}{p_1} \left( 1 + \frac{\delta}{p_1} \right) - 1 \right)].
\]

But from (2.14), under optimization, we have

\[
M_1 = \omega_1 h_1 \left( \frac{p_2}{p_1} \right).
\]

Hence, the above expression can be written as

\[
P_1^o + F_1 = (P_1 + F_1) - \frac{V \omega_1 h_1 p_2}{p_1} \left( 1 - \frac{\delta}{p_1} \right) - \frac{V \omega_1 h_1 p_2}{p_1} \left( 1 - \frac{\delta}{p_1} \right) + \frac{V \omega_1 h_1 p_2}{p_1} \left( 1 - \frac{\delta}{p_1} \right) + \frac{V \omega_1 h_1 p_2}{p_1} \left( 1 - \frac{\delta}{p_1} \right)
\]
\[\begin{align*}
= (P_1 + F_1) + \frac{V h_1 \omega_1 p_2 \delta}{p_1^3} - \frac{V \delta \omega_1 p_2}{p_1^2} \left( 1 - \frac{\delta}{p_1} \right) \\
= (P_1 + F_1) + \frac{V \delta \omega_1 p_2}{p_1^2} \left( \frac{h_1 + \delta}{p_1} - 1 \right) \\
= (P_1 + F_1) - \frac{V \delta \omega_1 p_2}{p_1^3} (c_1 - \delta) \\
\text{i.e. } (P_1^0 + F_1) - (P_1 + F_1) = - \frac{V \delta \omega_1 p_2}{p_1} (c_1 - \delta).
\end{align*}\]

If \( \delta < 0 \), the above quantity is negative.

If \( \delta > 0 \), then also the above quantity is negative since the reduction in price can not be greater than the cost of production. This means that if Brand-1 deviates from optimal policy his profit goes down.

(ii) Change in competitor's price level.

Let us suppose that the new price of opponent is

\[P_2^0 = P_2 + \delta.\]

Therefore, we have,

\[P_1^0 + F_1 = \text{V} h_1 \left[ u \frac{\alpha_1 a_1}{\alpha_1 a_1 + \alpha_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega_1 \frac{p_2 + \delta - p_1}{p_1} \right].\]
\[ (P_1 + F_1) = \frac{Vh_1 \omega_1 \delta}{p_1} \]

i.e. \((P^o_1 + F_1) - (P_1 + F_1) = Vh_1 \frac{\omega_1 \delta}{p_1}\).

Here the difference only depends on \(\delta\),

since \(V > 0\), \(h_1 > 0\), \(p_1 > 0\), \(\omega_1 > 0\).

This means that if the opponent increases its price (i.e. \(\delta > 0\)), the profit of a brand under consideration will increase but at the same time, if opponent is decreasing its price (i.e. \(\delta > 0\)), the profit of given brand decreases.

2. Analysis of Advertising Budget Sensitivity:

   (i) Change in the Advertising expenditure level of given brand.

Let \(a_1^o = a_1 + \delta\)

Therefore, we have,

\[ P^o_1 + F_1 = Vh_1[u \frac{\alpha_1^o a_1 + \alpha_1 \delta}{\alpha_2 a_1 + \alpha_1 \delta + \alpha_2 a_2} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2}]

+ \omega_1(\frac{p_2 - p_1}{p_1}) - (a_1 + \delta) - s_1 \]
\begin{align*}
\Psi & = \Psi_0 \left[ \alpha_1 a_1 + \frac{\alpha_1 \delta}{D} \left( 1 + \frac{\alpha_1 \delta}{D} \right)^{-1} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} \\
& \quad + \omega_1 \left( \frac{p_2 - p_1}{p_1} \right) \right] - (a_1 + \delta) - s_1.
\end{align*}

Considering \( D = \alpha_1 \alpha_1 + \alpha_2 a_2 \), we have

\begin{align*}
P_1 + F_1 & = \Psi_0 \left[ \alpha_1 a_1 + \frac{\alpha_1 \delta}{D} \left( 1 - \frac{\alpha_1 \delta}{D} \right) \left( \frac{\alpha_1 \delta}{D} \right)^2 \right] \\
& \quad + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega_1 \left( \frac{p_2 - p_1}{p_1} \right) - (a_1 + \delta) - s_1
\end{align*}

(Ignooring the higher powers of \( \delta \)).

\begin{align*}
& = (p_1 + F_1) + \Psi_0 u \left[ \frac{\alpha_1 \delta}{D} + \frac{\alpha_1 a_1 + \alpha_1 \delta}{D} \right] \\
& \quad \left( - \frac{\alpha_1 \delta}{D} - \left( \frac{\alpha_1 \delta}{D} \right)^2 \right) \right] - \delta \\
& = (p_1 + F_1) + \Psi_0 u a_1 \delta \left[ 1 + \frac{\alpha_1 a_1 + \alpha_1 \delta}{D} \right] \\
& \quad \left( -1 + \left( \frac{\alpha_1 \delta}{D} \right) \right) \right] - \delta \\\n& = (p_1 + F_1) + \Psi_0 u a_1 \delta \left[ 1 + \frac{\alpha_1 a_1 + \alpha_1 \delta}{D} \right] \\
& \quad \left( -1 + \left( \frac{\alpha_1 \delta}{D} \right) \right) \right] - \delta \left( -1 + \left( \frac{\alpha_1 \delta}{D} \right) \right) \frac{\Psi_0 u a_2 a_2}{D^2}.
\end{align*}
Since from (2.13), under optimization condition we have, \( \frac{Vh_1 u_{1 \alpha_1} a_2 a_2}{D^2} = 1 \) and hence we get,

\[
P^0_1 + F_1 = (P_1 + F_2) + \frac{Vh_1 u_{1 \alpha_1} \delta}{D^3} \left[ 1 + \frac{\alpha_1 a_1 + \alpha_1 \delta}{D} \right. \\
\left. \quad \cdot (-1 + \frac{\alpha_1 \delta}{D}) - \frac{\alpha_2 a_2}{D} \right]
\]

\[
= (P^0_1 + F_1) + \frac{Vh_1 u_{1 \alpha_1} \delta}{D^3} \left[ - \frac{\alpha_1 \delta}{D} + \frac{(\alpha_1 a_1 + \alpha_1 \delta)(\alpha_1 \delta)}{D^2} \right] 
\]

\[
= (P^0_1 + F_1) + \frac{Vh_1 u_{1 \alpha_1} \delta}{D^3} \left[ - \frac{D + \alpha_1 \delta}{D} \right] 
\]

i.e. \( (P^0_1 + F_1) - (P_1 + F_1) = -\frac{Vh_1 u_{1 \alpha_1} \delta}{D^3} (a_2^2 a_2 - a_1 \delta) \).

This is a negative quantity if \( a_2^2 a_2 > a_1 \delta \).

This suggests that under given condition if Brand-1 deviates from its optimal policy, its profit goes down.

(ii) Change in the competitor's advertising expenditure level.

Let us suppose that new advertising budget of opponent is \( a^0_2 = a_2 + \delta \).
Therefore, we have

\[ P_1^o + F_1 = \text{Vh}_1 \left[ u \frac{\alpha_1 a_1}{\alpha_1 a_1 + \alpha_2 a_2 + \alpha_2 \delta} + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega \frac{(p_2 - p_1)}{p_1} \right] - a_1 - s_1 \]

\[ = \text{Vh}_1 \left[ u \frac{\alpha_1 a_1}{D} (1 - \frac{\alpha_2 \delta}{D}) + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega \frac{(p_2 - p_1)}{p_1} \right] - a_1 - s_1 \]

(where \( D = \alpha_1 a_1 + \alpha_2 a_2 \))

\[ = \text{Vh}_1 \left[ u \frac{\alpha_1 a_1}{D} \left(1 - \frac{\alpha_2 \delta}{D}\right) + \left(\frac{\alpha_2 \delta}{D}\right)^2 \right] + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega \frac{(p_2 - p_1)}{p_1} \] - a_1 - s_1

(ignoring higher powers of \( \delta \)).

i.e. \( P_1^o + F_1 = (P_1 + F_1) + \text{Vh}_1 u \left[ \frac{\alpha_1 a_1}{D} \left(1 - \frac{\alpha_2 \delta}{D}\right) + \left(\frac{\alpha_2 \delta}{D}\right)^2 \right] \]

\[ + v \frac{\beta_1 s_1}{\beta_1 s_1 + \beta_2 s_2} + \omega \frac{(p_2 - p_1)}{p_1} \] - a_1 - s_1

i.e. \( (P_1^o + F_1) - (P_1 + F_1) = \text{Vh}_1 u \left( \frac{\alpha_1 a_1}{D} \right) \frac{\alpha_2 \delta}{D} (-1 + \frac{\alpha_2 \delta}{D}) \)
\[ (a,a) (a^d) = - V h_1 u \frac{(a_1 a_1 + a_2(a_2 - d))}{D^3}. \]

Here the difference depends only on the value of \( \delta \) since \( V > 0, u > 0, a_1, a_2 > 0, a_2 > \delta \). This means that if competitor is increasing his advertising budget (i.e. \( \delta > 0 \)), the profit of brand under consideration will go down and if he is decreasing his budget level (i.e. \( \delta < 0 \)) the profit of brand under consideration will increase.

2.4.1.5 Explanation through hypothetical problem:

Let it be assumed that two brands are competing in the market in which total sales volume of the product is 1000 units. Let Brand-1 be the brand under consideration and Brand-2 be the opponent brand. Let advertising effectiveness ratio, selling effectiveness ratio and cost ratio be denoted by

\[ \alpha = \frac{a_1}{a_2}, \quad \beta = \frac{b_1}{b_2}, \quad \lambda = \frac{c_1}{c_2} \]

respectively. The solutions can be expressed in terms of only these parameters and thus for optimum solution one needs to know only the relative values.
Table 2.1 gives the optimum values for the various sets of values of $\lambda$, $\alpha$ and $\beta$ given that $c_1 = 10$, $\omega_1 = 0.6$, $\omega_2 = 0.5$, $u = 0.3$, $v = 0.2$ and $\alpha_i = \beta_1$, $i = 1, 2$.

In this case the condition (2.37) can be written as

$$\frac{4Q}{(Q+1)^2} \left[ \omega_1 \left( \frac{1}{\lambda} \right) + \omega_2 (\lambda) \right] > u$$

and it can be shown that for both the values of $\lambda = 0.5$ and $\lambda = 1$, the inequality is satisfied.
Table - 2.1

Advertising Budget and Price determination for various sets of values of $\alpha$, $\beta$ and $\lambda^*$.  

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$p_1$</th>
<th>$a_1$</th>
<th>$P_1 + F_1$</th>
<th>$P_2 + F_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>26.7</td>
<td>800</td>
<td>10333.33</td>
<td>4000.00</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td></td>
<td>900</td>
<td>1800</td>
<td>10443.34</td>
<td>3194.89</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td></td>
<td>950</td>
<td>1900</td>
<td>10560.00</td>
<td>2474.07</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td></td>
<td>1000</td>
<td>2000</td>
<td>10680.00</td>
<td>1836.90</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td></td>
<td>1050</td>
<td>2100</td>
<td>10803.34</td>
<td>1260.92</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td></td>
<td>1100</td>
<td>2200</td>
<td>10925.00</td>
<td>0741.11</td>
</tr>
</tbody>
</table>

* As we are only interested in advertising budget determination, selling expenditure is not discussed here.
Table 2.1 contd...

