CHAPTER IV

SECOND ORDER RESONANCE AND FREQUENCY TIME STRUCTURE
OF VLF EMISSIONS

4.1.1 INTRODUCTION

It is well known that a gyrating electron, in presence of a uniform magnetic field interacts very strongly with a whistler mode wave when the frequency seen by the particle is equal to its own gyrofrequency (See Section 3.1.1 of Chapter III). The effect of nonuniform magnetic field on the particle motion is of great importance and this has been discussed for whistler mode wave by Kulkarni and Das (1971), Das (1972), Nunn (1971b), and Laird (1972).

In presence of a nonuniform magnetic field, the particle suffers the second order resonance when the change in the doppler shifted frequency is equal to that of its own gyrofrequency. The effect of the second order resonance on the particle motion and the effect of the currents of these particles on the VLF generation mechanism, in case of whistler mode
wave packet, have been discussed by Nunn (1974b, 1974) and Das and Kulkarni (1975). On the basis of detailed trajectory, it is seen that the second order resonant particles become more stably trapped (Nunn 1974b, 1974, Das 1972). The theory presented in this chapter is as follows. First, we shall discuss the particle motion in the presence of non-uniform magnetic field and then the region of second order resonance is estimated qualitatively. The equations, governing the time development of the wave field in presence of the resonant particle current, are derived and on the basis of these the detailed study of these resonant particle currents is given. Using these currents, the self consistent solution of the amplitude growth and frequency time structure in VLF emissions are summarised.

4.1.2 **LINEAR APPROXIMATIONS**

Here we shall discuss briefly the particle motion in presence of a weakly non-uniform magnetic field and study some of the characteristics of
whistler resonance. Using the unperturbed trajectory in presence of nonuniform magnetic field, the change in the parallel velocity of the particle because of the whistler mode wave is given by

\[
\delta V_h = -\frac{R}{\omega} Re \left( \int_{t_1}^{t_2} e^{-i P(t')} dt' \right)
\]

(1)

where

\[
P(t') = \int_{0}^{t} \left[ \omega - kv_h(t') + \Re \{ \mathcal{E}(t') \} \right] dt'
\]

and the other quantities are defined in the section 4.3. Re represents the real part of the integral, and \( V_h \) is taken to be nearly a constant and justification for this is given in the section 3.1.6.

Since, \( P \) is a complicated function of \( z, V_{||}, R \) and time, we carry out the approximate integration of (1) using the method of stationary phase. Assuming that this stationary phase occurs at \( t = t_0 \), we expand \( P \) in terms of Taylor series around \( t = t_0 \), as shown below:
\[ P(t) = P(t_0) + (t-t_0) \frac{dP}{dt} \bigg|_{t=t_0} + \frac{(t-t_0)^2}{2} \frac{d^2P}{dt^2} \bigg|_{t=t_0} + \frac{(t-t_0)^3}{6} \frac{d^3P}{dt^3} \bigg|_{t=t_0} + \cdots \]  

We also assume that the successive derivatives are smaller and therefore the terms higher than the second derivative are neglected. The phase remains nearly stationary around the point where \( \frac{dP}{dt} = 0 \).

The physical significance of the terms \( \frac{dP}{dt} \) and \( \frac{d^2P}{dt^2} \) is as follows. Remembering that the \( P \) represents the angle between \( V_\perp \) and \( E \), the electric vector of whistler mode wave, one can say that \( P \) remains stationary when \( \frac{dP}{dt} = 0 \). Using the definition of \( P \), it is easy to show that \( \frac{dP}{dt} = \omega - kV_\parallel - \Omega d(z) \) which when becomes zero, satisfies the condition for whistler resonance.
Using (2) in (1) the integration is straightforward and the integrated value is given by

$$
\delta \nu = - \frac{Rkv_0}{\omega} \sqrt{\frac{2r}{p}} \cos (p_0 - \nu_0)
$$

where

$$
\dot{p} = \frac{d^2 p}{dt^2}
$$

When the contribution of the terms higher than the second derivative, is negligible, it is seen from (3) that the integral explodes when \( \dot{p} = 0 \).

Using the definition of \( P \) one can show that

$$
\frac{d^2 p}{dt^2} = \frac{d}{dt} \delta \nu - \left( \frac{\omega - R_0}{R} \right)^2 .
$$

This defines the second order resonance which occurs when the rate of change in Doppler shifted frequency of the electron is equal to that of its own gyrofrequency. This phenomenon is studied in detail in the following few sections. If \( \frac{d^2 p}{dt^2} = 0 \) and \( \frac{d^3 p}{dt^3} \neq 0 \), the integration of eq.(1) has to
be carried out by retaining the third order term in eq. (2). The changes in $V_{\parallel}$ in this case have been discussed by Das (1972).

