TECHNIQUES OF THE STUDY OF COSMIC RAY TIME VARIATIONS

I.1. Comparison of cosmic ray detectors

Studies of the time variations in cosmic ray intensity are carried out by continuous monitoring of cosmic ray intensity using a variety of techniques in various energy ranges from a few MeV to $\sim 10^{21}$ eV. Cosmic rays of energies $< 10^9$ eV have been mainly studied by balloon borne and satellite detectors, whereas the extensive air shower technique is used for studying particles above $10^{12}$ eV. In the intermediate energy range, the ground based ionization chambers, Geiger-Müller telescopes, plastic scintillation telescopes and neutron monitors have been extensively used over the past few decades for the continuous monitoring of the cosmic ray intensity. The lower energy limit for these detectors is set by the atmospheric absorption of the cosmic ray particles. In addition to simplicity and relative ease in the operation of the ground based detectors, they have a further advantage, that the geomagnetic field acts as a momentum spectrum-analyser of primary cosmic rays resulting in an increase in cutoff rigidity from poles to equator. The ground based detectors measure the intensity of the secondary particles produced by the interaction of primary cosmic rays in the earth's atmosphere. Ionization chambers, Geiger-Müller telescopes and plastic scintillation telescopes respond basically to the hard component ($\mu$-mesons) in
the atmosphere, whereas 92% of the counting rate in a neutron monitor is due to nucleonic component of the secondary cosmic rays (Huges and Marsden, 1966). The mean rigidity response of neutron monitors is therefore lower than that of the meson monitors. The mean rigidity response is defined as

\[
\text{Mean rigidity response} = \frac{\int_{P_c}^{\infty} S(P) \frac{dN}{dP} \, dP}{\int_{P_c}^{\infty} S(P) \, dP} \quad \ldots (2.1)
\]

where \( P_c \) is the geomagnetic cutoff-rigidity of the station, \( \gamma' \) the spectral exponent of the variation under study, \( S(P) \) the specific yield function (Lockwood and Webber, 1967) and \( \frac{dN}{dP} \) is the primary differential rigidity spectrum of cosmic rays. For \( \gamma' = 1 \), as in the case of eleven year variation, the mean rigidity response for a neutron monitor at sea level varies from \( \sim 10 \) GV at the geomagnetic latitude \( \sim 55^\circ \) to \( \sim 41 \) GV near the equator (Trivandrum, India).

In comparison the meson monitor at equator has a mean rigidity response \( \sim 47 \) GV. The mean rigidity response further increases to \( \gtrsim 80-100 \) GV for an underground meson monitor depending upon the depth below ground. Thus all these detectors are supplementary to each other in the study of cosmic ray intensity variations in the different rigidity ranges.

Among all the ground based detectors, neutron monitors are found to be the most suitable for the study of solar modulation of the cosmic ray intensity. This is because of their lower rigidity response as well as due to easy
correction for the atmospheric effects. Study of the spatial anisotropies in the cosmic ray intensity by such ground based detectors needs analysis of data from the monitors distributed at various latitudes and longitudes forming a grid all over the earth. To meet this requirement, a large number of neutron monitors have been set up at various places during the IGY and later.

II.2. Atmospheric pressure correction to total counting rates in a neutron monitor

The intensity of secondary cosmic radiation in the atmosphere undergoes changes due to the variations in the atmospheric pressure. An increase in the atmospheric pressure results in a greater absorption of secondary cosmic rays and hence a decrease in their intensity at ground and vice versa. The pressure corrected counting rate \( N_{\text{corr}} \) in a neutron monitor is therefore given, to a first approximation, by the formula (Elliot, 1952),

\[
N_{\text{corr}} = N_{\text{obs}} (1 - \beta \Delta p) \quad \ldots(2.2)
\]

where \( N_{\text{obs}} \) is the observed counting rate, \( \Delta p \) is the change in pressure from standard mean atmospheric pressure at the station and \( \beta \) is the pressure coefficient having a negative value. Precisely, the relation is

\[
N_{\text{corr}} = N_{\text{obs}} \exp \left[ - \beta (p_{\text{obs}} - \bar{p}) \right] \quad \ldots(2.3)
\]

where \( p_{\text{obs}} \) is the atmospheric pressure observed at the time
of $N_{\text{obs}}$ and $\bar{p}$ is the standard mean atmospheric pressure of the station.