<table>
<thead>
<tr>
<th>Cost Ratio</th>
<th>Adverting efficiency ratio</th>
<th>Selling efficiency ratio</th>
<th>Optimum Price</th>
<th>Optimum Advertising expenditure</th>
<th>Optimum net contribution margins</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>Brand-1 $p_1$</td>
<td>Brand-2 $p_2$</td>
<td>Brand-1 $a_1$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>0.5</td>
<td>26.7</td>
<td>26.7</td>
<td>1111.00</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>1172.00</td>
<td>1172.00</td>
<td>1171.87</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>1211.00</td>
<td>1211.00</td>
<td>1413.04</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>1234.50</td>
<td>1234.50</td>
<td>1645.93</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>1246.54</td>
<td>1246.54</td>
<td>1869.87</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1250.00</td>
<td>1250.00</td>
<td>2083.33</td>
</tr>
</tbody>
</table>
2.4.1.6 Remarks:

The interesting and suggestive results are obtained for advertising expenditure determination from this case when the optimum outlay and profit equations are examined closely. This reveals that:

(i) the ratio of the optimal advertising expenditures of the two brands is equal to the ratio of their variable costs.

(ii) the relative effectiveness of a brand's advertising is of no significance in determining the brand's optimum advertising expenditure.

(iii) a relatively small increase in the effectiveness of a brand's advertising effort can cause large increase in the brand's profits by adapting an appropriate advertising strategy.

(iv) the determination of advertising budget depends on the advertising effectiveness, cost of competing firms, total sales volume and relative weightage of advertising, selling and price of market share.
2.4.2 Case 2: Competition between any number of brands:

Here all assumptions are the same as discussed in the previous case except that two other assumptions are at the basis of case 2, which differ from the analogous assumptions for case 1. These differing assumptions are:

1) Instead of assuming that only two brands are competing in the market, case 2 generalizes to any number of brands competing.

2) Each competitor's share of market depends on his relative effectiveness of advertising effort, selling effort and the relative proportion of the deviation of his price from that of the average price of all opponents.

Despite these differences in assumptions, the equations representing case 1 and case 2 are very similar in structure.

2.4.2.1 Problem formulation:

The profit function for ith brand, \( i = 1, 2, \ldots, n \) in this case is given by
\[ P_i = V M_i h_i - a_i - s_i - \frac{\sum_{j=1}^{n} a_j q_j}{\sum_{j=1}^{n} \beta_j s_j} \]

if \( \sum_{i=1}^{n} a_i > 0 \),
\[ \sum_{i=1}^{n} s_i > 0, \]

= 0 if \( a_i = 0 \) and \( s_i = 0 \), \( i = 1, 2, \ldots, n \),

\[ \ldots (2.52) \]

where, for ith brand,
\[ P_i = \text{Profit} \]
\[ V = \text{Fixed market potential} \]
\[ M_i = \text{Market share of the brand which is a function of price, advertising effort and selling effort such that} \]
\[ M_i = u \frac{\alpha_i a_i}{\sum_{j=1}^{n} \alpha_j q_j} + v \frac{\beta_i s_i}{\sum_{j=1}^{n} \beta_j s_j} + \omega_i \frac{(\bar{p}_i - p_i)}{p_i} \]

\[ \sum_{i=1}^{n} M_i = 1, \quad 0 < M_i < 1 \]

\[ p_i = \text{Price} \]
\[ \bar{p}_i = \text{average price of all competitors} \]

i.e.
\[ \sum_{j=1}^{n} \frac{p_j}{(n - 1)} \]
\[ j \neq 1 \]
u, v and \( \omega_i, i = 1, 2, \ldots, n \) are known constants, such that

\[
2\omega - u - v > 0 \hspace{1cm} \ldots (2.53)
\]

where \( \omega = \min [\omega_1, \omega_2, \ldots, \omega_n] \).

The problem here is to find equilibrium points \( (p, a, s) \) for each competitor in the sense that if any competitor deviates from the equilibrium values, his payoff goes down.

2.4.2.2 Problem Solution:

Advertising Budget Determination:

For \( a_i > 0, i = 1, 2, \ldots, n \), the necessary and sufficient conditions for maximum profit of \( i \)th brand are the same as stated by results (2.4), (2.5) and (2.6), for \( i = 1, 2, \ldots, n \).

From (2.52), we have

\[
\frac{\partial p_i}{\partial a_i} = \frac{V_{u_i} a_i \left( \sum_{j=1}^{n} a_j \right)}{\left( \sum_{j=1}^{n} a_j \right)^2} - 1 \hspace{1cm} \ldots (2.54)
\]
From (2.52), we have
\[ \frac{\partial p_i}{\partial p_i} = VM_i + Vh_i(\omega_i - \frac{\overline{p_i}}{p_i}) , \quad i = 1, 2, \ldots, n. \] ....(2.55)

From (2.54), we have
\[ \frac{2}{\delta a_i} = -2uVh_i\alpha_i \left( \sum_{j=1}^{n} \alpha_j a_j \right) \] ....(2.56)

Similarly from (2.55), we have
\[ \frac{2}{\delta p_i} = -2uVh_i c_i \left( \frac{\sum_{j=1}^{n} \alpha_j a_j}{p_i} \right) \] ....(2.57)

From (2.54) and (2.56), we get,
\[ \frac{2}{\delta a_i} = \frac{Vua_i}{\delta p_i} \left( \sum_{j=1}^{n} \alpha_j a_j \right) \] ....(2.58)
From (2.54), it can be observed that the necessary condition \( \frac{\partial p_i}{\partial a_i} = 0 \) is satisfied only if

\[
Vuh_i \alpha_i \left( \sum_{j=1, j \neq i}^{n} \alpha_j a_j \right) \quad \frac{n}{(\sum_{j=1}^{n} \alpha_j a_j)^2} = 1, \quad \ldots \quad (2.59)
\]

and from (2.55), condition \( \frac{\partial p_i}{\partial p_i} = 0 \) is satisfied only if

\[
M_i = \frac{h_i w_i b_i}{p_i^2}, \quad \ldots \quad (2.6c)
\]

From (2.56) and (2.57) it can be observed that sufficient condition (i) stated in (2.5) is satisfied since \( \alpha_i, a_i, h_i, w_i \) are all positive for \( i = 1, 2, \ldots, n \).

From (2.6), sufficient condition (ii) is satisfied only if

\[
-2Vuh_i \alpha_i^2 \left( \sum_{j=1, j \neq i}^{n} \alpha_j a_j \right) \quad \frac{n}{(\sum_{j=1, j \neq i}^{n} \alpha_j a_j)^3} \quad (\sum_{j=1}^{n} \alpha_j a_j)^3 \quad \frac{Vuh_i \left( \sum_{j=1, j \neq i}^{n} \alpha_j a_j \right)}{p_i^3} > \left( \sum_{j=1}^{n} \alpha_j a_j \right)^2.
\]
(Using the results (2.56), (2.57) and (2.58) in (2.6)).

On further simplification, the above inequality can be rewritten as

\[
\frac{4h_i w_i p_i c_i}{p_i^3} > u \frac{(\Sigma \alpha_j a_j)}{j=1} \quad \ldots \ldots (2.61)
\]

Hence from (2.4) and (2.6) optimum \( p_i \) and \( a_i \) of Brand-1 for \( i = 1, 2, \ldots, n \) in this case should satisfy the relations stated in (2.69), (2.60) and (2.61).

2. Simultaneous Optimization Conditions:

Conditions on Advertising Expenditure:

The system of equations obtained by setting the expressions of (2.54) to zero for each \( i \) can be solved for \( a_1, a_2, \ldots, a_n \) and yield the following information.
From (2.59), we have,

\[ \frac{V_{uh_i} \alpha_i \left( \sum_{j=1}^{n} \alpha_j a_j \right)}{\left( \sum_{j=1}^{n} \alpha_j a_j \right)^2} = 1 \]

i.e. \( V_{uh_i} \alpha_i \left( \sum_{j \neq i} \alpha_j a_j \right) = \left( \sum_{j=1}^{n} \alpha_j a_j \right)^2 \left( \frac{1}{\alpha_i h_i} \right) \).

\[ \ldots \ldots (2.62) \]

There is one such equation for each brand and summing these \( j \) equations yields

\[ V_{u(n-1)} \left( \sum_{j=1}^{n} \alpha_j a_j \right) = \left( \sum_{j=1}^{n} \alpha_j a_j \right)^2 \left( \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} \right) \]

i.e. \[ \sum_{j=1}^{n} \alpha_j a_j = \frac{(n-1)V_u}{\frac{n}{\sum_{j=1}^{n} \alpha_j h_j}} \]

\[ \ldots \ldots (2.63) \]

and from (2.59), we have,

\[ \frac{V_{uh_i} \alpha_i \left( \sum_{j=1}^{n} \alpha_j a_j \right)}{\left( \sum_{j=1}^{n} \alpha_j a_j \right)^2} = 1 \]

which can be rewritten as
\begin{align*}
Vuh_1 a_i \left( \sum_{j=1}^{n} \alpha_j a_j \right) & \quad \frac{\sum_{j=1, j \neq 1}^{n} \alpha_j a_j}{\sum_{j=1}^{n} \alpha_j a_j} = \sum_{j=1, j \neq 1}^{n} \alpha_j a_j + \alpha_i a_i \\
\text{and hence} \\
\alpha_i a_i &= \frac{Vuh_1 a_i \left( \sum_{j=1}^{n} \alpha_j a_j \right) - \left( \sum_{j=1}^{n} \alpha_j a_j \right) \left( \sum_{j=1}^{n} \alpha_j a_j \right)}{\left( \sum_{j=1}^{n} \alpha_j a_j \right) - \left( \sum_{j=1}^{n} \alpha_j a_j \right)} \\
&= \left( \sum_{j=1}^{n} \alpha_j a_j \right) \left[ \sum_{j=1, j \neq 1}^{n} \alpha_j a_j - \sum_{j=1}^{n} \alpha_j a_j \right].
\end{align*}

Considering (2.63) in the above expression and on further simplification we get,

\begin{align*}
\sum_{j=1, j \neq 1}^{n} \alpha_j a_j &= \left( \alpha_i a_i \right) \left( \frac{n-1}{n} \right) \left( \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} \right) \\
&= \frac{\alpha_i h_i - \left( \frac{n-1}{n} \right) \left( \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} \right)}{\alpha_i h_i - \left( \frac{n-1}{n} \right) \left( \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} \right)}.
\end{align*}

Adding \( \alpha_i a_i \) to both the sides of the above equation, we get,
\[
\sum_{j=1, j \neq i}^{n} \alpha_j a_j + \alpha_i a_i = \frac{(\alpha_i a_i) \left( \frac{n-1}{n} \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} \right)}{\Sigma \frac{1}{\alpha_j h_j} - \frac{n-1}{n}} + \alpha_i a_i
\]

which gives

\[
\sum_{j=1}^{n} \alpha_j a_j = \frac{(\alpha_i a_i) \left( \alpha_i h_i \right)}{(\alpha_i h_i) - \frac{n-1}{n} \Sigma \frac{1}{\alpha_j h_j}}
\]

\[
\Sigma \frac{1}{\alpha_j h_j} - \frac{n-1}{n}
\]

But from (2.63),

\[
\sum_{j=1}^{n} \alpha_j a_j = \frac{(n-1)Vu}{\Sigma \frac{1}{\alpha_j h_j}}
\]

Using (2.63) in (2.64), we get,

\[
\frac{(n-1)Vu}{\Sigma \frac{1}{\alpha_j h_j}} = \frac{(\alpha_i a_i) \left( \alpha_i h_i \right)}{(\alpha_i h_i) - \frac{n-1}{n} \Sigma \frac{1}{\alpha_j h_j}}
\]

\[
\text{i.e. } \alpha_i a_i = Vu \left( \frac{n-1}{\Sigma \frac{1}{\alpha_j h_j}} \right) \left[ 1 - \frac{n-1}{\Sigma \frac{1}{\alpha_j h_j}} \right]
\]
\[ A_1 = 1 - \frac{(n - 1)}{\sum_{j=1}^{n} \frac{a_j h_j}{a_{j+1} h_{j+1}}} \]

where

\[ A_1 = \frac{1}{\sum_{j=1}^{n} \frac{a_j h_j}{a_{j+1} h_{j+1}}} \]