The exact equations of motion for a particle in presence of whistler mode wave and the non-uniform magnetic field are quite complex and their analytical treatments are found to be difficult, therefore the computer study of the particle motion, in this case has been done by Kulkarni and Das (1971). It is seen that $V_{\parallel}$ and $V_{\perp}$ increase very rapidly in the initial period of resonance interaction and later attain steady values. The velocities $V_{\parallel}$ and $V_{\perp}$ then oscillate around a certain mean value and show similar results as that of trapped oscillations. It is inferred that these oscillations are due to the presence of the non-uniform magnetic field.

Similar equations are studied by Gladis (1973) for positive particles in presence of the ion-cyclotron waves and it is shown that the effect of second order resonance leads to particle
diffusion in velocity space. This diffusion may lead to energising the particles and their subsequent precipitation into the auroral zones. Therefore it is interesting to study the effect of second order resonance on the particle motion which has been done in detail in the next section.

4.2 **EQUATION OF MOTION OF THE PARTICLE:**

The motion of the particle in presence of a non-uniform asymmetric ambient magnetic field is given by

\[ m \frac{d\vec{V}}{dt} = -e \left( \vec{E} + \frac{\vec{J} \times \vec{B}}{c} \right) - (\vec{\mu} \cdot \nabla) \vec{B} \]  

(4.1)

where \( |\vec{\mu}| = \frac{1}{2} \frac{m v^2}{B} \) (The magnetic moment of the particle).

For whistlers, whose wave fields are given as in the equation (3.2), the equation (4.1) for
the components $V_\perp$, $V_\parallel$, and $P$ of the particle, can be written as

$$\frac{dV_\perp}{dt} = -\frac{eE}{m\omega} (1 - \frac{kV_\parallel}{\omega}) \cos P + \frac{V_\parallel V_\perp}{2B} \frac{dB}{dz} \quad (4.2)$$

$$\frac{dV_\parallel}{dt} = -\frac{eEk}{m\omega} V_\perp \cos P - \frac{V_\parallel^2}{2B} \frac{dB}{dz} \quad (4.3)$$

$$\frac{dP}{dt} = -k(V_\parallel - V_{\omega \parallel}) + \frac{eE}{mV_\parallel} (1 - \frac{kV_\parallel}{\omega}) \sin P \quad (4.4)$$

where $V_{\omega \parallel} = \frac{\omega - \omega_0}{k}$; $P = \omega t - kZ - \phi$;

$V_\perp^2 = V_x^2 + V_y^2$ and $\phi = -\tan^{-1} \left( \frac{V_y}{V_x} \right)$

These equations (4.2), (4.3), and (4.4) are solved in the right hand co-ordinate system with $z$ axis, which is parallel to the geomagnetic field line. Our aim is to study the particle motion in presence of nonuniform magnetic field and to explain some of the structures in the VLF emissions. Therefore, we have assumed that the nonuniform field is that of a dipole situated at the centre of the earth and its $z$ dependence is
given by

\[ B = B_0 \beta = B_0 (1 + \eta z^2) \quad (4.5a) \]

where \( \eta = \frac{4.5}{(LR_e)^2} \) and \( R_\oplus \) is the radius of the earth, which we have taken to be 6.4 x 10^8 cm.

The whistler and VLF emissions which are propagating along the geomagnetic field line, are assumed to follow the cold plasma dispersion relation which is given below:

\[ \frac{c^2 k^2}{\omega^2} = \frac{\omega_p^2}{\omega(\Omega_e - \omega)} \quad (4.6) \]
4.3 **DIMENSIONAL ANALYSIS**

Since the equations in the dimensionless parameters are found more advantageous in carrying out the computer calculations, we would like to define the dimensionless parameters in the following way.

We shall assume the co-ordinates such that the zero of the co-ordinate system lies on the equatorial plane of the geonagnetic field and therefore, when $z = 0$, $\beta = 1$ (4.5). For any given whistler we can define $\omega$ and $k$ at $z = 0$ such that this whistler satisfies the dispersion relation (4.6). Using these parameters of $\omega$, $k$ and $L_o$ at $z = 0$, we would like to define the dimensionless parameters.

$$ t' = \omega t, \quad \psi' = \frac{k \psi}{\omega} , \quad \psi' = \frac{k \psi}{\omega} , \quad z' = k z $$

$$ R = \frac{eE}{m \omega^2} , \quad k' = \frac{k}{k} , \quad \alpha = \frac{\omega}{\omega}, \quad \omega' = \frac{\omega}{\omega} $$

$$ \gamma' = \frac{\eta}{k^2} = \frac{4.5 z'^2}{(L R^2) k^2} \quad \Omega_o = \frac{eB(z = 0)}{mc} $$
By making use of these dimensionless parameters, we can rewrite the equations (4.2), (4.3) and (4.4). However, we are dealing with these dimensionless parameters in this complete chapter, and therefore, for convenience, the primes are omitted. These equations then become

\[
\frac{dv}{dt} = -R \left( 1 - \frac{k_{V_0}}{\omega} \right) \cos \rho + \frac{v_{\perp} v_{\parallel}}{2 \beta} \frac{\partial \beta}{\partial z} \tag{4.7}
\]