Various methods [Mathews, 1959; McCracken and Johns, 1959; and Lindgren, 1962], suggested for determining the pressure coefficient from the above equation, suffer from the drawback that the primary cosmic ray intensity at the top of the atmosphere is constantly changing. The method of successive differencing used by Lapointe and Rose (1962), Bachélet et al. (1965) and Griffiths et al. (1966) is, however, considerably more reliable in this regard. The method is as follows:

If $N_1$ is the counting rate on day 1 having daily mean pressure $p_1$, then the counting rate $\bar{N}_1$ on that day for the mean standard pressure $\bar{p}$ at the station can be found from the relation,

$$ N_1 = \bar{N}_1 \cdot \exp \left[ - \beta \cdot (p_1 - \bar{p}) \right] \quad \ldots \quad (2.4) $$

Similarly on the subsequent day 2, we would have,

$$ N_2 = \bar{N}_2 \cdot \exp \left[ - \beta \cdot (p_2 - \bar{p}) \right] \quad \ldots \quad (2.5) $$

and hence

$$ \ln \left( \frac{N_1}{\bar{N}_2} \right) = \ln \left( \frac{\bar{N}_1}{\bar{N}_2} \right) - \beta \cdot (p_1 - p_2) \quad \ldots \quad (2.6) $$

If the periods are chosen when the primary intensity does not change appreciably from one day to the next, then $\bar{N}_1 \approx \bar{N}_2$ and $\ln \left( \frac{N_1}{\bar{N}_2} \right)$ is approximately zero. With this assumption, $\beta$ can be determined by means of a regression
analysis of \( \ln \left( \frac{N_1}{N_2} \right) \) on \( (p_1 - p_2) \).

McCracken and Johns (1959), Bachelet et al. (1965) and Carmichael et al. (1968), Carmichael and Bercovitch (1969) have studied the latitude and altitude dependence of the pressure coefficient for neutron monitors. Figure 2.1 shows the latitude variation of the pressure coefficient at sea level obtained by Carmicheal and Bercovitch. The coefficient increases from about 0.94 percent per millimeter of Hg at about 16 GV cutoff rigidity to about 1.04 percent per millimeter of Hg at about 2 GV and then onward it remains constant up to the poles. Likewise, the pressure coefficient is found to increase up to an atmospheric depth of about 600 gm cm\(^{-2}\), and thereafter it decreases up to the sea level. The increase in the pressure coefficient with increasing latitude and atmospheric depth is understood in terms of the reduction in the energy and hence absorption length of the particles contributing significantly to the counting rate in the monitor. The anomalous decrease in the pressure coefficient below 600 gm cm\(^{-2}\) has been explained by Carmichael and Bercovitch (1969) and Singh et al. (1970) as due to obliquely incident cosmic rays having an apparently smaller absorption length, which contribute more at higher altitudes than near sea level.
Figure 2.1 The latitude and altitude dependence of atmospheric pressure coefficients for neutron monitors (Carmichael and Bercovitch, 1969).
II.3. Geomagnetic field and asymptotic cones of acceptance:

The geomagnetic field, to a first approximation, is that due to a magnetic dipole located near the centre of the earth. The equivalent dipole would be offset by 436 km from the centre of the earth, displaced toward the Pacific ocean. Further, it is tilted with respect to the earth's rotation axis by approximately 11° (handbook of Geophys, USAF Geophysics Research Directorate, the Macmillan Company, 1960). The north geomagnetic pole is located in Greenland at 81.0°N, 84.7°W in the geographic system of coordinates. The corresponding south geomagnetic pole is in Antarctica at 75.0°S, 120.4°E. The geomagnetic latitude \( \lambda_m \) and longitude \( \psi_m \) (east), of a point having geographic latitude \( \phi \) and longitude \( \phi \), are given by

\[
\sin \lambda_m = \cos 78.3^\circ \cdot \cos \phi \cdot \cos (\phi - 291^\circ) + \sin 78.3^\circ \cdot \sin \phi \quad \ldots (2.7)
\]

and

\[
\cos \psi_m = \left[ \sin 78.3^\circ \cdot \cos \phi \cdot \cos (\phi - 291^\circ) - \cos 78.3^\circ \cdot \sin \phi \right] / \cos \lambda_m \quad \ldots (2.8)
\]

The geomagnetic dipole has a magnetic moment of \( 8.06 \times 10^{25} \) gauss cm\(^3\).