Conditions on Price:

From (2.60), we have

\[ M_i = \frac{h_i \omega_i p_i}{p_i} \]

Considering summation over all \( n \) values, result (2.60) can be written as

\[ \sum_{j=1}^{n} M_j = \sum_{j=1}^{n} \frac{h_j \omega_j}{p_j} \]

But sum of the market share of all brands is always one and hence we have,

\[ \sum_{j=1}^{n} \frac{h_j \omega_j}{p_j} = 1 \]

From (2.52), we have
Using the above result in (2.60), we get,

\[
M_i = u \frac{\sum_{j=1}^{n} a_j a_j}{\sum_{j=1}^{n} \beta_j s_j} + v \frac{\sum_{j=1}^{n} \beta_j s_j}{\sum_{j=1}^{n} \beta_j s_j} + \omega_i \frac{p_i - p_i}{p_i}
\]

\[
\sum_{j=1}^{n} a_j a_j + \sum_{j=1}^{n} \beta_j s_j = \frac{h_i \omega_i p_i}{p_i} - \omega_i \frac{p_i - p_i}{p_i}
\]

\[
= \omega_i \left[ \frac{p_i}{p_i^2} (h_i - p_i) + 1 \right]
\]

\[
= \omega_i \left( 1 - \frac{p_i}{p_i^2} \right) \quad ...(2.67)
\]

Considering summation of over j for all n brands
the above result can be written as

\[
u + v = \sum_{j=1}^{n} \omega_j - \sum_{j=1}^{n} \frac{\omega_j c_j}{2}
\]
i.e. \[ \bar{\pi} \frac{n}{j=1} \frac{w_j c_j}{p_j^2} = \frac{n}{j=1} \frac{w_j - u - v}{p_j^2} \]

Equating results (2.66) and (2.68), we get,

\[ \begin{align*}
\bar{\pi} \frac{n}{j=1} \frac{w_j h_j w_j}{p_j^2} &= \bar{\pi} \frac{n}{j=1} \frac{w_j c_j}{p_j^2} \\
\frac{n}{j=1} \frac{w_j - u - v}{p_j^2} &= 1 \\
\end{align*} \]

\[ \text{.....(2.69)} \]

i.e. \[ \bar{\pi} \frac{n}{j=1} \frac{w_j}{p_j^2} \left( \frac{Q_n h_j - c_j}{\frac{n}{j=1} \frac{w_j - u - v}{p_j}} \right) = 0. \]

Since \( w_j > 0, \ p_j > 0, \) for \( j = 1, 2, ..., n, \) the above condition is possible only if

\[ h_j - \frac{c_j}{Q_n} = 0, \]

where \( Q_n = \frac{n}{j=1} \frac{w_j - u - v}{p_j}, \)

i.e. \[ h_j = \frac{c_j}{Q_n}. \]

\[ \text{.....(2.70)} \]

From (2.70), it can be easily shown that

\[ p_j = (1 + \frac{1}{Q_n}) c_j \] (since \( h_j = p_j - c_j). \)
3. Joint Conditions:

Using the above results (2.70) and (2.71) in (2.65), we get, optimum advertising budget as

\[
\alpha_i a_i = Vu \frac{(n - 1)}{\sum_{j=1}^{n} \frac{u_j}{\alpha_j c_j}} \left[ 1 - \frac{(n - 1)}{1 + \sum_{j=1}^{n} \frac{\alpha_i c_i}{\alpha_j c_j}} \right]
\]

[using the result (2.70) in (2.65)]

and hence we have,

\[
a_i = \frac{Vu}{n \sum_{j=1}^{n} \frac{1}{\alpha_j c_j}} \frac{(n - 1)}{\alpha_i a_i} a_i^o \quad \ldots \quad (2.72)
\]

where,  
\[
a_i^o = 1 - \frac{n - 1}{1 + \sum_{j=1}^{n} \frac{\alpha_i c_i}{\alpha_j c_j}}
\]

4. Sufficient Condition under optimization:

Under optimization, sufficient condition (2.61), can be given by
Considering results (2.63), (2.65), (2.70), (2.71) in (2.61),

where

\[ A_i = 1 - \frac{n - 1}{1 + \sum_{j=1}^{n} \frac{\alpha_i c_i}{\alpha_j c_j}}. \]

Therefore the above inequality can be written as

\[ \frac{4 Q_n}{(1 + Q_n)^2} \left( \frac{\omega_i c_i}{c_i} \right) > u \left[ 1 - \left( 1 - \frac{n - 1}{1 + \sum_{j=1}^{n} \frac{\alpha_i c_i}{\alpha_j c_j}} \right) \right]. \]

i.e. \[ \frac{4 Q_n}{(1 + Q_n)^2} \left( \frac{\omega_i c_i}{c_i} \right) > u - \frac{n - 1}{1 + \sum_{j=1}^{n} \frac{\alpha_i c_i}{\alpha_j c_j}} \]

Sales Budget Determination:

Since \( s_i > 0, \ i = 1, 2, \ldots, n, \) the necessary and sufficient conditions for the maximum profit of the \( i \) th competitor are the same as stated by the result (2.44).

Proceeding according to earlier case, we can derive similar results.
Using conditions (2.44), it can be shown that

\[ s_i^o = \frac{V_u Q}{n \beta_i} \frac{(n - 1)}{\Sigma_{j=1}^{n} \beta_j c_j} B_i^o \quad \ldots \quad (2.74) \]

where \( B_i^o = 1 - \frac{(n - 1)}{1 + \Sigma_{j=1}^{n} \frac{p_j c_j}{1 + \Sigma_{j=1}^{n} p_j c_j}} \).

It can be shown that sufficient condition (ii) in this case is satisfied if

\[ \frac{\beta_i s_i}{\Sigma_{j=1}^{n} \beta_j s_j} > \frac{4w_i h_i p_i c_i}{p_i^3} \quad \ldots \quad (2.75) \]

Adding results (2.61) and (2.75), we get,

\[ u \frac{\alpha_i c_i}{\Sigma_{j=1}^{n} \alpha_j c_j} + v \frac{\beta_i s_i}{\Sigma_{j=1}^{n} \beta_j s_j} > u + v - \frac{8w_i h_i p_i c_i}{p_i^3} \]

Adding \( \frac{( - p_i - p_i)}{p_i^3} \) on both the sides, we obtain a new condition as

\[ M_i > (u + v - w_i) + \frac{w_i p_i}{p_i} \left( 1 - \frac{8h_i c_i}{p_i^2} \right) \quad \ldots \quad (2.76) \]
Optimization of Profit:

From (2.52), profit function for Brand-1 can be given by

\[
P_i = V\left[ u \frac{a_i a_i}{n} + v \frac{\beta_i s_i}{p_i} + w_i \frac{(p_i - p_i)}{p_i} \right] h_i
\]

\[= \delta_1 + \delta_2 + \delta_3 - F_i\]

where

\[
\delta_1 = V u h_i \frac{a_i a_i}{n} \sum_{j=1}^{n} \alpha_j a_j - a_i
\]

\[
\delta_2 = V u h_i \frac{\beta_i s_i}{n} \sum_{j=1}^{n} \beta_j s_j - s_i
\]

\[
\delta_3 = w_i \left( \frac{\bar{p}_i - p_i}{p_i} \right) h_i
\]

Now optimum value of \( \delta_1 \) can be given by

\[
\delta_1^o = V u h_i \frac{a_i a_i^o}{n} \sum_{j=1}^{n} \alpha_j a_j - a_i^o
\]
Considering (2.70) and (2.72) in the above expression, we get,

\[ \delta_1^o = \frac{V_{\mu_1}^o}{Q_n} \left[ c_i - \frac{1}{\alpha_i} \sum_{j=1}^{n-1} \frac{1}{\alpha_j c_j} \right] \]

\[ = \frac{V_{\mu_1}^o c_i}{Q_n} \left( 1 - \frac{n-1}{\sum_{j=1}^{n} \frac{1}{(\alpha_i c_i) \sum_{j=1}^{n-1} \frac{1}{\alpha_j c_j}} } \right) \]

\[ = \frac{V_{\mu_1}^o c_i}{Q_n} \]

\[ \text{where } \lambda_1^o = 1 - \frac{n-1}{\sum_{j=1}^{n} \frac{1}{(\alpha_i c_i) \sum_{j=1}^{n-1} \frac{1}{\alpha_j c_j}} } \]

Similarly, optimum value of \( \delta_2 \) can be given by

\[ \delta_2^o = \frac{V_{\mu_1}^o B_i^c}{Q_n} \]

\[ \text{where } B_i^c = 1 - \frac{n-1}{\sum_{j=1}^{n} \frac{1}{(\beta_j s_j) \sum_{j=1}^{n-1} \frac{1}{\beta_j s_j}} } \]

\[ \text{......(2.77)} \]

and Optimum value of \( \delta_3 \) is given by
\[ \delta_3^o = \frac{V_{\text{uh}i}^o}{Q_n} \quad w_i \left( \frac{p_i^o - p_i^o}{p_i^o} \right). \]

From (2.71), we have

\[ p_i^o = (1 + \frac{1}{Q_n}) c_i \]

and

\[ p_i^o = (1 + \frac{1}{Q_n}) c_i^o. \]

Using these results in the above expression, we get,

\[ \delta_3^o = \frac{V_{\text{uc}i} w_i}{Q_n} \frac{(c_i^o - c_i)}{c_i} \quad \ldots (2.78) \]

Using results (2.75), (2.77) and (2.78) in the profit function we get optimum profit as

\[ p_i^o = \delta_1^o + \delta_2^o + \delta_3^o - F_i \]

\[ = \frac{V_{c_i}}{Q_n} \left[ uA_i + vB_i + w_i \frac{(c_i^o - c_i)}{c_i} \right] - F_i \]

where \( \overline{c}_i \) is the average product cost of all opponents and

\[ \overline{c}_i = \sum_{i=1}^{n} c_i \frac{1}{(n-1)}. \]
2.4.2.3 Advertising Budget determination Procedure:

The advertising budget equilibrium values are obtained by similar procedure discussed by Mills\(^\text{(10)}\) which consists of finding optimum values for two competitors and extending it to \(n\) competitors successively.

In the procedure discussed here, it is assumed that for \(t = 2\), there exist \(a_1^0\) and \(a_2^0\) that satisfy equation (2.72). At each stage, it is tested whether the incoming competitor can profit more by spending on advertising effort. The procedure can be explained by the following steps.

Algorithm:

1. Relabel the competitor so that
   \[ a_1^c_1 > a_2^c_2 > \ldots > a_n^c_n \quad (2.80) \]

2. Find an equilibrium point \( (a_1^0, a_2^0, \ldots, a_t^0) \) where \(t\) represents the number of competitors who spend on advertising effort and \(a_i^0\)'s satisfy equation (2.72). Initially consider \(t = 2\).
3. If \( t < n \), test whether
\[
\alpha_{t+1} c_{t+1} > \frac{t - 1}{\sum_{j=1}^{t} \frac{1}{\alpha_j c_j}}.
\]
\( \ldots \quad (2.81) \)

4. If yes, go to step (2) and replace \( t \) by \( (t + 1) \).

5. If no, \( (a_1^0, a_2^0, \ldots, a_t^0, 0, \ldots, 0) \) is the equilibrium solution.

**Theorem 1**: Given a profit function defined by (2.52), there exists a unique competitive equilibrium point which can be determined by the above algorithm.