\[
\frac{dv_{\perp}}{dt} = - \frac{R k_{V_0}}{\omega} \cos \rho - \frac{v_{\perp}^2}{2 \beta} \frac{\partial \beta}{\partial z} \tag{4.8}
\]

\[
\frac{d\rho}{dt} = -R \left( v_{\parallel} - v_{\perp} \right) + \frac{R}{v_{\perp}} \left( 1 - \frac{k_{V_0}}{\omega} \right) \sin \rho \tag{4.9}
\]

and the magnetic field is given by

\[
B = B_0 (1 + \eta z^2) \tag{4.5b}
\]
4.4 NUMERICAL ESTIMATES FOR DIMENSIONAL ANALYSIS

In this section we would like to give typical values of dimensional parameters which are used in the forthcoming sections. The particle density distribution along the field line is found by using Smith's model (which is popularly known as gyro-frequency model) which states that the ratio of the density to the geomagnetic field is a constant. Therefore, the plasma frequency $\omega_p$ is related to $\Omega_e$ by

$$\omega_p^2 = C_1 \Omega_e$$

(4.10)

where $\Omega_e$ is electron gyrofrequency and $C_1$ is a constant of proportionality which is approximately equal to $1.02 \times 10^7$ rad/sec. For $L = 3$, using (4.10) and (4.6), the values of $\bar{\omega}$ and $\bar{k}$ can be found. $\Omega_e$ for this value of $L$, is of the order $2.4 \times 10^5$ rad/sec. The numerical estimates of $\bar{\omega}$, $\bar{k}$, $v_{ph}$ and $\gamma$ for different values of $\alpha$ are given on the next page.
For $\alpha = 0.5$

\[ \overline{\omega} = 1.2 \times 10^5 \text{ rad/sec} \]
\[ \bar{k} = 5.2 \times 10^{-5} \text{ cm}^{-1} \]
\[ \nu_{ph} = \frac{\overline{\omega}}{\bar{k}} = 2.3 \times 10^9 \]
\[ \gamma = 5 \times 10^{-10} \]

and for $\alpha = 0.1$

\[ \overline{\omega} = 2.4 \times 10^4 \text{ rad/sec} \]
\[ \bar{k} = 1.7 \times 10^{-5} \text{ cm}^{-1} \]
\[ \nu_{ph} = 1.4 \times 10^9 \text{ cm/sec} \]
\[ \gamma = 4 \times 10^{-9} \]

Using the expression (4.10), the dispersion relation (4.6) can be rewritten in the following form

\[ k^2 = \frac{\omega(1-\alpha)}{\left(1 - \frac{\omega^2}{\beta}\right)} \]  \hspace{1cm} (4.11)
4.5 Domain of Second Order Resonance

The gyrosoront particle experiences the second order resonance, when the rate of change in doppler shifted frequency seen by the particle is equal to that of its gyrofrequency. These particles, then satisfy the following relations.

\[ V_{ii} = V_{\text{res}} = \left( \omega - \frac{\beta}{c} \right) / k \]  \hspace{1cm} (4.12)

and

\[ \frac{dV_{ii}}{dt} = \frac{dV_{\text{res}}}{dt} \] \hspace{1cm} (4.13)

Assuming that \( \omega \) is nearly a constant and making use of (4.5b), (4.8), (4.12) and (4.13), we obtain the second order resonance condition which is given by

\[ \frac{\alpha k \nu}{\omega} \sigma = \frac{2 \eta v_{ii} Z}{\alpha k} - \frac{v_{ii} \omega (1 - \beta \kappa)}{k \beta^2 (1 - \omega^2) / \beta} \gamma Z \]

\[ - \frac{v_{ii}^2 - \eta Z}{\beta^2} \]  \hspace{1cm} (4.14)
From the dispersion relation we know that for the whistler of known frequency the resonant velocity, $V_{\text{res}}$, is a constant at a given point along the magnetic field line. However, the particle which is undergoing the resonance may not necessarily experience the second order resonance because the condition (4.14) depends on the local gradient of magnetic field, particle energy and its pitch angle, amplitude of the whistler mode wave and also on the phase angle, $\beta$, subtended by perpendicular velocity. This phase angle, $\beta$, which is very important in studying the resonant interactions, is also a complicated function of various parameters mentioned above. Therefore, we would like to determine the domain of second order resonance interaction from the equation (4.10). For resonant particles, we replace $V_{\parallel}$ by $V_{\text{res}}$ and then like to study whether the solution for $\beta$ exists for all values of pitch angle $\alpha_0$ and $z$. This is not easily seen from (4.10), therefore some numerical computations are done.
For clarity, we have shown in Fig. 4.1, the plot of $P$ vs. $Z$ satisfying the equation (4.14). For the given amplitude of the wave pulse $R = 0.00002$ and for the pitch angle of the particle ($\alpha_0 = 62^\circ$), the graph has been plotted for different gradients of magnetic field. From this figure, it is seen that, for decreasing gradient of magnetic field, the second order resonance region shrinks towards the equator.