The geomagnetic field produces two major effects on the cosmic ray particles. Firstly, it acts as a momentum spectrum analyser and, at each geomagnetic latitude, the particles arrive only above a threshold magnetic rigidity
from a particular zenithal and azimuthal direction. This effect has been extensively used for understanding the rigidity spectrum of various cosmic ray intensity variations. The second consequence of the geomagnetic field on cosmic rays is the change in the asymptotic direction of approach of the particles in the interplanetary space from the direction of incidence on the surface of the earth, so that the intensity observed at any station at a particular local time has to be related to a different direction in space. The correction becomes complicated since a ground based monitor records the secondaries in the atmosphere produced by primary cosmic ray particles of all the rigidities from the geomagnetic cutoff to infinity and the geomagnetic bending for each rigidity is different. Both these effects are discussed in detail below.

The cutoff-rigidities of cosmic ray particles in the centered dipole geomagnetic field have been worked out by Stömer (1936), Lemaitre and Vallarta (1936, a, b) and Alpher (1950). At a geomagnetic latitude $\lambda_m$, the cosmic ray particles arriving from a zenith angle $\theta$ and azimuth $\phi$ are subject to a cutoff-rigidity given by (Alpher, 1950):

$$P_c = \frac{300 \times M \cos^4 \lambda_m}{a^2 \left[ 1 + \sqrt{1 - \sin \theta \cdot \cos \phi \cdot \cos^3 \lambda_m} \right]^2}$$

in units of GV where $M$ is the earth's magnetic moment in Gauss and $a$ is the earth's radius in Stömer unit (C)
defined as

\[ 1 \text{ Störmer Unit} = C = \left(\frac{\text{Magnetic moment of the earth}}{\text{rigidity of particle}}\right)^{\frac{1}{2}} \]

\[ = \left(\frac{M}{\eta c/Ze}\right)^{\frac{1}{2}} \]

\[ \ldots (2.10) \]

In the above expression, \( p \) is the momentum, \( Ze \) is the charge of the particle and \( c \) is the velocity of light.

For vertically incident particles, the expression for cutoff-rigidity reduces to

\[ P_c = 14.9 \cos ^4 \chi_m \]

\[ \ldots (2.11) \]

The reduction in cutoff-rigidity from equator to poles results in an increase in the integral cosmic ray intensity with increasing latitude as is verified from the latitude surveys conducted by Carmichael et al. (1965) and in earlier surveys quoted by Webber (1962). The surveys also show that the increase in the intensity continues only up to \( \sim 55^\circ \) geomagnetic latitude and there onwards it remains constant. The constancy of the intensity above \( \sim 55^\circ \) latitude, known as the "latitude knee" results from the atmospheric absorption of all the primary particles having rigidity less than \( \sim 1.5 \text{ GM} \).

Therefore any reduction in the cutoff-rigidity beyond \( 55^\circ \) geomagnetic latitude does not increase the integral cosmic ray intensity. The latitude surveys have shown a disagreement with the expected latitude distribution of the cosmic ray intensity at various longitudes for a centered dipole geomagnetic field. This has led to the study of the influence of higher moments of the earth's magnetic field on
Figure 2.2 Comparison of observed and calculated cosmic ray equator.

- Observed - Katz et al. (1953)
- Calculated - Kellogg and Schwartz (1959)
- Quenby and Webber (1959)
cutoff-rigidities [Quenby and Webber, 1959; and Wehk, 1961],
which has shown better agreement with the results from the
latitude surveys and has revealed a longitudinal dependence
in the cutoff-rigidities. The maximum cutoff-rigidity of
\( \sim 17 \text{ GV} \) is observed in the Indian zone near the geomagnetic
equator. Figure 2.2 shows the comparison of the observed
cosmic ray equator (Katz et al., 1958) and the calculated
ones by Kellog and Schwartz (1959) using terms up to Octopoles
of the earth's magnetic field as well as by Quenby and
Webber (1959) using spherical harmonics of the earth's
magnetic field up to sixth degree. The deformed geomagnetic
field under the pressure of solar wind also leads to a
change in the cutoff-rigidity at a particular station at
different local times as shown by Ahluwalia and McCracken
(1965), Razdan and Summers (1965) and Smart et al. (1969).