**Proof**: 

1. To show existence of competitive equilibrium point.

   **Case-1**: When \( t = 2 \).

   From (2.72), we have,
   \[
   \alpha_i a_i = \frac{V_i}{Q_i} = \frac{(n - 1)}{n} \left[ 1 - \left( \frac{n - 1}{n} \right) \right].
   \]

   For \( n = 2 \), the above expression can be rewritten as
\[ a_i a_i^0 = \frac{\text{Vu}}{Q_2} \left( \frac{1}{\alpha_i c_1} + \frac{1}{\alpha_2 c_2} \right) \left[ 1 - \frac{2}{1 + \sum_{j=i}^{\alpha_i c_i} 1 + \sum_{j=1}^{\alpha_j c_j} 1} \right] \]

where \( Q_2 = \sum_{i=1}^{2} \omega_i - u - v, \)

which is always positive due to condition (2.53).

For \( i = 1, \) the result (2.79), can be written as

\[ a_1 a_1^0 = \frac{\text{Vu}}{Q_2} \left( \frac{a_1 a_2}{(\alpha_1 c_1 + \alpha_2 c_2)} \right) \left[ 1 - \frac{1}{1 + \frac{a_1^2 c_2}{\alpha_2 c_2}} \right] \]

\[ a_1^0 = \frac{\text{Vu}}{Q_2} c_1 \left( \frac{a_1 a_2}{(\alpha_1 c_1 + \alpha_2 c_2)} \right)^2. \]

Similarly it can be shown that

\[ a_2^0 = \frac{\text{Vu}}{Q_2} c_2 \left( \frac{a_1 a_2}{(\alpha_1 c_1 + \alpha_2 c_2)} \right)^2. \]

Since \( a_1, u, c_1, V \) are all positive for \( i = 1,2, \)

and from (2.53) it can be said that, \( Q_2 > 0, \)

we have, \( a_i^0 > 0, \) for \( i = 1,2. \)
Case-2: when \( t < n \).

(i) Since \( t-1 < n \), we make the use of result (2.81) and test whether

\[
\alpha_t c_t > \frac{t-2}{t-1} \sum_{j=1}^{t-1} \frac{1}{\alpha_j c_j} > t-2.
\]

Rewriting this equation we obtain,

\[
\alpha_t c_t \sum_{j=1}^{t-1} \frac{1}{\alpha_j c_j} > t-1.
\]

Adding 1 to both the sides,

\[
\alpha_t c_t \left[ \frac{1}{\alpha_t c_t} + \sum_{j=1}^{t-1} \frac{1}{\alpha_j c_j} \right] > t-1
\]

and therefore we get,

\[
\alpha_t c_t \sum_{j=1}^{t-1} \frac{1}{\alpha_j c_j} > t-1
\]

i.e.

\[
\alpha_t c_t > \frac{t-1}{t} \sum_{j=1}^{t-1} \frac{1}{\alpha_j c_j}
\]

i.e.

\[
\frac{t-1}{(\alpha_t c_t) \sum_{j=1}^{t} \frac{1}{\alpha_j c_j}} < 1
\]
and hence, \( A_t^0 = 1 - \frac{t - 1}{t \sum_j \frac{1}{\alpha_j c_j}} > 0 \).

With \( A_t^0 > 0 \), it can be seen from (2.72), that \( a_t^0 > 0 \).

From result (2.80) it can be observed that \( a_t^c_t < a_j c_j \) if \( j \leq t \).

This means that

\[
\frac{(t - 1)}{\sum_j \frac{1}{\alpha_j c_j}} > \frac{(t - 1)}{\sum_j \frac{1}{\alpha_j c_j}}
\]

i.e. \( 1 - \frac{(t - 1)}{\sum_j \frac{1}{\alpha_j c_j}} < 1 - \frac{(t - 1)}{\sum_j \frac{1}{\alpha_j c_j}} \)

i.e. \( A_j^0 < A_t^0 \) and from this it follows that \( A_j^0 > 0 \), since we have shown that \( A_t^0 > 0 \). That is \( A_j^0 > 0, \; a_j^0 > 0 \) when \( j \leq t \).

Hence \( a_1^0, a_2^0, \ldots, a_t^0 \) is an equilibrium point at each stage, using result (2.72) in the algorithm.
(ii) If the condition (2.81) is not satisfied i.e. we have,
\[ \sum_{j=1}^{\infty} \alpha_k < \frac{(-1)^k \alpha_k}{t} \], for \( k = t+1, t+2, \ldots n \).

From (2.53),
\[ \frac{dP_i}{\delta a_i} = \sum_{j=1}^{n} \alpha_j a_j \]

Therefore, we have,
\[ \frac{dP_i}{\delta a_i} \bigg|_{a_i=0} = \frac{V_h_i \alpha_i}{\sum_{j \neq i}^{n} \alpha_j a_j} - 1 \quad \ldots (2.82) \]

But from (2.63),
\[ \sum_{j=1}^{n} \alpha_j a_j = \frac{(n-1)V_u}{\sum_{j=1}^{n} \frac{1}{\alpha_j h_j}} \]

Considering this result in (2.82), we get,
\[ \frac{dP_i}{\delta a_i} \bigg|_{a_i=0} = \frac{V_h_i \alpha_i}{\frac{V_u(n-1)}{\sum_{j=1}^{n} \frac{1}{\alpha_j h_j}}} - 1 \quad \ldots (2.83) \]
Using the result \((2.70)\) in the above expression, we get,

\[
\frac{\partial P_i}{\partial a_i} \bigg|_{a_i=0} = \frac{a_i c_i}{(n-1)} - 1 \quad \ldots \ldots (2.84)
\]

Therefore at the point \((a_1^o, a_2^o, \ldots, a_t^o, 0, \ldots, 0)\), we use \((2.72)\), and get,

\[
\frac{\partial P_k}{\partial a_k} = \frac{(\sum_{j=1}^{t} \frac{1}{\alpha_j c_j}) \frac{V_u}{Q_t}}{(t-1)} - 1 < 0,
\]

where \(Q_t = \sum_{j=1}^{t} w_{j-u-v}\)

and the second derivative is negative by \((2.56)\). Hence \(P_k\) assumes its maximum at \(x_k = 0, k > t+1\) and therefore \((a_1^o, a_2^o, \ldots, a_t^o, 0, \ldots, 0)\) is an equilibrium point.

2. To establish uniqueness of equilibrium point:

Let us suppose that \(\tilde{a}\) is the equilibrium point for which \(a_j^c = 0\) and \(a_s^o > 0\) and hence
using the result (2.80), it can be said that
\[ \alpha_j c_j < \alpha_s c_s. \]

For no equilibrium point \( a^o \), let \( a_j^o = 0 \), \( a_s > 0 \) and \( \alpha_j c_j > \alpha_s c_s \).

Let us suppose that there exists another equilibrium point \( a'^o \) such that \( a_j^o > 0 \), \( j < t \), \( a_s = 0 \), \( s > t \) and for some \( j, s \), \( \alpha_j c_j > \alpha_s c_s \) \((j < t < s)\).

Therefore \( (a_1^o, a_2^o, \ldots, a_t^o) \) satisfy the necessary and sufficient condition for the brands 1, 2, ..., t and we have,

\[
\alpha_j^o = \frac{V u}{Q t} \frac{(t - 1)}{t} [1 - \frac{t - 1}{t} \frac{1}{\alpha_j c_j}] > 0,
\]

that is \( \alpha_j^o > 0 \).

Now if we consider derivative of \( P_s^o \) where \( a_s = 0 \), using result (2.82) and (2.63) we get,

\[
\frac{\partial P_s}{\partial a_s} = \frac{V u}{Q t} \frac{\alpha_s c_s}{(t - 1) \sum_{j=1}^{t-1} \frac{1}{\alpha_j c_j}} - 1.
\]
From result (2.85) it can be observed that \( P_s \) is not maximum at \( a_0 = 0 \) and hence \( a_0^0 \) is not an equilibrium point.

2.4.2.4 Explanation Through Hypothetical Problem:

Let us suppose that four brands, called \( X_1, X_2, X_3 \) and \( X_4 \) are competing in a market in which total sales of the product is fixed and it is 10,000 units.

For equal advertising and selling efficiency (i.e. \( \alpha_i = \beta_i \), \( i = 1, 2, \ldots, n \)) the following information is given:

<table>
<thead>
<tr>
<th>Brand</th>
<th>Production Cost ((c_i)) (in Rs.)</th>
<th>Advertising efficiency ((\alpha_i))</th>
<th>Price constant ((\omega_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>4.00</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>2.75</td>
<td>0.60</td>
<td>0.60</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>3.90</td>
<td>0.40</td>
<td>0.55</td>
</tr>
<tr>
<td>( X_4 )</td>
<td>3.50</td>
<td>0.45</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Advertising constant \((u) = 0.4\)
Selling constant \((v) = 0.3\)
Now, according to the algorithm, the first step is to label each brand. The brand with the largest product of advertising efficiency and cost of production is labelled number 1, the brand with the next largest product is labelled number 2, and so on until the brand with the smallest margin has the highest number. The results are as follows:

<table>
<thead>
<tr>
<th>Brand Number</th>
<th>Brand Name</th>
<th>Product ((a_i c_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(X_1)</td>
<td>2.000</td>
</tr>
<tr>
<td>2</td>
<td>(X_2)</td>
<td>1.650</td>
</tr>
<tr>
<td>3</td>
<td>(X_4)</td>
<td>1.575</td>
</tr>
<tr>
<td>4</td>
<td>(X_3)</td>
<td>1.560</td>
</tr>
</tbody>
</table>

The second step of the algorithm states that the problem should be considered as if there are only two brands competing in the market and we start with brands \(X_1\) and \(X_2\).

From (2.72), a given brand’s optimum advertising expenditure is given by

\[
a^{\phi}_{i} = \frac{Vu}{Q_n a_i} \left[ \frac{n}{\sum_{j=1}^{n} \frac{1}{\alpha_j c_j}} \left( \frac{n-1}{\sum_{j=1}^{n} \frac{1}{ \alpha_j c_j}} \right) \right]
\]
Considering the given information and using the result (2.72) for \( n = 2 \), the optimum advertising expenditure for Brand \( X_1 \) and \( X_2 \) can be obtained as

\[
\begin{align*}
& a_1^o = \text{Rs. 26423.21} \quad \text{and} \quad a_2^o = \text{Rs. 18165.96}.
\end{align*}
\]

But these expenditure are optimal only for these two brands which we can verify by using the ratio rule discussed in case-1.

From result (2.34), we have

\[
\frac{a_1^o}{a_2^o} = \frac{c_1}{c_2},
\]

which gives us

\[
\frac{26423.21}{18165.95} = \frac{4}{2.75} = 1.4545.
\]

Thus these expenditures are only optimal if two brands \( X_1 \) and \( X_2 \) are competing.

Step-3 of the algorithm says that if the number of brands in step-2 is less than the total number of competing brands, a test should be made and the brand with the largest product \( (a_1^o c_1) \) is
selected from among those brands for which no advertising budget has yet been calculated.

Then, we determine whether the product of this selected brand is greater than the value of test expression indicated by (2.81).

Now, no budget has been calculated as yet for brand $x_4$ and $x_3$, but of these, Brand $x_4$'s product is larger and hence test for Brand $x_4$ is made.

From (2.81) we test whether

$$a_{t+1} c_{t+1} > \frac{t - 1}{2} \sum_{j=1}^{t} \frac{1}{\alpha_j t_j}$$

i.e. $1.575 > \frac{2 - 1}{1} \frac{1}{.5} + \frac{1}{.6061}$

i.e. $1.575 > 0.9041$.

This means that Brand $x_4$'s value exceeds the value of test expression and hence return to step - 2 considering Brands $x_1$, $x_2$ and $x_4$. We calculated here the optimal advertising expenditure by using the result (2.72), which gives us

$a_1^o = Rs. 12270.08$, $a_2^o = Rs. 7298.82$ and $a_4^o = Rs. 8668.23$. 
The only brand remaining for which no advertising budget has been determined is Brand-\(x_3\) and hence we turn to the test for Brand-\(x_3\).