The equation (4.11) also suggests qualitatively that the particles will not undergo second order resonance with the whistler of given amplitude at all points along the field line. Therefore, it is essential to know how the region of second order resonance depends on the amplitude of waves with a given gradient of the ambient magnetic field for which numerical computation of (4.11) has been carried out. Fig. 4.2 shows the second order resonance curves of the particle with known pitch angle, energy and for different values of $R$. This shows that the second order resonance region tends to confine itself around the equator and with decreasing amplitude of
Fig 4.1

\[ R = 0.0002 \]
\[ \beta = 1 + \eta^2 \]

\[ \eta = 10^{-8} \ (\xi = 62) \]
\[ \eta = 10^{-9} \ (\xi = 62) \]
\[ \eta = 10^{-10} \ (\xi = 62) \]
\[ B = B_0 \beta (1 + \eta z^2) \]

\[ \eta = 10^2 \]

\[ \text{Fig 4.2} \]
a wave pulse this region shrinks towards the equator.

We would also like to mention that the second order resonance is important around the equator as seen in the Fig. 4.1 and 4.2 and therefore any theory of VLF emissions based on this phenomena suggests that the equatorial region of the magnetosphere plays an important role which is in support with the observations.

4.6 TIME DEVELOPMENT OF WAVE FIELDS AND THE FREQUENCY CHANGE

The general dispersion relations and the growth of the electromagnetic waves are based on the finite amplitude for the wave where the growth is considered to be infinitesimally small. Here, we shall study the waves of finite amplitudes whose growth terms are substantial to alter their initial amplitudes. Using this non-linear behaviour, we
would like to discuss the phase and amplitude modulation of the wave pulse.

One can derive the general relationship between the wave amplitudes and currents for the electromagnetic waves (\( \nabla \cdot \vec{B} = 0 \)) from Maxwell's equations

\[
\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}
\]

and

\[
\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]

which is given by

\[
\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial \vec{J}}{\partial t}
\]

(4.15)

However, the current density, \( \vec{J} \), that occurs on the right hand side of (4.15) contains the contribution of cold plasma current and the resonant particle current. We can eliminate the cold plasma
current by using the equation of motion (4.1) for cold electrons and the part of current due to the resonant particles is retained as it is. Therefore (4.15) becomes

\[
\left( \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial}{\partial t} - i \Omega_e \right) - \frac{\omega_p^2 \cdot e}{c^2} \hat{E} \right) \vec{E} = \frac{4\pi}{c^2} \left( \frac{\partial}{\partial t} - i \Omega_e \right) \frac{\partial \vec{J}_{\text{res}}}{\partial t}
\]

(4.16)

where \( \vec{J}_{\text{res}} \) is the resonant particle current which will be discussed in the next section. The term \( \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \) is essentially due to the displacement current and in case of whistlers this term is so small that this can be neglected. The equation (4.16) is now considered for whistlers propagating along \( z \) direction for which \( \vec{E} \) and \( \vec{J}_{\text{res}} \) are in the plane perpendicular to the direction of propagation and therefore this can be rewritten as
Here we shall assume that the variation of $E$ and $J_{\text{res}}$ are given by $E = E(z,t) \exp\left( i (\omega_0 t - k_0 z) \right)$ and $J_{\text{res}} = J_{\text{res}}(z,t) \exp\left( i (\omega_0 t - k_0 z) \right)$. The amplitudes of $E$ and $J_{\text{res}}$ are dependent on space and time and in particular we expect that the resonant particle current will increase with $Z$. However, it is seen that in the case of a narrow wave pulse these quantities depend very weakly on $Z$ and $t$ (Nunn 1971b).

We shall retain, for convenience, the terms which vary as $R^2$ and variations of order $R$ and higher are neglected. We can easily see that the terms $\frac{dk}{dt}$ and $\frac{d\omega}{dt}$ are of the order $R$ which are too small to be included. Essentially, $k_0$ and $\omega_0$ remain nearly constant and also, the
changes in the magnitude of $E$ and $J_{\text{res}}$ are such that

$$\frac{\partial}{\partial t} E(z,t) < \omega_0 E(z,t), \quad \frac{\partial J_{\text{res}}(z,t)}{\partial t} < \omega_0 J(z,t)$$

$$\frac{\partial E(z,t)}{\partial z} < k_0 E(z,t), \quad \frac{\partial J_{\text{res}}(z,t)}{\partial z} < k_0 J(z,t)$$

Therefore, we have neglected second and higher order derivatives w.r.t space and time and the slow variation in the current which is of the first order is also neglected. Then the equation (4.17) can be written as

$$k_0^2 \frac{\partial E(z,t)}{\partial t} - 2k_0 \omega_0 \frac{\partial E(z,t)}{\partial z} + i k_0^2 \omega_0 E(z,t)
- i \Omega e \left( 2i k_0 \frac{\partial}{\partial z} E(z,t) + k_0^2 E(z,t) \right)
- \frac{\omega_0^2}{c^2} \left( \frac{\partial}{\partial t} E(z,t) + i \omega_0 E(z,t) \right)
= \frac{4\pi}{c^2} \omega_0 (\omega_0 - \Omega e) J_{\text{res}}$$