A cosmic ray detector on the ground responds in varying
degrees to primary cosmic ray particles of different rigidities
impinging on the top of the atmosphere from different
directions. These primary cosmic ray particles of different
rigidities above the geomagnetic cutoff come from different
asymptotic directions outside the deflecting geomagnetic
field. Therefore, in order to find out the mean asymptotic
direction of approach of the cosmic ray particles contrib-
buting significantly to the counting rate in a ground
based detector, a combined weightage is assigned to every
accessible asymptotic direction in terms of the primary
rigidity spectrum and the zenithal dependence of the counting rate in the monitor. Asymptotic cone of acceptance of a ground based cosmic ray detector is then defined by the solid angle containing the asymptotic directions of approach of primary cosmic ray particles that contribute significantly to its counting rate [LaPointe and Rose, 1961; McCracken, 1962]. The mean asymptotic direction $A_m$ is given by

$$A_m = \frac{\int_{P_c}^{\infty} P^{-\gamma} S(P) A(P) dP}{\int_{P_c}^{\infty} P^{-\gamma} S(P) dP}$$

where $A(P)$ is the asymptotic direction of approach for a particle of rigidity $P$, $P^{-\gamma}$ is the primary differential rigidity spectrum, $P_c$ the geomagnetic cutoff-rigidity at the station and $S(P)$ the specific yield function (Treiman, 1952; and Lockwood and Webber, 1967).

The calculations of LaPointe and Rose (1961) and McCracken (1962) reveal that for monitors located at geographic latitudes less than $\pm 55^\circ$, the asymptotic cone of acceptance lies to the east of the station meridian and within $\pm 10^\circ$ of the equatorial plane. The asymptotic cones of very high latitude monitors make an appreciable angle with the equatorial plane. Figure 2.3 a,b,c show the mean asymptotic latitude, longitudinal shift in the mean asymptotic direction from station-meridian and the width of the
Figure 2.3a,b The mean asymptotic latitude and longitudinal shift in the mean asymptotic direction from vertical for neutron monitors at various geomagnetic latitudes (Lapointe and Rose, 1961).
Figure 2.3c The width of the asymptotic cones of acceptance of neutron monitors (Lapointe and Rose, 1961).

Figure 2.3d The actual asymptotic cones of acceptance for different neutron monitors (McCracken, 1962).
asymptotic cone of acceptance at various geographic latitudes as calculated by Lapointe and Rose (1961) for a simple dipole geomagnetic field. Figure 2.3 d, shows explicitly, the asymptotic cones of acceptance for different stations (McCracken, 1962), taking into account the spherical harmonics up to sixth degree in the geomagnetic field. It is seen that the longitudinal width of the asymptotic cones decreases from \( \sim 50^\circ \) near the equator to \( \sim 20^\circ \) near the poles, while the eastward shift in the mean asymptotic direction decreases from \( \sim 55^\circ \) near the equator to \( \sim 30^\circ \) near the poles. The higher harmonics in the geomagnetic field make a considerable contribution in the asymptotic coordinates of stations at very high geographic latitudes. For different stations at such high latitudes, the longitudinal shifts in the asymptotic direction can differ by more than three hours. At lower latitudes, this difference due to deviations from dipole geomagnetic field is smaller. All these factors make middle latitude stations the most suitable for the study of cosmic ray anisotropies in the equatorial plane.

II.4. Variational coefficients

The effect of the characteristics of asymptotic cones of acceptance (described in the last section) on the spatial anisotropy in cosmic ray intensity, has been discussed by Rao et al. (1963). The fractional change in the counting rate of a ground based monitor due to particles arriving
in the solid angle $\Omega_i$ at the top of the atmosphere is given by

$$\frac{dN (\Omega_i)}{N} = \int W(P) \frac{\Delta J_i(P)}{J_0(P)} \frac{Y(\Omega_i, P)}{Y(4\pi, P)} \ dP \quad ... (2.13)$$

where $J_i(P)$ is the differential cosmic ray rigidity spectrum within $i^{th}$ solid angle $\Omega_i$ and is expressed as $J_i(P) = J_0(P) + \Delta J_i(P)$ where $J_0(P)$ is the "average" differential spectrum while the change $\Delta J_i(P)$ is different from one $\Omega_i$ to the next. $N$ is the average counting rate of the monitor over all directions, $W(P)$ are the coupling coefficients discussed by Dorman (1957). $Y(\Omega_i, P)$ are the weighting functions (derived from the zenithal and azimuthal response of the monitor) integrated over the zenith angle and azimuthal angle in the solid angle $\Omega_i$. $Y(4\pi, P)$ is its average over all the solid angles in the space.