This test reveals that

\[ 1.56 > \frac{2}{1.741C} \]

i.e. \(1.56 > 1.1488\).

This means that the product of Brand-\(x_3\) exceeds the value of the expression and hence we now focus on all the four brands \(x_i\), \(i = 1, 2, 3, 4\) and obtain optimum advertising expenditure as well as profit. Using the results (2.71), (2.72) and (2.73) we get the following information:
Table 2.2

Optimum advertising expenditure, optimum price and optimum net contribution margin of Brands $x_1$, $x_2$, $x_3$, $x_4$ (in h₅).

<table>
<thead>
<tr>
<th>Brand</th>
<th>Optimum advertising expenditure $a_1$</th>
<th>Optimum price $p_1$</th>
<th>Optimum net contribution margin $P_1 + F_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>7106.63</td>
<td>6.8572</td>
<td>18285.71</td>
</tr>
<tr>
<td>$x_2$</td>
<td>3785.54</td>
<td>4.4058</td>
<td>15428.57</td>
</tr>
<tr>
<td>$x_3$</td>
<td>4622.76</td>
<td>6.6858</td>
<td>8190.48</td>
</tr>
<tr>
<td>$x_4$</td>
<td>4273.32</td>
<td>6.0001</td>
<td>10213.33</td>
</tr>
</tbody>
</table>

If all the Brands compete at the price and advertising expenditure stated in the Table 2.2, all can maximize their profit.

**Note:** For all the four Brands it can be shown that sufficient condition (2.61) under optimization is satisfied.
2.4.2.5 Remarks:

Analysis of the optimum solution for this case reveals that the conclusion derived from case-1 also holds for case-2. In addition we obtain from case-2, the following interesting results:

1. The total advertising expenditure for the given brand varies directly with the level of brand's production cost, its advertising efficiency and brand's sales volume.

2. The total amount of advertising expenditure varies directly with the number of competing brands. The larger the number of brands, the greater will be this total.

3. If a brand's product of advertising effectiveness and cost of production in a particular market does exceed the value of the expression discussed in (2.74), the seller can not operate profitably under conditions of competitive equilibrium in that market.

Model 2:

2.5 Budget determination in a varying market:

This model differs from Model-1, however, in this market size is not assumed to be fixed. Instead,
the market is assumed to be changing with the amount growth depending upon the total amount of effective advertising expenditure for the product.

Many efforts have been made to link advertising with sales volume. In this model we have considered the view of Till Donald(121) and have visualized possible cases. But the general structure of this model and all assumptions except the assumption of varying market are identical to that of the previous model.

2.5.1 Case - l : Diminishing returns Market Response.

2.5.1.1 Problem Formulation :

Here we assume that total market is not fixed but increases with decreasing returns as aggregate advertising efforts increase and it can be indicated by

\[ V' = V \cdot \delta \left( \sum_{j=1}^{n} \alpha_j a_j \right)^x \]  

(2.86)

where \( V \) is the fixed market, \( \delta \) is the positive constant and \( x \) is an exponent, such that \( 0 < x < 1 \). Also here it is assumed that total amount of effective
advertising expenditure of the product is greater than one

\[ \sum_{j=1}^{n} \alpha_{j}a_{j} > 1. \]

The profit function for the \( i \)th brand in this case is given by

\[
P_{i} = V' M_{i} h_{i} - a_{i} - s_{i} - F_{i} \quad \text{if } \sum a_{i} > 0, \sum s_{i} > 0,
\]

\[ 0 \quad \text{if } a_{i} = 0 \text{ and } s_{i} = 0, \text{ for } i=1,2,...,n, \]

\[ \text{.....(2.87)} \]

where \( M_{i}, h_{i}, a_{i}, s_{i} \) and \( F_{i} \) are the same as defined earlier.

Using (2.86) in (2.87) we get,

\[
P_{i} = (V' \delta(\sum_{j=1}^{n} \alpha_{j}a_{j})^x \left[ u \frac{\alpha_{i}a_{i}}{\sum_{j=1}^{n} \alpha_{j}a_{j}} + v \frac{\beta_{i}s_{i}}{\sum_{j=1}^{n} \alpha_{j}a_{j}} + w_{i} \frac{(\bar{p}_{i} - p_{i})}{p_{i}} \right] h_{i} - a_{i} - s_{i} - F_{i} \]

\[ \text{.....(2.88)} \]
2.5.1.2 Problem Solution:

Necessary and Sufficient Conditions:

For \( a_1 > 0 \), \( i = 1,2, ..., n \) the necessary and sufficient conditions for maximum Pay-off of \( i \)th brand are the same as in previous cases.

From (2.4), the necessary conditions can be given by

\[
\frac{\delta P_i}{\delta a_i} = \frac{\delta}{\delta x} VxA \alpha_i \gamma_1 h_1 + V\delta A^x \left( \frac{\sum_{j=1}^{n} a_j a_j}{A^2} \right) h_1 - 1
\]

where \( A = \sum_{j=1}^{n} \alpha_j a_j \)

[using (2.88) in (2.4)]

and

\[
\frac{\delta P_i}{\delta P_i} = V\delta A \left( \nu_i - \frac{w_i P_i}{P_i^2} \right) = 0 \quad \ldots (2.90)
\]

From (2.89), we have

\[
\frac{\delta^2 P_i}{\delta a_i^2} = - \frac{\delta}{\delta x} V\alpha_i^2 h_i A^x \left[ (1-x)M_i + u(2-x) \frac{\sum_{j=1}^{n} \alpha_j a_j}{\sum_{j=1}^{n} \alpha_j a_j} \right]
\]

\[
= - \frac{\delta}{\delta x} V\alpha_i h_i A^x H_i \quad \ldots \ldots (2.91)
\]

* As usual it is assumed here that \( j = 1,2, ..., n \).
where \( H_i = (1-x)M_i + u(2-x) \sum_{j \neq i}^{n} \frac{\alpha_i a_j}{A} \).

Since \( 0 < x < 1 \), we have \( H_i > 0 \) and hence

\[
\frac{2}{\delta a_i} \frac{\delta P_i}{\delta a_i} < 0.
\]

From (2.89), we have,

\[
\frac{2}{\delta a_i} \frac{\delta P_i}{\delta a_i} = \delta V x A \alpha_i M_i + \delta V x A \alpha_i h_i \left( \frac{-w_i P_i}{p_i^2} \right) + \delta V \delta u x_i \left( \sum_{j \neq i}^{n} \alpha_j a_j \right) x^{-2}
\]

\[
= \delta V x A \alpha_i F_i
\]

\[
= \frac{\delta^2 P_i}{\delta a_i \delta p_i}
\]

\[
\ldots \ldots (2.92)
\]

where \( F_i = (M_i - \frac{w_i P_i}{p_i^2} h_i) x + u \sum_{j \neq i}^{n} \frac{\alpha_j a_j}{A} \).

From (2.90), we have,

\[
\frac{2}{\delta p_i} \frac{\delta P_i}{\delta p_i} = - \frac{2w_i P_i c_i V'}{p_i^3}
\]

\[
\ldots \ldots (2.93)
\]
Since \( w_i, c_i, p_i, \bar{p}_i, V' \) are all positive for
for \( i = 1, 2, \ldots, n \), \( \frac{\delta^2 p_i}{\delta \bar{p}_i^2} < 0 \).

This result is the same as (2.57) because \( V' \) is
not a function of price.

From (2.91) and (2.93), it can be observed that
sufficient condition (i) is satisfied but sufficient
condition (ii) is satisfied only if
\[
\left( \frac{\delta p_i}{\delta a_i^2} \right)^2 \left( \frac{\delta p_i}{\delta a_i^2} \right) > \left( \frac{\delta p_i}{\delta p_i \delta a_i} \right)^2.
\]

Therefore, considering results (2.91), (2.92), (2.93)
in the above inequality, we get,
\[
\frac{2x_i h_i w_i p_i c_i}{3 p_i} > (F_i)^2 \quad \ldots \ldots (2.94)
\]

From (2.4) and (2.6) it can be said
that optimum \( p_i \) and \( a_i \) of Brand-i, for \( i=1,2,\ldots,n \)
in this case should satisfy the relations stated in
(2.89), (2.90) and (2.94).

Simultaneous Optimization:

From (2.90), we have,
\[ \delta V_{xA} \alpha_i M_i h_i + V \delta A \left( \alpha_i \frac{\sum_{j \neq i} a_j a_j}{A^2} \right) h_i = 1 \]

\[ i.e. \delta V_{xA} M_i + \delta V_u A \left( \sum_{j \neq i} \frac{1}{\alpha_j h_j} \right) = \frac{1}{\alpha_i h_i} . \]

\[ \ldots \ldots \ldots (2.95) \]

Considering summation on both the sides we get,

\[ \delta V_{xA} \sum_{j=1}^{n} M_j + \delta V_u A (n-1)A = \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} . \]

Since \( \sum_{j=1}^{n} M_j = 1 \), the above equation can be rewritten as

\[ \delta V_{xA} \sum_{j=1}^{n} M_j + \delta V_u A (n-1)A = \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} . \]

Considering

\[ G = \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} , \text{ and on further simplification we get,} \]

\[ A = \frac{\frac{G}{\delta V[x + (n-1)u]}}{1/(x-1)} \]

\[ i.e. A = \left( \frac{G}{\delta V[x + (n-1)u]} \right)^{1/(x-1)} \ldots \ldots (2.96) \]
To obtain expression for $\alpha_1 a_1$:

From (2.95), we have,

$$
\delta VxAM_1 + \delta VuA (\Sigma_{j \neq i} \alpha_j a_j) = \omega_i,
$$

where $\omega_i = \frac{1}{\alpha_i h_i}$

i.e. $\Sigma_{j \neq i} \alpha_j a_j = \frac{1}{\delta VuA x^{-2}} [\omega_i - \delta VxA x^{-1} M_i]$.

Adding $\alpha_1 a_1$ on both the sides, we get,

$$
A = \frac{1}{\delta VuA x^{-2}} [G_i - \delta VxA x^{-1} M_i] + \alpha_1 a_1.
$$

Dividing both the sides of the above expression by $A$, we get,

$$
1 = \frac{1}{\delta VuA x^{-1}} [G_i - \delta VxA x^{-1} M_i] + \frac{\alpha_1 a_1}{A}
$$

$$
= [\frac{G_i}{\delta VuA x^{-1}} - \frac{xM_i}{u}] + \frac{\alpha_1 a_1}{A}.
$$

Considering the value of $A$, we get,
\[ 1 = \left[ \frac{G_i}{G} \left( x + (n-1)u \right) - x M_i \right] + \frac{\alpha_i a_i}{A} \]

i.e., \( \alpha_i a_i = A \left[ 1 - \frac{1}{u} \left( \frac{G_i(x+(n-1)u)}{G} - x M_i \right) \right]. \)

but \( M_i = u \frac{1}{n} \sum_{j=1}^{n} a_j a_j + v \frac{1}{n} \sum_{j=1}^{n} \beta_j a_i + w_i \frac{(\bar{p}_i - p_i)}{p_i} \)

and hence,

\[ \alpha_i a_i = A \left[ 1 - \frac{1}{u} \left( \frac{G_i(x+(n-1)u)}{G} - x \left( \frac{\alpha_i a_i}{A} + \bar{m}_i' \right) \right) \right], \]

where

\[ \bar{m}_i' = v \frac{1}{n} \sum_{j=1}^{n} \beta_j s_j + w_i \frac{(\bar{p}_i - p_i)}{p_i}. \]

Hence on further simplification it can be shown that

\[ \alpha_i a_i = \frac{A}{(1-x)} \left[ 1 - \frac{1}{u} \left( x + (n-1)u - x M_i' \right) \right]. \]

.......