(4.18)
As seen in the next section, $J_{\text{res}}$ is complex which we shall represent as $J_{\text{res}} = J_r + iJ_i$, where subscript $r$ and $i$ represent the real and imaginary parts respectively. Because of this, the wave amplitude $E(z,t)$ also becomes complex, which can be represented by

$$E(z,t) = |E(z,t)| e^{i \phi(z,t)} \tag{4.19}$$

where $|E(z,t)|$ is the magnitude of the wave amplitude and $\phi(z,t)$ is the phase part. Using (4.19) in (4.18), and separating the real and imaginary parts, we get

$$\frac{\partial E}{\partial t} + \frac{2kE^2(\omega_c - \omega)}{c^2 k_o^2 + \omega^2} \frac{\partial E}{\partial z} = \frac{4\pi \omega_c^2 (\omega_c - \omega)}{c^2 k_o^2 + \omega^2} J_i \tag{4.20}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{2kE^2(\omega_c - \omega)}{c^2 k_o^2 + \omega^2} \frac{\partial \phi}{\partial z} = \frac{4\pi \omega_c^2 (\omega_c - \omega)}{c^2 k_o^2 + \omega^2} \frac{J_i}{E} \tag{4.21}$$

These are the equations which show the slow evolutions of the amplitude and phase of whistler mode.
wave pulse and its fine structures. We would like to rewrite (4.20) and (4.21) in dimensionless form, in consistent with the particle motions considered in the previous sections, which can then be written as follows (Nunn 1971b, 1974, Das and Kulkarni, 1975)

$$\left( \frac{\partial}{\partial t} + v_c \frac{\partial}{\partial z} \right) R = \frac{4\pi \omega_b (\omega_0 - \beta k)}{\epsilon k c + \omega_p^2} \Gamma_r \tag{4.22}$$

$$\left( \frac{\partial}{\partial t} + v_c \frac{\partial}{\partial z} \right) \Phi = \frac{4\pi \omega_b (\omega_0 - \beta k)}{\epsilon k c + \omega_p^2} \frac{\Gamma_i}{R} \tag{4.23}$$

(The primes are omitted for convenience)

where $\Gamma_r$ and $\Gamma_i$ are real and imaginary parts of current which are given by $\Gamma_r = \frac{4\pi e k}{m \omega_b^2} \frac{\Gamma_r}{\omega_b^2}$ and $\Gamma_i = \frac{4\pi e k}{m \omega_b^2} \frac{\Gamma_i}{\omega_b^2}$ and $v_g$ is the group velocity.

In the next few sections, we have essentially studied the equations (4.22) and (4.23).
4.7 RESONANT PARTICLE CURRENTS

The resonant particle current occurs on the right hand side of the equation (4.22) and (4.23) and this current is known to provide the source of energy for electromagnetic radiation. Therefore, the detailed description of this current is essential and also useful in analysing the properties of VLF emissions.

In the interaction region, the Liouville theorem is valid and therefore we can write the particle distribution as

$$f_{\text{res}}(w, \mu) = f_0(w + \Delta w, \mu + \Delta \mu)$$

(4.24)

where $w$ is the particle energy and $\mu$ is its magnetic moment, which is given by $m \nu_0 / B$.

$\Delta w$ and $\Delta \mu$ are changes in particle energy and the magnetic moment due to resonant interaction.

The right hand side of (4.24) can be expanded in terms of Taylor's series and we shall retain only the first order terms in $\Delta w$ and $\Delta \mu$. Therefore, the change in the resonant particle distribution $f_{\text{res}}$.
is given by
\[ f_1 = F_{\text{res}}(W, M) - F_0(W, M) \]
\[ = \frac{\partial F_0}{\partial W} \Delta W + \frac{\partial F_0}{\partial M} \Delta M \]

where, \( F_0(W, M) \) is a suitable initial distribution of the particles which is time independent.

Using this value of \( f_1 \) and the equations of motion (4.7) and (4.8) for finding \( \Delta W \) and \( \Delta M \), we can write the resonant particle current to be

\[ J_{\text{res}} = J_{x_2} + i J_{x_1} \]
\[ = \text{constant} \int \int |V_x| |V_y|^2 \left[ \frac{\partial F_0}{\partial W} + \left(1 - \frac{\partial x}{\partial W} \right) \frac{\partial F_0}{\partial M} \right] \]
\[ \times \int \int \mathcal{R} \cos \rho_0 \, d\tau \, d|V_x| \, dP \]