It is assumed that the variational spectrum of the anisotropy $\frac{\Delta J_i(P)}{J_0(P)}$ of the primary cosmic rays is a power law of the form $AP^Y$ where $A$ is a function of the asymptotic direction and $P$ the rigidity in GV. So, the observed anisotropy $\frac{dN (\Omega_i)}{N}$ can be expressed as

$$\frac{dN (\Omega_i)}{N} = A'(\Omega_i, P) \quad ... (2.14)$$

where

$$A'(\Omega_i, P) = \int W(P) \frac{Y(\Omega_i, P)}{Y(4\pi, P)} \ dP \quad ... (2.15)$$
\( v(\Omega_1, \psi) \) is called the "variational coefficient" of the monitor for the solid angle \( \Omega_1 \) and the spectral index \( \psi \).

Let \( A \), the amplitude of the anisotropy, be expressed in terms of the asymptotic latitude \( \lambda \) and the asymptotic longitude \( \psi \) as

\[
A = f(\lambda) \cdot \cos \lambda \quad \ldots(2.16)
\]

The expression for the anisotropy observed at the ground based detector due to particles coming in the solid angle \( \Omega_1 \) can then be written as

\[
\frac{dN(\Omega_1)}{N} = f(\psi) \cdot v(\Omega_1, \psi) \cdot \cos \lambda \quad \ldots(2.17)
\]

Summing over all the solid angles \( \Omega_1 \), we get

\[
\frac{dN(\psi_j)}{N} = f(\psi_j) \left[ \sum_{i} v(\Omega_1, \psi) \right] \cdot \cos \lambda \quad \ldots(2.18)
\]

\[
= f(\psi_j) \cdot v(\psi_j, \psi) \quad \ldots(2.19)
\]

where \( \frac{dN(\psi_j)}{N} \) is the anisotropy due to particles arriving from the \( j \)th asymptotic longitude \( \psi_j \) and \( v(\psi_j, \psi) \) is the corresponding total variational coefficient.

Let the anisotropy be expanded as an arbitrary function of the angle \( \eta \) in the form of a Fourier series

\[
f(\eta) = J_0(\beta) \sum_{m=1}^{\infty} \alpha_m \cdot \cos m(\eta - C_m) \quad \ldots(2.20)
\]

where \( \alpha_m \) and \( C_m \) are the arbitrary amplitude and phase constants. \( C_m \) is also the asymptotic direction of viewing from which the maximum of \( m \)th harmonic is seen. The
Figure 2.4 The angles employed to specify the asymptotic direction of viewing in equation 2.21.
asymptotic angle \( \eta \) is measured east of the earth-sun line (figure 2.4) and can be expressed as

\[
\eta = \psi + 15T - 180^\circ \quad \text{(2.21)}
\]

where \( \psi \) is the asymptotic longitude of the station and \( T \) is the local time measured in hours. Thus

\[
f(\psi) = J_0(p) \left[ \sum_{m=1}^{\infty} \alpha_m \cos m (\psi + 15T - 180^\circ - C_m) \right] \quad \text{(2.22)}
\]

Substituting this expression for \( f(\psi) \) in the equation defining the observed fractional change (anisotropy) in the counting rate due to change \( \Delta J(\psi_j) \) in the primary intensity from the asymptotic direction \( \psi_j \), and then summing over all the asymptotic directions, we have

\[
\frac{\Delta N_N}{N_N(\text{Total})} = \sum_j V(\psi_j, \phi) \sum_{m=1}^{\infty} \alpha_m \cos \left[ m (\psi + 15T - 180^\circ - C_m) \right] \quad \text{(2.23)}
\]

\[
= \sum_{m=1}^{\infty} \alpha_m B_m \cos \left[ m (15T - C_m - 180^\circ) + \delta_m \right] \quad \text{(2.24)}
\]

where

\[
B_m^2 = \left[ \sum_j V(\psi_j, \phi) \sin(m \psi_j) \right]^2 + \left[ \sum_j V(\psi_j, \phi) \cos(m \psi_j) \right]^2 \quad \text{(2.25)}
\]

and

\[
\tan \delta_m = \frac{\sum_j V(\psi_j, \phi) \sin (m \psi_j)}{\sum_j V(\psi_j, \phi) \cos (m \psi_j)} \quad \text{(2.26)}
\]