As market potential is not a function of price, condition \( \frac{\partial F_i}{\partial p_i} = 0 \), will give us the same results as discussed earlier.
Therefore from \((2.7C)\) and \((2.71)\), we have

\[ h_j^o = \frac{c_j}{Q_n} \]

where \( Q_n = \sum_{j=1}^{n} \omega_j - u - v \) and

\[ p_j^o = \left(1 + \frac{1}{Q_n}\right)c_j. \]

Considering these values in results \((2.96)\), we get,

\[ A^o = \left[ -\frac{\frac{1}{Q_n}}{\delta V} \left[ x + (n - 1)u \right] \right] \]

which should be greater than 1 according to condition \((2.86)\).

Similarly, considering these values in \((2.97)\), we get,

\[ \alpha_i a_i = \frac{A^o}{X - x} \left[ 1 - \frac{1}{u} \left( \frac{x + (n - 1)u}{\sum_{j=1}^{n} \frac{1}{c_j \alpha_j c_j}} - xM_i^o \right) \right] \]

where \( M_i^o = v \sum_{j=1}^{n} \frac{\beta_i s_i}{\sum_{j=1}^{n} \beta_j s_j} + \omega_i \left( \bar{p}_i - p_i^o \right) \)

\[ = v \left[ 1 - \frac{\left( n - \bar{c}_i \right)}{\sum_{j=1}^{n} \frac{1}{\beta_j c_j}} + \omega_i \left( \bar{c}_i - c_i \right) \right] \]

\[ \beta_i c_i \sum_{j=1}^{n} \frac{1}{\beta_j c_j} \]

\[ \bar{c}_i \]

\[ \omega_i c_i \]

\[ \sum_{j=1}^{n} \frac{1}{\beta_j c_j} \]

\[ \bar{p}_i - p_i^o \]
and hence considering (2.130) in (2.99),

\[ a_i a_i^0 = \frac{A^0}{(1 - x)} \left[ 1 - \frac{1}{u} \left[ \frac{x + (n - 1)u}{n} \sum_{j=1}^{\infty} \frac{1}{\alpha_j c_j} \right] \right. \]

- \left. x(n - 1) + \frac{(c_i - c_i)}{\sum_{j=1}^{\infty} \frac{1}{\beta_j c_j}} \right].

\[ \ldots(2.101) \]

2.5.1.3 Advertising Budget determination procedure:

Here \( a_i a_i^0 > 0 \) only if it can be observed that

\[ 1 - \frac{1}{u} \left[ \frac{x + (n - 1)u}{n} \sum_{j=1}^{\infty} \frac{1}{\alpha_j c_j} - x M_i^\circ \right] > 0 \]

i.e. \[ 1 > \frac{1}{u} \left[ \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} + x(1 - M_i^\circ) \right] \]

i.e. \[ 1 + a_i c_i \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} > \frac{(n - 1)u}{u - x(1 - M_i^\circ)} \]

i.e. \[ a_i c_i > \frac{1}{\sum_{j=1}^{n} \frac{1}{\alpha_j c_j}} \left[ \frac{(n - 1)u}{u - x(1 - M_i^\circ)} - 1 \right] \]
where \[ O_i = \frac{u(n - 2) + (1 - M_i^0)}{u - x(1 - M_i^0)} \].

Since
\[ u(n - 2) + x(1 - M_i^0) > 0, \quad \text{as } 0 < x < 1, \]
\[ 0 < M_i^0 < 1, \quad u > 0, \quad \text{and } n > 2, \]
it can be observed that \( O_i \) is positive only if \( M_i^0 > 1 - \frac{1}{x} \).

Hence, the algorithm in this case can be stated as the case-2 of the previous model but with some little changes as follows:

1. Arrange the competitors so that
\[ \alpha_1 \cdot 1 > \alpha_2 \cdot 2 > \ldots > \alpha_n \cdot n. \]

2. Find an equilibrium point \((a_1^0, a_2^0, \ldots, a_t^0)\)
where \( t \) represents the number of competitors who spend on advertising effort and \( a_i^0 \)'s satisfy the equation \((2.101)\). Initially we consider \( t = 2 \).

3. If \( t < n \), test whether
\[ \alpha_t+1 \cdot t+1 > \frac{1}{t} \sum_{j=1}^{t} \frac{1}{C_j \cdot c_j} \left[ \frac{1}{t} \right] \left[ \frac{(t - 1)}{u - x(1 - M_{t+1}^0)} - 1 \right] \]
\[ \ldots \quad \text{(2.102)} \]
where

\[ \mu_{t+1}^0 = v(1 - \frac{t}{\beta_{t+1}^{\sum} \frac{1}{j=1} \frac{1}{\beta_{j} c_{j}}}) + \frac{(\bar{c}_{t+1} - c_{t+1})}{c_{t+1}} \]

4. If yes, go to step-2 and replace t by t+1.

5. If no, \((a_1^0, a_2^0, \ldots, a_t^0, \ldots, a_{\infty}^0)\) is the equilibrium solution

**Theorem 2**: Using the above mentioned algorithm, we can show that there exists a unique competitive equilibrium point for a game defined by (2.88), for which

\[ \mu_i^0 > 1 - \frac{u}{x}, \quad i = 1, 2, \ldots, n \quad \ldots(2.103) \]

The proof of this theorem runs parallel to that of the proof given in Theorem 1. The restriction on u and x defined by (2.103), guarantees an equilibrium point when \( t = 2 \); otherwise the arguments are identical.
2.5.1.4 Optimization of Profit:

From (2.88), we have,

\[ p_1^o + f_i = h_i^o v_i^o \left[ u \frac{a_i^o}{\lambda^o} + v \frac{s_i^o}{\beta_i^o} \right] + w_i \frac{(p_i^o - p_i)}{p_i} = a_i^o - s_i^o \]

\[ = \delta_1^o + \delta_2^o + \delta_3^o \quad \ldots \ldots (2.104) \]

where

\[ \delta_1^o = h_i^o v_i^o \left( u \frac{a_i^o}{\lambda^o} \right) - a_i^o \]

\[ = \frac{V}{Q_n} c_i (A^o)^x \left( u \frac{a_i^o}{\lambda^o} \right) - a_i^o \quad \ldots \ldots (2.105) \]

\[ \delta_2^o = h_i^o v_i^o \left( v \frac{s_i^o}{\beta_i^o} \right) = s_i^o \]

\[ = \frac{V}{Q_n} c_i v(A^o)^{x-1} \left( u \frac{a_i^o}{\lambda^o} \right) \cdot \beta_i^o B_1 \quad \text{[Using the result (2.74)]} \]

\[ \ldots \ldots (2.106) \]
where

\[ B_i = 1 - \frac{\sum_{j=1}^{n} \frac{1}{\beta_j s_j}}{(\beta_i c_i) \sum_{j=1}^{n} \frac{1}{\beta_j s_j}} \]

\[ \delta_3^o = h_i^o v^{o^1} w_i \left( \frac{p_i^o - p_i^o}{p_i^o} \right) \]

\[ = \frac{\delta V c_i(A^o)^x}{\bar{Q}_n} w_i \left( \frac{c_i^o - c_i}{c_i} \right). \]

\[ \text{......(2.107)} \]

Considering (2.105), (2.106) and (2.107) in (2.104), we get

\[ p_i^o + f_i = \frac{c_i V_0}{\bar{Q}_n} (A^o)^{x-1} \left[ u \frac{a_i a_i^o}{A^o} + v b_i^2 + w_i \frac{(c_i^o - c_i)}{c_i} \right] \]

\[ - a_i^o. \]

\[ \text{......(2.108)} \]

Using (2.98) and (2.101) in the above expression, the optimum value of net contribution margin can be obtained.
2.5.2 Case-2: Quadratic Market Response:

2.5.2.1 Problem Formulation:

In this case it is assumed that market is not fixed but as advertising efforts increase, up to a certain limit market increases and then it starts decreasing.

This can be indicated by total market

\[ V' = \hat{V} [\beta_2 (\sum_{j=1}^{n} a_j a_j) - \beta_3 (\sum_{j=1}^{n} a_j a_j)^2] \quad \ldots \ldots (2.109) \]

where \( \hat{V} \) is the fixed sales volume, \( \beta_2 \) and \( \beta_3 \) are positive constants for the product under consideration such that the total amount of effective advertising expenditure of all brands is less than the ratio \( \frac{\beta_2}{\beta_3} \).

i.e. \( \frac{\beta_2}{\beta_3} > \sum_{j=1}^{n} a_j a_j \).

The profit function for \( i \)th brand in this case is given by

\[ p_i = \hat{V} [\beta_2 (\sum_{j=1}^{n} a_j a_j) - \beta_3 (\sum_{j=1}^{n} a_j a_j)^2] \]
\[ \frac{\partial P_i}{\partial a_i} = V[\beta_2 \alpha_i - 2\beta_3 A \alpha_i]M_i h_i + V(\beta_2 A - \beta_3 A)^2 \]

\[
\left( u_{a_i} \frac{\alpha_i a_i}{A} \right) h_i - i \quad \text{......(2.111)}
\]

where \( A = \sum_{j=1}^{n} \alpha_j a_j \).

The necessary condition (2.4) in this case can be obtained by equating (2.111) with respect to zero which gives

\[ V\alpha_i M_i h_i (\beta_2 - 2\beta_3 A) + V(\beta_2 A - \beta_3 A)^2 u_{a_i} h_i \left( \frac{\sum_{j=1}^{n} \alpha_j a_j}{A} \right) = 1. \]

\[ \text{......(2.112)} \]
Similarly from (2.90) we have

$$\frac{\partial P_i}{\partial p_i} = V(M_i - \frac{\omega_i p_i}{p_i^2})$$

and hence the necessary condition shows that

$$\frac{V'M_i}{V'} = \frac{\omega_i p_i}{p_i^2}$$

...(2.113)

where \( V' = V(\beta_2 A - \beta_3 A^2) \).

From (2.111), we have,

$$\frac{\partial p_i}{\partial a_i} = -2V\beta_2 \alpha_1 M_1 h_i + V\alpha_1(\beta_2 - 2\beta_3 A)h_i \left( u\alpha_i \frac{a_j}{A} \right)$$

$$+ V(\beta_2 A - \beta_3 A^2)h_i \left( u\alpha_i \frac{a_j}{A} \right)$$

$$+ \frac{n}{A} \sum_{j=1}^{n} \alpha_j a_j$$

$$= -2V\alpha_1 h_i [\beta_2 M_1 + \beta_3 \frac{A}{A}]$$

$$< 0,$$

since \( V, c_i, a_i, h_i, M_i \) and \( \beta_2, \beta_3 \) are all positive, for \( i = 1, 2, \ldots, n. \)
From (2.111), we have,

$$\frac{2}{\delta p_i \delta a_i} = Vp_i (\mu_2 - 2\beta_3 A) M_i + \omega_1 h_i (\beta_2 - \beta_3 A)(-\frac{\omega_i p_i}{p_i})$$

$$+ VA(\beta_2 - \beta_3 A) \omega_1 (\frac{j=1}{\sum a_j a_j})$$

Also from (2.93),

$$\frac{\delta^2 p_i}{\delta p_i} = -\frac{2\omega_i p_i c_i V'}{3}$$

$$< 0,$$

since \(\omega_i, p_i, c_i, V'\) are all positive for \(i = 1, 2, \ldots, n\).

The sufficient condition (2.6) can be stated as

$$\left(\frac{\delta^2 p_i}{\delta a_i} \right) \left(\frac{\delta^2 p_i}{\delta^2} \right) > \left(\frac{\delta^2 p_i}{\delta p_i \delta a_i} \right).$$

Considering results (2.114), (2.115) and (2.116) in the above condition and on further simplification this inequality yields,
Hence, using (2.4) and (2.6), it can be said that optimum $p_i$ and $a_i$ for brand-$i$, $i = 1, 2, \ldots, n$ should satisfy the relations stated in (2.111), (2.113) and (2.117).