Here, we are considering a set of trapped 'Simple' particles put at phase \( P = P_0 \) and we would like to approximate the time integration as the product of
average trapped particle current and the trapping period which we have found to be quite adequate. The approximate trapping period is given by

$$\tau_r = \frac{4\pi}{R^2 \kappa \left( \frac{|v_{ii} \sin \theta_0|}{\omega} \right)}$$

Using this value of trapping period in (4.25) and $V_\perp = |v_{ii}| e^{i\Phi}$, the real and imaginary parts of currents can be obtained as

$$J_r = \text{constant } R^{3/2} \int \left( \cos^2 \theta_0 \left( |\sin \theta_0| \right)^2 \right)^{1/2} \left| v_{ii} \right|^2$$

$$\left[ \frac{\partial F_0}{\partial \theta} + \left( 1 - \frac{k v_{ii}}{\omega} \right) \frac{\partial F_0}{\partial \theta_0} \right] \tau_r \left| v_{ii} \right| \partial \rho$$

and

$$J_1 = \text{constant } R^{3/2} \int \left( \cos^2 \theta_0 \left( |\sin \theta_0| \right)^2 \right)^{1/2}$$

$$\left[ \frac{\partial F_0}{\partial \theta} + \left( 1 - \frac{k v_{ii}}{\omega} \right) \frac{\partial F_0}{\partial \theta_0} \right] \tau_r \left| v_{ii} \right| \partial \rho$$

To obtain the current densities (4.26) and
(4.27), we would like to assume that the distribution of the particles is given by

\[ F_0 \propto W^{-3/2} \log \left( \frac{\sin \alpha_p}{\sin \alpha_0} \right) \]  

(4.28)

(\( \alpha_0 \) is the loss cone angle) and the effective trap size of these particles is \( R_2^{1/2} (|\sin \alpha_0|)^{1/2} \).

From the current expressions (4.26) and (4.27), it is also seen that as \( \rho_p \to \pi_2 \), \( J_y \) becomes very small whereas \( J_i \) shows a maximum.

Similarly as \( \rho_p \to 0 \), for \( Z < 0 \), current changes its sign.

4.8 NUMERICAL PROCEDURE

The analytical solutions of coupled equations (4.22), (4.23), (4.26), (4.27) and using particle distribution (4.28) are rather involved, therefore, these equations are solved numerically by using the computer IBM 360/44. We have assumed a set of
trapped particles with different pitch angles $\alpha_0$. In this, the integration of (4.26) and (4.27) is simplified and the integration procedure used is the trapezoidal rule.

Now the equations (4.22), (4.23), (4.26) and (4.27) are solved simultaneously. The equations (4.22) and (4.23) are solved by using a standard implicit scheme which is unconditionally stable. This method is based on the finite differences and any instability found in the solutions is entirely due to non-linearity in the mathematical equations. The implicit scheme which is used in solving (4.22) and (4.23) is given below:

$$\frac{\partial R}{\partial t} = -V_g \frac{\partial R}{\partial z} + \frac{4\pi \omega_0 (\omega_0 - \beta_0)}{c^2 k_0^2 + \omega_p^2} I_y$$

$$\frac{\partial \phi}{\partial t} = -V_g \frac{\partial \phi}{\partial z} + \frac{4\pi \omega_0 (\omega_0 - \beta_0)}{c^2 k_0^2 + \omega_p^2} \frac{I_i}{R}$$

Using the finite differences, these equations are written as
\[ R_{I, J+1} = R_{I, J} + \frac{\Delta t}{2} \left[ \frac{\xi - V_q \left( \frac{R_{I+1, J} - R_{I-1, J}}{2 \Delta Z} \right)}{2 \Delta Z} + \frac{4 \pi \omega_0 \left( \omega_0 - \beta \phi \right)}{C^2 R_0^2 + \omega_p^2} \frac{I_{I, J}}{R_{I, J}} \right] \]

\[ - \frac{R_{I-1, J-1}}{2 \Delta Z} \] 

\[ + \frac{4 \pi \omega_0 \left( \omega_0 - \beta \phi \right)}{C^2 R_0^2 + \omega_p^2} \frac{I_{I+1, J+1}}{R_{I+1, J+1}} \] 

and

\[ \phi_{I, J+1} = \phi_{I, J} + \frac{\Delta t}{2} \left[ \frac{\xi - V_q \left( \frac{\phi_{I+1, J} - \phi_{I-1, J}}{2 \Delta Z} \right)}{2 \Delta Z} + \frac{4 \pi \omega_0 \left( \omega_0 - \beta \phi \right)}{C^2 R_0^2 + \omega_p^2} \frac{I_{I, J}}{R_{I, J}} \right] \]

\[ + \frac{4 \pi \omega_0 \left( \omega_0 - \beta \phi \right)}{C^2 R_0^2 + \omega_p^2} \frac{I_{I, J+1}}{R_{I, J+1}} \] 

(4.29)

(4.30)

where \( I \) and \( J \) refer to grid points in \( Z \) and time respectively. \( \Delta Z \) is the step size in \( Z \) and \( \Delta t \) is the step size in time. Here, the \( Z \) derivatives are replaced by central finite differences. Since \( J + 1 \)th value is appearing on both sides of equality
sign of equations (4.29) and (4.30), these equations are to be solved by the method of iteration, till the required accuracy in $R_{i, j+1}$ is reached.