(\( \alpha_m B_m \) and \(-m C_m + \delta_m\)) are the amplitude and phase constants of the \( m^{\text{th}} \) harmonic of the cosmic ray intensity variation at the detector. The universal time of maximum
The intensity of the $m$th harmonic is given as
\[ I_m = \frac{180 \ m + m \ C_m - \delta_m}{15 \ m} \text{ hours} \quad \ldots (2.27) \]

The local time of maximum for a detector at a geographic longitude $L^\circ$ is given by
\[ t_m = \frac{190 \ m + m \ C_m - (\delta_m - m \ L)}{15 \ m} \text{ hours} \quad \ldots (2.28) \]

The quantity $\frac{\delta_m - m \ L}{15 \ m}$ is also known as the "geomagnetic bending" of the cosmic ray flux.

II. 5. Rigidity dependence of cosmic ray anisotropies.

The method of studying rigidity dependence of the cosmic ray anisotropy developed by Rao et al. (1963) is described below. The variational spectrum is expressed as
\[ \frac{\delta J(P)}{J(P)} = F(\lambda), \quad \text{a. e. for } P_L < P < P_u \quad \ldots (2.29) \]
\[ = 0 \quad \text{for } P < P_L \text{ and for } P > P_u \quad \ldots (2.30) \]

where $P_L$ and $P_u$ are the lower and upper limits of rigidity beyond which isotropy prevails, $a$ is the amplitude of the anisotropy in the free space, $\gamma$ is the spectral index and $F(\lambda)$ is the dependence of the anisotropy on the latitude $\lambda$.

The spectral exponent $\gamma$ is determined by varying the parameters in the above equation until the variance among the theoretically normalized anisotropy values in the interplanetary space, obtained from various stations is minimized. Using this method, Rao et al. (1963) and Mori (1968) have
shown that for the average diurnal variation, with $0 < \mathbf{P} < 9 \text{ GV}$, the spectral index $\gamma$ is $\sim 0$. The upper cutoff-rigidity of diurnal variation in the cosmic ray intensity is strongly dependent upon the level of solar activity (Agrawal and Rao, 1969; Jacklyn et al., 1970).

II.6. Numerical techniques

II.6 a) Harmonic analysis:

To determine the diurnal and semi-diurnal variation in the cosmic ray intensity, the data is subjected to Fourier (harmonic) analysis. Following is a general description of a Fourier series and the scheme of calculating the various harmonic coefficients.

Any periodic function $y = f(\theta)$ can be expressed in the form of a trigonometric Fourier series,

$$Y = f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta) \cdots (2.31)$$

The function has a period $2\pi$. The coefficients in the series are given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cdot d\theta \cdots (2.32)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cdot \cos(n\theta) \cdot d\theta \cdots (2.33)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cdot \sin(n\theta) \cdot d\theta \cdots (2.34)$$
The series may be written in the alternative form as,

\[ Y = f(\phi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} r_n \cos(n \phi - \phi_n) \] ....(2.35)

where \( r_n = \sqrt{a_n^2 + b_n^2} \) ....(2.36)

\[ \cos \phi_n = \frac{a_n}{r_n} \] ....(2.37)

\[ \sin \phi_n = \frac{b_n}{r_n} \] ....(2.38)

i.e. \( \phi_n = \tan^{-1} \frac{b_n}{a_n} \) ....(2.39)

The condition \( n \phi = \phi_n \) then gives the argument for the maximum contribution from the harmonic amplitude \( r_n \).