Simultaneous Optimization:

From (2.112), we have:

$$V\alpha_i h_i M_i (\beta_2 - 2\beta_3 A) + V(\beta_2 A - \beta_3 A)^2$$

$$\sum_{j=1}^{n} \alpha_j a_j$$

$$u(\frac{j \neq i}{A^2}) = 1.$$

This can be rewritten as:

$$(\beta_2 - 2\beta_3) M_i + (\beta_2 - \beta_3 A) u(\frac{j \neq i}{A}) = \frac{1}{V\alpha_i h_i}.$$
Considering summation 1 to n on both the sides of the above equation, we get,

\[(\beta_2 - 2\beta_3 A) \sum_{j=1}^{n} M_j + (\beta_2 - \beta_3 A)u \frac{(n - 1)A}{A} = \frac{1}{V} \sum_{j=1}^{n} \frac{1}{a_j h_j} \cdot \]

Since \( \sum_{j=1}^{n} M_j = 1 \), the above expression can be written as,

\[(\beta_2 - 2\beta_3 A) + (\beta_2 - \beta_3 A)u(n - 1) = \frac{1}{V} G \]

where \( G = \sum_{j=1}^{n} \frac{1}{a_j h_j} \) and hence we have,

\[\mu_2(1 + u(n - 1)) \cdot \frac{G}{V} = \phi_3(2 + u(n - 1)) \]

i.e., \( A = \frac{1}{\beta_2(2 + u(n - 1))} \left[ \mu_2(1 + u(n - 1)) \cdot \frac{G}{V} \right] \).

\[\ldots(2.113)\]

To obtain the expression for \( a_i a_j \):

From (2.112), we have,

\[(\beta_2 - 2\beta_3 A)M_i + u(\beta_2 A - B_3 A^2)(\sum_{j=1}^{n} \frac{a_j a_j}{A^2} = \frac{1}{V} G_i \]

where \( G_i = \frac{1}{a_i h_i} \).
\[ u(\beta_2 - \beta_3A) \left( \sum_{j=1}^{n} \alpha_j a_j \right) = \frac{1}{V} G_1 - (\beta_2 - 2\beta_3A) M_1 \]

i.e. \[ n \sum_{j=1}^{n} \alpha_j a_j = \frac{A}{u(\beta_2 - \beta_3A)} \left[ \frac{1}{V} G_1 - (\beta_2 - 2\beta_3A) M_1 \right] \]

Adding \( \alpha_i a_i \) on both the sides of the above expression, we get,

\[ \alpha_i a_i + \sum_{j=1}^{n} \alpha_j a_j = \frac{A}{u(\beta_2 - \beta_3A)} \left[ \frac{1}{V} G_1 - (\beta_2 - 2\beta_3A) M_1 \right] + \alpha_i a_i \]

i.e. \[ \sum_{j=1}^{n} \alpha_j a_j = \frac{A}{u(\beta_2 - \beta_3A)} \left[ \frac{1}{V} G_1 - (\beta_2 - 2\beta_3A) \right] \]

\[ = \left( \frac{\alpha_i a_i}{A} + M'_1 \right) + \alpha_i a_i \]

where \( M'_1 = V \frac{\beta_1 a_i}{n} \sum_{j=1}^{n} \beta_j a_j + \omega_i \frac{(\bar{p}_i - p_i)}{p_i} \).

The above expression can be rewritten as
\[
A = \frac{A}{u(\beta_2 - \beta_3 A)} \left[ \frac{1}{V} G_1 - (\beta_2 - 2\beta_3 A)M_1' \right]
\]

\[
= \frac{(\beta_2 - 2\beta_3 A)}{\beta_2 - \beta_3 A} \alpha_1 a_1 + \alpha_1 a_1
\]

i.e. \[
\frac{\beta_3 A}{\beta_2 - \beta_3 A} \alpha_1 a_1 = A_1 \left[1 - \frac{1}{u(\beta_2 - \beta_3 A)} \left( \frac{1}{V} G_1 \right) - (\beta_2 - 2\beta_3 A)M_1' \right]
\]

Dividing both the sides of the above equation by \(A\), we get,

\[
\alpha_1 a_1 = \frac{\beta_2 - \beta_3 A}{\beta_3} \left[1 - \frac{1}{u(\beta_2 - \beta_3 A)} \left( \frac{1}{V} G_1 \right) - (\beta_2 - 2\beta_3 A)M_1' \right]
\]

...(2.119)

Considering result (2.118), we get,

\[
\beta_2 - \beta_3 A = \frac{1}{(k + 1)} \left[ \beta_2 + \frac{G}{V} \right]
\]

where \(k = 1 + u(n - 1)\)

and \(\beta_2 - 2\beta_3 A = \frac{1}{(k + 1)} \left[ - \beta_2 (k - 1) + \frac{2G}{V} \right]\)
Using the above results in (2.119) and on further
simplification we get,

\[ \alpha_i a_i = \frac{1}{\beta_3 u(k + 1)} \left[ \left( u \beta_2 + \frac{G_i}{V} \right) - (k + 1) \frac{G_i}{V} \right] \\
- \beta_2 (k - 1) M_i \left( 1 + \frac{G_i}{V} \right) + 2 \left( \frac{G_i}{V} M_i \right) \]

\[ = \frac{1}{\beta_3 u(k + 1)} \left[ \left( u + 2 M_i \right) \frac{G_i}{V} + u \beta_2 \right] - \beta_2 (n - 1) M_i \left( 1 + \frac{G_i}{V} \right) \\
- (k + 1) \frac{G_i}{V} \]

\[ = \frac{1}{\beta_3 u(k + 1)} \left[ \left( u + 2 M_i \right) \frac{G_i}{V} + u \beta_2 \right] - \beta_2 (2 - n M_i) \left( 1 + \frac{G_i}{V} \right) \\
- (u(n - 1) + 2) \frac{G_i}{V} \] ...(2.120)

As market potential is not a function of price, in
this case

\[ \frac{dP_i}{dP_i} = 0, \] will give us the same results
as discussed earlier and hence we consider the results
(2.70) and (2.71), in (2.118) and (2.120), and obtain
optimum budget for advertising efforts as follows:

From (2.118), optimum value of \( A^0 \) is given by

\[ A^0 = \frac{1}{\beta_3 (k + 1)} \left[ \beta_2 k - \frac{G_i}{V} \right] \]
where \( G^0 = \sum_{j=1}^{n} \frac{1}{\alpha_j h_j} \).

But from (2.70), \( h_j^0 = \frac{r_j}{s_j} \) and hence,

\[
A^0 = \frac{1}{\beta_3(k + 1)} \left[ \beta_2 k - \frac{Q_n}{V} \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} \right].
\]

From the above expression, we have,

\[
\beta_2 - \beta_3 A^0 = \beta_2 - \frac{1}{(k + 1)} \left[ \beta_2 k + \frac{Q_n}{V} \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} \right]
\]

\[
= \frac{1}{(k + 1)} \beta_2 + \frac{Q_n}{V} \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} > 0, \text{ since } \beta_2, \alpha_i, c_i, V, Q_n, u \text{ are all positive for } i = 1, 2, \ldots, n.
\]

Hence under optimization the total amount of effective advertising expenditure of all brands is less than the ratio \( \frac{\beta_2}{\beta_3} \). From (2.119), we have optimum value of \( a_1^0 \) as

\[
a_{1i}^0 = \frac{1}{\beta_3 u(k + 1)} \left[ (u + 2m_i^0) \frac{G_i^0}{V} + u \beta_2 (2 - nM_i^0) \right]
\]

\[- (u(n - 1) + 2) \frac{G_i^0}{V} \] \quad \ldots (2.122)
where \( G_i = \frac{Q_n}{\alpha_i c_i} \), \( i = 1, 2, \ldots, n \) and \( G = \sum_{j=1}^{n} G_i \).

\( M_i^0 \) is the same as discussed earlier since market potential is considered as independent of sales effort as well as price.

Hence from (2.74), we have

\[
M_i^0 = v \left[ 1 - \frac{(n-1)}{n} \right] + \omega_1 \frac{(c_i - c)}{c_i}.
\]

2.5.2.3 Advertising Budget determination Procedure:

From (2.121), it could be observed that \( \alpha_i a_i^0 \) is positive only if

\[
(u + 2M_i^0) \frac{Q_n}{V} \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} + u \beta_2(2 - nM_i^0) > 0,
\]

since \( \beta_2 \), \( u \) are positive.

Further simplification of the above inequality will yield,

\[
(u + 2M_i^0)Q_n \sum_{j=1}^{n} \frac{1}{\alpha_j c_j} + u \beta_2 V(2 - nM_i^0) > \frac{[u(n-2) + 2(1-M_i^0)]Q_n}{\alpha_i c_i}.
\]
i.e. \( \alpha_i c_i > \frac{[u(n - 2) + 2(1 - \alpha_i^0)]}{(u + 2\alpha_i^0) \sum_{j=1}^{n-1} \frac{1}{\alpha_j c_j} + \alpha_i c_i v(2 - n M_i^0)} \)

and hence the algorithm can be stated as:

1. Arrange the competitors so that
   \( \alpha_1 c_1 > \alpha_2 c_2 > \ldots > \alpha_n c_n \).

2. Find an equilibrium point \((a_1^0, a_2^0, \ldots, a_t^0)\), where \( t \) represents the number of competitors who spend on advertising effort and \( a_i^0 \)'s satisfy the equation (2.122), where initially we consider \( t = 2 \).

3. If \( t < n \), we test whether
   \[
   \alpha_{t+1} c_{t+1} > \frac{[u(t - 1) + 2(1 - M_{t+1}^0)]}{(u + 2M_{t+1}^0) \sum_{j=1}^{t} \frac{1}{\alpha_j c_j} + \alpha_i c_i v(2 - (t+1) M_{t+1}^0)}
   \]

where \( M_{t+1}^0 = v(1 - \frac{t}{t+1}) + \omega_{t+1} \frac{(c_{t+1} - c_t)}{c_t} \).

If market share only depends on the advertising effort i.e. \( M_{t+1}^0 = 0 \), the above inequality can be given by

\[
\frac{(c_{t+1} - c_t)}{c_t}.
\]
\( \alpha_{t+1}^{c} t+1 > \frac{t + 1}{\sum_{j=1}^{t} \frac{1}{a_{j}^{c_j}} - 2^p_2 V} \).

4. If inequality stated in step-3 is satisfied; go to step-2 and replace \( t \) by \( t+1 \).

5. If it is not satisfied, \( (a_1^0, a_2^0, \ldots, a_t^0; 0 \ldots 0) \) is the equilibrium solution.

Theorem - 3: Using the above mentioned algorithm, we can show that there exists a unique competitive equilibrium point for a game defined by (2.122), for which the arguments are identical as discussed in the previous cases.

2.5.2.4 Optimization of Profit:

From (2.110), we have,

\[
P_i^o = h_i V \left[ u \frac{a_1^o a_i^o}{a_j^o} + v \frac{\beta_i^o s_i^o}{\beta_j^o s_j^o} + \omega_i \frac{(p_i^o - p_1^o)}{p_i^o} \right] \\
= \delta_i^o + \delta_2^o + \delta_3^o - F_i
\]
where \( \delta_1^o = h_i^o V \beta_2^o - \beta_3^o ) ( u \frac{\alpha_i a_i}{\bar{A}^o} ) - a_i^o \\
= h_i^o ( VN^o ) v_1 a_1^o - a_i^o \quad \ldots \ldots (2.123) \\
\]

where \( N^o = \beta_2^o - \beta_3^o A \)

\[
\delta_2^o = h_i^o V^o v \frac{\beta_i s_i^o}{\Sigma j=1 \beta_j s_j} - s_i^o .
\]

The expression is the same as that in the previous case and hence from result (2.76),
\[
\delta_2^o = \frac{c_i ( VN^o ) v B_i^o}{Q_n^o} \quad \ldots \ldots (2.124)
\]

where \( B_i^o = 1 - \frac{(n - 1)}{\Sigma j=1 \beta_j s_j} \).

