APPENDIX I

NUMERICAL SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION FOR THE CURRENT FUNCTION.

The equation (Eqn. 15, Chapter VI) is technically termed as a second-order elliptic partial differential equation in \( \Psi \)

\[
f_1 \frac{\partial^2 \Psi}{\partial r^2} + 2 f_2 \frac{\partial^2 \Psi}{\partial r \partial \theta} + f_3 \frac{\partial^2 \Psi}{\partial \theta^2} + f_4 \frac{\partial \Psi}{\partial r} + f_5 \frac{\partial \Psi}{\partial \theta} + f_6 = 0
\]  

(1)

where \( f_1 \) to \( f_6 \) are coefficients depending upon parameters which in turn depend upon \( r \) and \( \theta \). The discriminant \( (f_2^2 - f_1 f_3) \) is negative and hence the equation is of elliptic type.

**Boundaries and boundary conditions:**

The current function is to be computed in the region \( 80^\circ \leq \theta \leq 100^\circ \) and \( 80 \text{ km} \leq (r - a) \leq 200 \). It is assumed that \( \Psi = 0 \) at these boundaries.

**Difference equations:**

The difference equations for various differential coefficients are written as,

\[
\left[ \frac{\partial^2 \Psi}{\partial r^2} \right]_{i,j} = \left( \Psi_{i+1,j} - 2 \Psi_{i,j} + \Psi_{i-1,j} \right) / h_r^2 
\]  

(1)

\[
\left[ \frac{\partial^2 \Psi}{\partial \theta^2} \right]_{i,j} = \left( \Psi_{i,j+1} - 2 \Psi_{i,j} + \Psi_{i,j-1} \right) / h_\theta^2 
\]  

(2)

\[
\left[ \frac{\partial^4 \Psi}{\partial r^2 \partial \theta^2} \right]_{i,j} = \left( \Psi_{i+1,j+1} - \Psi_{i+1,j} - \Psi_{i,j+1} + \Psi_{i-1,j+1} \right) / 4 h_r h_\theta 
\]  

(4)

\[
\left[ \frac{\partial \Psi}{\partial r} \right]_{i,j} = \left( \Psi_{i+1,j} - \Psi_{i,j} \right) / h_r \lor \left( \Psi_{i,j} - \Psi_{i,j-1} \right) / h_r 
\]  

(5)

\[
\left[ \frac{\partial \Psi}{\partial \theta} \right]_{i,j} = \left( \Psi_{i,j+1} - \Psi_{i,j} \right) / h_\theta \lor \left( \Psi_{i,j} - \Psi_{i,j-1} \right) / h_\theta 
\]  

(6)
where $h_r$ and $h_\theta$ are the step sizes in $r$ and $\theta$.

$i = 1, 2, \ldots, n + 1$ and $j = 1, 2, \ldots, m + 1$; $n$ and $m$

being the number of intervals in the ranges for $r$ and $\theta$,

i.e., $n = \frac{(r_{n+1} - r_1)}{h_r}$ and $m = \frac{(\theta_{m+1} - \theta_1)}{h_\theta}$. At the

outset, the values of $\Psi_{i,j}$ are initialized by,

$$\Psi_{i,j} = 0,$$ for $i = 2$ to $n$ and $j = 2$ to $m$.

We then write,

$$F_{i,j} = f_{1,i,j} \left( \frac{2}{h_r} \frac{\partial^2 \Psi}{\partial r^2} \right)_{i,j} + 2 f_{2,i,j} \left( \frac{1}{h_r \frac{\partial}{\partial r}} \right)_{i,j} + f_{3,i,j} \left( \frac{1}{h_\theta \frac{\partial}{\partial \theta}} \right)_{i,j}$$

$$+ f_{4,i,j} \left( \frac{\partial \Psi}{\partial r} \right)_{i,j} + f_{5,i,j} \left( \frac{\partial \Psi}{\partial \theta} \right)_{i,j} + f_{6,i,j}$$

(7)

The derivatives are replaced by the corresponding difference equations given above.

**Method of successive over-relaxation:**

In general, if the coefficient of $\Psi_{i,j}$ which is
determined by the diagonal elements of the coefficient
matrix, is predominant over the other coefficients, then
the convergence of the equations to a reasonable solution
is fast. To ensure the dominance of the diagonal elements
we make the right choice of the difference equations for
$(\partial \Psi / \partial r)$ and $(\partial \Psi / \partial \theta)$, so that the coefficient of
$\Psi_{i,j}$ is maximum. For example, if the coefficient of $\partial \Psi / \partial r$
($f_4$) is positive, then we choose the first choice for the
difference equation of $\partial \Psi / \partial r$, $(\Psi_{i+1,j} - \Psi_{i,j}) / h_r$
because the coefficients $\Psi_{i,j}$ are negative in the difference
equations of higher order differential coefficients.
Now we write,
\[ F_{i,j} - Q_{i,j} \psi_{i,j} = 0 \]  \hspace{1cm} (8)
where \( F_{i,j} \) is the equation \( F_{i,j} \) minus the terms containing \( \psi_{i,j} \); and \( Q_{i,j} \) is the coefficient of \( \psi_{i,j} \).
\[ \therefore \psi_{i,j} = \frac{F_{i,j}}{Q_{i,j}}, \text{ or} \]
\[ \psi_{i,j} = \frac{C}{Q_{i,j}} \frac{F_{i,j}}{Q_{i,j}} + (1 - C) \psi_{i,j} \]  \hspace{1cm} (9)
where \( C \) is an arbitrary constant. For the method of successive over-relaxation used to solve the elliptic type of equations, \( C \) is generally taken to be greater than 1.

We first substitute the values of \( F_{i,j}, Q_{i,j} \) and the initial values of \( \psi_{i,j} \) and calculate new \( \psi_{i,j} \). These new \( \psi_{i,j} \) are compared with the initial \( \psi_{i,j} \) and it is checked whether the relative error \( \frac{\psi_{i,j} \text{ (new)} - \psi_{i,j} \text{ (inj)}}{\psi_{i,j} \text{ (new)}} < \) computed/each time is less than a certain number which determines the accuracy of the numerical results. Till the \( \psi_{i,j} \) 'converge' to this accuracy, the 'iteration' is continued.

We achieved fast convergence by choosing \( C = 1.7 \) and for accuracy number \( 0.5 \times 10^{-3} \); this number assures that the results are accurate to the first three significant digits.
Furthermore, the physical validity of the numerical results can be verified (i) by changing the boundary conditions, (ii) by changing the boundaries, (iii) by changing the step sizes in \( r \) and \( \theta \) and (iv) by changing the accuracy. It is found that the choice of the boundaries \( r = 80 \) and \( 200 \text{ km} \) and \( \theta = 80^\circ \) to \( 100^\circ \) with \( \psi = 0 \) at all the boundaries and with \( h_r = 2 \text{ km} \) and \( h_\theta = 0.5^\circ \) gives fairly accurate and physically valid results.