A periodic function having a period different from \( 2\pi \) can be reduced to the above form in the following manner. If \( t \) is the independent variable and the given function \( Y = f(t) \), we write

\[ t = k + m\phi \] ....(2.40)

If limits for \( t \) are \( g \) and \( h \), and \( \phi \) is to have limits from 0 to \( 2\pi \) we have

\[ g = k + 0 \text{ and } h = k + 2\pi m \] ....(2.41)

This gives \( k = g \) and \( m = (h-g)/2\pi \). The derived transformation is therefore,

\[ t = g + \frac{(h-g)}{2\pi} \phi \] ....(2.42)

or \( \phi = 2\pi \left( \frac{t-g}{h-g} \right) \) ....(2.43)

In cosmic ray studies, the data is normally analysed for periods of 24 hours (one day) or its higher harmonics.
If $t$ is the time in hours, we have for hourly data, $g = 1$ and $h = 24$, while the direction of the anisotropy is to be measured from $t = 0$ hours. The argument of the harmonic series is

$$\Theta = 2 \pi r \left( \frac{t-1}{23} \right) + 0.5 \quad \ldots \ldots \ldots (2.44)$$

The numerical scheme to find the harmonic coefficients is described by Scarborough (1966).

In the case of hourly data, the period of $360^\circ$ is divided into 24 intervals of $15^\circ$ width each. The standard error $\sigma_{r_n}$ in the amplitude $r_n$ of the $n^{th}$ harmonic is then

$$\sigma_{r_n} / \sqrt{12} \quad \text{where} \quad \sigma$$

is the standard error in the hourly counting rate. The standard error in the time of maximum of the $n^{th}$ harmonic is

$$\sigma_{\phi_n} = \frac{\sigma_{r_n}}{r_n}$$

In the case of bi-hourly data (interval $30^\circ$) having standard error $\sigma$, the standard error in the amplitude of the $n^{th}$ harmonic is given by

$$\sigma_{r_n} = \sigma / \sqrt{6}$$

while the standard error in the time of maximum is given by the same formula namely,

$$\sigma_{\phi_n} = \frac{\sigma_{r_n}}{r_n}$$

II.6 b) Correction of harmonic coefficients for secular changes in the data:

The experimental data used for harmonic analysis may contain a systematic linear variation or periodic secular variation of large periods. Such variations, to some extent,
distort the results of harmonic analysis. These effects may be eliminated as follows:

1) Linear variation:

The extreme ordinates at the end of the period are interpolated in the intermediate ordinates and these differences are then subtracted from the data. This removes the linear trend in the data.

2) Secular variations:

The secular variations are determined by the method of moving averages and are filtered out by subtracting from the data.

Let $x_1, x_2, x_3 \ldots, x_n$ be a series of data spaced at equal intervals $\tau$ and let the total period be $T = n\tau$.

The successive sums in the sliding intervals are formed as

$$
\sum_{i=1}^{n} x_i, \quad \sum_{i=2}^{n+1} x_i, \quad \sum_{i=3}^{n+2} x_i,
$$

Mean values of these sums corresponding to the central terms $\frac{n}{2}, (\frac{n}{2} + 1), \ldots$ form a series,

$$
\bar{x}_{n/2} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \bar{x}_{(n/2 + 1)} = \frac{1}{n} \sum_{i=2}^{n+1} x_i.
$$

These represent the secular variations. If the period $T$ is divided in even number of intervals, the value $\bar{x}_{n/2}$ actually corresponds to $x_{(n/2 + \frac{1}{2})}$. The actual value of
\[ \bar{x}_{n/2} \text{ is then given by} \]
\[ \left[ \bar{x}_{n/2} + \bar{x}_{(n/2 - 1)} \right] / 2 . \]

Subtracting these values from the actual data at corresponding intervals of time leaves only, those variation of period T in the data, which may be calculated by harmonic analysis.

II.6 c) Method of superposed Epoch or the Chree analysis

for small variations:

This method is especially useful in detecting small non-periodic as well as periodic variations. Values of the observed quantity corresponding to onset of these small phenomena observed at different times are entered in a vertical column numbered zero. The values succeeding this epoch are consecutively written in the columns 1, 2, 3 ... etc. The preceding values are entered in the columns -1, -2, -3 ... etc. The table contains n lines if n phenomena are superposed. The mean value of the observed quantity in each column is found and plotted against the column number. Presence of the phenomena is clearly then, brought out, with onset in the column zero. The stray variations in the other columns are averaged out resulting in their total or partial cancellation.