Similarly
\[
\delta_3^o = h_i^o V_i^o w_i \frac{( p_i^o - p_i^o )}{p_i} \ldots
\]

Using the result (2.78), we get
\[
\delta_3^o = \frac{c_i}{Q_n^o} ( VN_i^o ) w_i \frac{( \bar{c_i} - c_i^o )}{c_i} \quad \ldots \ldots (2.125)
\]

Considering (2.123), (2.124) and (2.125) in (2.110), we get,
\[ p_0^o + F_1 = \frac{c_i V}{Q_n} N \left[ u(\alpha_i \delta_i) + v \beta_i + \omega_i \left( \frac{c_i - c_i^o}{c_i^o} \right) \varepsilon_i \right]. \]  
\[ \ldots (2.126) \]

Using results (2.121) and (2.122) in the above expression, the optimum net contribution in this case can be obtained.

2.5.3 **Case-3 : Saturation Market Response:**

2.5.3.1 **Problem Formulation:**

In this case it is assumed that market is not fixed, but as advertising efforts increase, market of the product increases up to a certain level and then it becomes constant or reaches saturation level.

Here the total market \((V')\) can be indicated by

\[ V' = V[\gamma - \delta (\sum_{j=1}^{n} \alpha_j \delta_j)^{-1}] \]  
\[ \ldots (2.127) \]

where \(V\) is the fixed sales volume, \(\gamma\) and \(\delta\) are the positive constants such that the total amount of effective advertising expenditure of all brands is greater than the ratio \(\frac{\delta}{\gamma}\).

i.e. \[ \sum_{j=1}^{n} \alpha_j \delta_j > \frac{\delta}{\gamma} \]

Using (2.127) in (2.87), we get,
\[ P_i = V(\gamma - \delta (\sum \alpha_j a_j))^{-1} \left[ u \frac{\alpha_j a_j}{\sum_{j=1}^{n} \alpha_j a_j} \right. \]
\[ + v \frac{\beta_i s_i}{n \sum_{j=1}^{n} \beta_j s_j} + \omega_i \frac{(p_i - p_{\bar{i}})}{p_{\bar{i}}} h_{i-a_{i} s_{i}-\bar{r}_{i}}. \]

...(2.128)

2.5.3.2 Problem Solution:

Necessary and sufficient conditions:

Here from (2.128), it can be observed that necessary conditions are given by

\[ \frac{\partial P_i}{\partial a_i} = V h_{i} a_{i} \delta A \bar{M}_1 + V(\gamma - \delta A)^{-1} h_{i} u \alpha_i (\sum_{j=1}^{n} \alpha_j a_j)^2 \]

\[ = 0 \]

...(2.129)

where \( A = \sum_{j=1}^{n} \alpha_j a_j \)

and

\[ \frac{\partial P_i}{\partial P_i} = V(\gamma - \delta A)^{-1} \bar{M}_1 - \frac{\omega_i P_i}{2 p_{\bar{i}}} = 0 \]

...(2.130)

From (2.129), we have

\[ \frac{\delta P_i}{\delta a_i} = -2V h_{i} a_{i} A^{-3} \left[ \delta v_i - \delta u \frac{(\sum_{j=1}^{n} \alpha_j a_j)}{A} \right] + u(\gamma - \delta A)^{-1} \]

\[ = 0 \]

...(2.131)
\[
\frac{\partial^2 p_i}{\partial a_i \partial p_i} = V_\alpha \partial A M_i + V h_\alpha a_i \partial A (- \frac{\partial a_i}{p_i}) \\
+ V(\gamma - \partial A) \sum_{j=1}^{n} -1 \alpha_j a_j + \frac{1}{A^2} \\
= \frac{\partial p_i}{\partial a_i \partial p_i} 
\]

...(2.132)

and from (2.130),
\[
\frac{\partial^2 p_i}{\partial p_i^2} = -2 \frac{\partial_a p_i c_i V'}{p_i^3}. 
\]

...(2.133)

It can be observed from (2.131), that sufficient condition:

\[
\frac{\partial}{\partial a_i} p_i < 0, \text{ is satisfied only if } \\
\frac{\partial}{\partial a_i} M_i + u \sum_{j=1}^{n} -1 \alpha_j a_j [\gamma - 2\partial A] > 0. \\
\]

From (2.133) it can be seen that condition \(\frac{\partial^2 p_i}{\partial p_i^2} < 0\)

is already satisfied since \(\omega_i, p_i, V', c_i\) are all positive.

And, the inequality stated in condition (2.6) is satisfied if
From (2.4), (2.5) and (2.6) it can be observed that optimum \( p_i \) and \( a_i \) of Brand-i, for \( i = 1, 2, \ldots, n \) should satisfy the relations stated in (2.129), (2.130), (2.131) and (2.134).

Simultaneous Optimization:

From (2.129), we have

\[
Vh_i a_i \delta A^{-2} M_i + V(\gamma - \delta A^{-1}) h_i u \alpha_i \left( \sum_{j=1}^{n} \alpha_j^a_j \right) = 1
\]

i.e. \( \delta A^{-2} M_i + (\gamma - \delta A^{-1}) u \left( \sum_{j=1}^{n} \alpha_j^a_j \right) = \frac{1}{V \alpha_j h_j} \).

As discussed earlier, there is one such equation for each brand and summing these \( j \) equations, we get,
\[ \delta A \sum_{j=1}^{n} M_j + (\gamma - \delta A) \frac{1}{A^2} u(n-1) A = \sum_{j=1}^{n} \frac{1}{V \alpha_j h_j} \]

i.e. \[ \delta A^{\frac{-1}{2}} + (\gamma - \delta A)^{\frac{-1}{2}} u(n-1) A = G \]

where \[ G = \sum_{j=1}^{n} \frac{1}{h_j \alpha_j} \]

i.e. \[ \delta + (\gamma A - \delta) u(n-1) = \frac{G}{V} A \]

On further simplification we get,

\[ \delta + \gamma A u(n-1) - \delta u(n-1) = \frac{G}{V} A \]

i.e. \[ \frac{2}{V} A - A(\gamma u(n-1) + 0.5[u(n-1) - 1]) = 0 \]

\[ \ldots \ldots (2.135) \]

The above equation (2.135) can be written in the form \[ a A^2 + b A + c = 0 \], the solution of which is already available. It is well known that roots of \( A \) are

\[ A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Since \( A > 0 \), we have,

\[ A = \frac{V u(n-1) + \Delta}{2 \frac{G}{V}} \]

\[ \ldots \ldots (2.136) \]
where \[ \Delta = \sqrt{(Y u(n-1))^2 + 4 \frac{G}{V} (1-u(n-1))} \].

To obtain the expression for \( \alpha_i a_i \):

From (2.129), we have

\[ \delta A M_i + (Y - \delta A^{-1}) u( \sum_{j \neq i}^{n} \alpha_j a_j ) A^{-2} = \frac{G_i}{V} \]

where \( G_i = \frac{1}{\alpha_i h_i} \)

\[ \begin{align*}
\text{i.e. } & \delta M_i + (Y - \delta A^{-1}) u( \sum_{j \neq i}^{n} \alpha_j a_j ) = \frac{G_i}{V} A^2 \\
\text{i.e. } & \sum_{j \neq i}^{n} \alpha_j a_j = \frac{1}{u(Y - \delta A^{-1})} \left( \frac{G_i}{V} A^2 - \delta M_i \right). 
\end{align*} \]

Adding \( \alpha_i a_i \) on both the sides we get

\[ A = \frac{1}{u(Y - \delta A^{-1})} \left[ \frac{G_i}{V} A - \delta \left( u \frac{\alpha_i a_i}{A} + M_i' \right) \right] + \alpha_i a_i \]

where \( N_i' \) is the same as defined earlier.

\[ \text{i.e. } A = \frac{1}{u(Y - \delta A^{-1})} \left[ \frac{G_i}{V} A^2 - \delta M_i' \right] + \alpha_i a_i \left[ \frac{Y A - 2 \delta}{Y_A - \delta} \right]. \]

On further simplification it can be shown that
As market potential is not a function of price, in this case also
\[
\frac{\partial P_i}{\partial P_i} = 0 \quad \text{will give us the same results as discussed earlier.}
\]
That is from (2.70) and (2.71), we get,
\[
h_j^o = \frac{c_j}{Q_n} \quad \text{and} \quad p_{j}^o = (1 + \frac{1}{Q_n}) c_j.
\]
We consider these values in results (2.136) and (2.137) and obtain optimum budget for advertising efforts.

From (2.135), optimum value of
\[
A^o
\]
is given by
\[
A^o = \frac{\gamma u(n-1) + \sqrt{[\gamma u(n-1)]^2 + 4 \frac{Q_n}{V} n \sum_{j=1}^{n} \frac{1}{\alpha_j c_j}}}{2 \frac{Q_n}{V} n \sum_{j=1}^{n} \frac{1}{\alpha_j c_j}} \quad \text{(2.138)}
\]
which should be greater than \(\delta/\gamma\) (according to (2.127)).
Similarly,

$$\alpha_i^o = \frac{A_0}{u(\gamma A_0 - 2\delta)} \left[ u(\gamma A_0^c - \delta) - \frac{Q_n}{\nu} \left( \frac{1}{\alpha_i^0} A_0^c \right) + \hat{c}_i^0 \right]$$

.....(2.139)

where

$$M_i^0 = v[1 - \frac{(n-1)}{n} \beta_i c_i \sum_{j=1}^{1} \frac{1}{\beta_j c_j} ] + \omega_i \frac{c_i^* - c_i}{c_i}.$$ 

The advertising budget determination procedure in this case can be discussed as previous cases.

Also, using results (2.70), (2.71), (2.138) and (2.139) in (2.128) maximum profit can be obtained similarly.

2.5.4 Remarks:

It can be observed from the analysis of the optimum solution for this model that all the conclusions derived from Model-1 do not hold for Model-2 which reveals the following results:

(i) For all the cases, the ratio of profit maximising advertising outlays for two brands is not equal to the ratio of their unit cost of production.
(ii) The relative effectiveness of a brand's advertising effort has significance in determining the brand's optimum advertising outlay.

(iii) The total advertising expenditure for the given brand depends upon the brand's sales volume, the brand's production cost, its advertising efficiency, the number of competitors in the market and market share of a product which depends upon remaining factors i.e. price and selling effort.

2.6 Conclusion:

Non-constant sum game theory approach has been used here and models have been developed for the analysis of three marketing mix elements of price, advertising efforts and selling efforts within the game theory context. The advertising term here is similar to the form indicated by Ackoff. Selling term is also assumed to be of similar form. The price effects in the model are based on the assumption that market share is linearly related to the relative proportion of difference in prices. These relationships are used to describe the market share effects of marketing mixes by each brand.
Since the solution of these game theory models are based on the use of multivariate calculus, the form of the reward function must be differentiable and must lead to solvable set of partial differential equations. We have examined here the existence of possible equilibria and derive the necessary conditions for an equilibrium.

The above discussed models can however be extended to the case where there are more than three affecting variables and market potential can be considered as a function of all these affecting variables. Also, it does not seem reasonable here to assume that market time period could be independent of the past, since it is equivalent to assuming no brand loyalty. If time factor is included here, more reliable results could be expected.

Statistical decision theory offers an alternative method of formulating the advertising budget decision problem. The probabilistic nature of both competitive situation and market response can be encompassed within this framework. The concept of consumer behaviour also can be useful in evolving advertising budget determination. Research can be
directed at linking the parameters of the consumer preference with the controllable marketing variables. At the same time, linking the effects of advertising budget with buyer behaviour would be helpful in obtaining the optimum determination of the advertising expenditure.