These equations (4.22), (4.23) and (4.26), (4.27) are solved simultaneously using the procedures for various equations which are discussed above. Whenever, the implicit equations are solved iteratively for the unknowns, the equations involving the unknowns are also solved simultaneously.

To start with, constant value of $R$ is taken at each point $Z$, along a given field line. Taking a set of trapped particles at different $P$ values such that these particles satisfy the second order resonance condition (4.14), the currents (4.26) and (4.27) are evaluated. Using these current values, the equations (4.22) and (4.23) are solved. This gives a new set of $R$ and $P$ values which certainly change the condition for a second order resonance (4.14), which will then lead to a new set of current values. Thus the whole cycle is repeated at every step in time and a set of self consistent solutions is obtained for a given initial pulse.
4.9 AMPLITUDE STRUCTURE IN VLF EMISSIONS

Here, we would like to discuss the amplitude structures of VLF emissions on the basis of the equations (4.22), (4.23), (4.26) and (4.27) which are solved numerically using the procedure discussed in the section 4.8. Fig. 4.3 shows the evolution of $|R|$ with time at different points in space for the pulse whose initial value is given by $R = 0.00002$, and $\alpha$ is equal to 0.5. The position of the pulse is at $L = 3$ where $\eta = 5 \times 10^{-10}$. This amplitude of the pulse is equivalent to $300 \mu V/m$ which is consistent with the observations of Cain et al. (1961). The trapping period is of the order 17 nsec. The figure shows the existence of peaks in the amplitude growth on both sides of the equator ($Z = 0$). The growth is seen to follow the resonant particle current distribution and the amplification is enhanced by a factor 3, at $t_3 = 24$ msec.

Fig. 4.4 represents the time evolution of a wave pulse of $\alpha = 0.1$ and $R = 0.0002$ at $L = 3$ and $\eta = 4.0 \times 10^{-9}$. The equivalent wave field is $\sim 300 \mu V/m$ and the trapping period is $\sim 18$ msec.
$\eta = 4 \times 10^{-9}$
$R = 0.0002 (E = 300 \mu V/m)$
$\alpha = 1$
$L = 3$

$Z = Z \times 100$

**Fig 4.4**
The time evolution of the pulse as seen in the plot shows the presence of peaks in growth on both sides of the equator. The growth is studied upto time 33 msec. The Figs. 4.5, 4.6 and 4.7 are for $\alpha = 0.1$ and $\eta = 10^{-10}$, which corresponds to a very large $L$ value and this may not be very realistic. But the general features seen in these figures are similar to Fig. 4.3 and Fig. 4.4.

We observe from these figures that, for the strong pulses, the amplifications become substantial. However, for weak pulses whose interaction region is very near to the equator, the amplification is found to be limited. It is seen that the emissions are also possible for weak pulses of long duration, which is supported by the observations.
\[ R = 1 + \eta z^2 \]
\[ \alpha = -1 \]
\[ \eta = 10^{-10} \]

\[ t = t_1 \]
\[ t = t_2 \]
\[ t = t_3 \]

**Fig 4.5**
\[ R = \eta z^2 + 1 \]
\[ \alpha = 0.0005 \]
\[ \beta = \eta z \]

\[ \eta = 10^{-10} \]

FIG. 4.6
4.10 FREQUENCY TIME STRUCTURE

The importance of existence of reactive part in the second order resonant particle current, which is finite everywhere, is to give the phase modulation to the wave pulse. This phase modulation also depends on the wave amplitude, therefore the combined effect of these two terms in the phase modulation is to give the frequency change to the initial pulse. Using this phase modulation one can study the frequency modulation on the basis of the equations \( \frac{2K_x}{\beta t} = \frac{\partial}{\partial \tau} \frac{\partial \Phi}{\partial \tau} \) and \( \frac{\partial \omega}{\partial t} = \frac{\partial^2 \Phi}{\partial t^2} \). Here, we shall discuss the frequency time structure on the basis of self-consistent solution of (4.22) and (4.23).

Frequency time structure in case of a wave pulse with \( \alpha = 0.5 \) and the amplitude \( R = 0.00002 \) is shown in the Fig. 4.8. This pulse is taken to be at \( L = 3 \), where \( \gamma = 5 \times 10^{-10} \). Here, it is seen that the initial pulse leads to frequency time structures similar to hooks, inverted hooks, risers, falling tones and a long enduring oscillations.
Fig V

\[ \eta = 5 \times 10^0 \]
\[ a = 0.0002 \]
\[ R = 0.00002 \] (E=300 \( \mu \)V/m)

\[ L = 3 \]
Similar structures are observed in case of triggered emissions by a Morse Code pulse. These structures strongly depend on the position of the pulse along given field line. At $L = 3$, a riser (Fig. 4.9a) is seen at $Z = 1100$ for the pulse of amplitude $R = 0.0002$ and $\alpha$ is equal to 0.1. The same pulse is seen to trigger a falling tone at $Z = -800$ (Fig. 4.9b). For a very weak pulse of amplitude $R = 0.000008$ and $\alpha = 0.1$, a riser is seen at $Z = 1000$ (Fig. 4.9c) and a falling tone at $Z = -2000$ (Fig. 4.9d).

Some studies are also made for relatively strong pulses, in which case the emissions are observed to be different in nature. For the pulse of $R = 0.0005$ and $\alpha = 0.1$, it is seen that the hooks are triggered at $Z = -7800$ and inverted hooks observed at $Z = 7800$ (Fig. 4.10a, b). Fig. 4.11 depicts various kinds of frequency time structures for $R = 0.00002$ and frequency $\alpha = 0.1$. However, the gradient $\eta = 10^{-10}$, corresponds to very large $L$ value, and seems to be unrealistic. Nevertheless the emissions do show a set of complex frequency time structures.
Fig 4.9 b

\[ R = 0.0002 \left( \epsilon = 300 \mu V/m \right) \]

\[ \alpha = 1 \times 10^{-4} \]

\[ \eta = 4 \times 10^{-9} \]

\[ L = 3 \]

\[ Z = 600 \]
4.11 FREQUENCY TIME STRUCTURES AS SEEN ON THE GROUND
AND BY SATELLITE

The VLF emissions and their frequency time structures in sections 4.8 and 4.10 are not adequate enough to be compared with the observations outside the VLF generating region. Therefore, it is essential to translate them into the region outside the VLF generation regions, with sufficient accuracy. This will also prove as a test to the present theory. However, the exact propagation characteristics of these emissions are not known, we assume that these emissions follow cold plasma dispersion relation of whistlers. As VLF emissions move away from the generating region, using the group delay, the changes in the frequency time structures are discussed.

The group delay for the wave propagation along the magnetic field line is given by

$$\tau = \int_{\text{Path}} \frac{ds}{V_g} \quad (4.26)$$
where

\[ v_g = \frac{2 \omega^2 (1 - \omega \alpha)^{3/2}}{(1 - \alpha)^{3/2}} \text{ group velocity.} \]

The integration \( \int ds \) has to be taken along the path of propagation which is along the invariant latitude \( \phi \) and we know that whistlers propagate faster for higher frequencies and slower at lower frequencies.

Using (4.26), each part of frequency time structure was projected far away from the generating region. It should be noted that \( v_g \) is not a constant but a function of space coordinate \( Z \).

By including the appropriate group delay, the frequency time structure of a riser at \( Z = 1100 \), is obtained at \( Z = 10000 \) (Fig. 4.12a). The delay time at different frequencies as they move away to a distance \( Z = 10000 \) is shown in Fig. 4.12a.

This figure resembles Fig. 4.9 except that the slope is changed at higher frequencies. It is also seen that the time delay for different
Fig 4.12
frequencies is found to be very small, therefore we expect that the general outline of the frequency structure is not much altered. Thus, the structures discussed here are more or less good enough to be compared with the observations but with small addition of appropriate time constant.

4.12 DISCUSSIONS

In this chapter we have discussed the effect of second order resonance in triggering the VLF emissions which is found to be dominant around the equitorial plane of the magnetosphere. This resonance leads to substantial amplification of the wave pulses and it is found that the phase modulation gives rise to the frequency structures. The growth is proportional to the initial amplitude and for moderately strong pulses, the growth occurs on both sides of the equator.

The studies for weak and strong pulses indicate that the weak amplitudes trigger simple frequency time structures like falling tones and these also have
only one peak in the amplitude growth. The strong pulses generate almost all types of complicated frequency time structures like hooks and inverted hooks. The risers are also triggered by these strong pulses. These theoretical results are supported by observations (Helliwell 1969).

The present theory invokes conditions which are similar to quasilinearity conditions, for example the amplitude changes are assumed to be small in one trapping period. However, it seems that the results are fairly good even for rapid variations of the amplitudes when amplitudes and trap sizes are increasing simultaneously.

It should also be noted that the present model is free of side bands whose existence may ruin the analyses discussed above. Therefore the present analysis are strictly true for emissions which are devoid of side bands. We know that, in case of homogeneous magnetic field, the phase oscillations of individual particles in the trap are a serious problem which may lead to lack of self-consistency. This self-consistency picture seems
to be improved due to non-uniform magnetic field which leads to more stable trapping.

We would like to conclude that the model described above is capable of giving rise to almost all the features of frequency time structures in VLF emissions. However, the different frequency time structures are generated at different points in space, which support the phenomenological theory of Helliwell (1967). The weak signals prefer to produce falling tones and strong signals give rise to nonlinear frequency time structures like hooks, inverted hooks, and the long enduring oscillations.