II.6 d)'Principle of least square' and regression analysis:

The 'Principle of least squares', first formulated by Legendre can be expressed as follows:
The most probable value of any observed quantity is such that the sum of the squares of the deviations of the observations from this value is least. For example, if \( x_1, x_2, \ldots, x_n \) are the observed values of any given quantity then the most probable value \( X \) is such that

\[
(x_1 - X)^2 + (x_2 - X)^2 + \ldots + (x_n - X)^2
\]

is least. If \( \bar{x} \) is the arithmetic mean of \( x_1, x_2, \ldots, x_n \), so that

\[
\sum_{i=1}^{n} x_i = n \bar{x}
\]  

\[(2.45)\]

i.e.

\[
\sum_{i=1}^{n} (x_i - \bar{x}) = 0
\]  

\[(2.46)\]

We have,

\[
\sum_{i=1}^{n} (x_i - X)^2 = \sum_{i=1}^{n} \left[ (x_i - \bar{x}) + (\bar{x} - X) \right]^2
\]  

\[(2.47)\]

\[
= \left[ \sum_{i=1}^{n} (x_i - \bar{x})^2 \right] + n (\bar{x} - X)^2
\]  

\[(2.48)\]

which is clearly least when \( X = \bar{x} \). This gives that the most probable value of a measured quantity is the arithmetic mean of the observations.

One of the other useful applications of the 'Method of least squares' is in fitting a curve to a set of
experimental data points. Let \( y_1, y_2, \ldots, y_n \) be the measured quantity \( y \) for the values \( x_1, x_2, \ldots, x_n \) of another quantity \( x \) and let a linear relation between the two (the method can be extended to any degree curve) variables \( x \) and \( y \) to be computed. The linear relation can be expressed as

\[
y = a_1 x + b_1
\]

where \( a_1 \) and \( b_1 \) are constants. The experimental values of \( y_i \) for different values of \( x_i \) will, in general, not be the same as those calculated from the above equation. The difference (error) will be \( (a_1 x_i + b_1 - y_i) \). To obtain the linear best fit to the data, the constants \( a_1 \) and \( b_1 \) are so chosen that the sum of squares of the 'errors' is least, that is,

\[
\sum_{i=1}^{n} (a_1 x_i + b_1 - y_i)^2 \text{ is least.}
\]

The partial differentiation with respect to \( a_1 \) and \( b_1 \) then gives

\[
\sum_{i=1}^{n} x_i (a_1 x_i + b_1 - y_i) = 0 \quad \ldots \quad (2.50)
\]

\[
\sum_{i=1}^{n} (a_1 x_i + b_1 - y_i) = 0 \quad \ldots \quad (2.51)
\]
Solution of these equations gives

\[
\begin{align*}
  a_1 &= \frac{\sum_{i=1}^{n} x_i y_i - \sum_{i=2}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2} \quad \text{....(2.52)} \\
  b_1 &= \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2} \quad \text{....(2.53)}
\end{align*}
\]

where \( a_1 \) is the slope of the line with respect to x-axis and \( b_1 \) is the intercept on the y-axis. The above treatment assumes that only \( y_i \) are subject to experimental errors. However, it may happen that \( x_i \) may also be experimentally obtained and are liable to error. In this case, a regression line of \( x \) on \( y \) is also to be found by similar method, which will give \( a_2 \) and \( b_2 \) as the coefficients in the relation

\[
x = a_2 y + b_2 \quad \text{....(2.54)}
\]

The quantity

\[
r = \sqrt{a_1 \cdot a_2} \quad \text{....(2.55)}
\]

is known as the regression coefficient between \( x \) and \( y \) and is a measure of the dependence between the two. The maximum value of the regression coefficient is unity and the relation between \( x \) and \( y \) is taken to be significant.
if \( r > 0.5 \). The standard errors in the coefficients \( a_1, b_1 \) and \( r \) are

\[
\hat{\sigma}_a = \sqrt{\frac{\sum_{i=1}^{n} (a_1 x_i + b_1 - y_i)^2}{n(n-2)}}
\]

\[
\hat{\sigma}_b = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 - (\frac{\sum_{i=1}^{n} x_i}{n})^2}{n(n-2)}}
\]

\[
\hat{\sigma}_r = \frac{1 - r^2}{\sqrt{n}}
\]

Some of the other complicated functional forms can be made suitable for linear regression analysis by slight modifications. For example, taking natural logarithm of both sides of the equation

\[ y = a e^{b x} \]

\[ \log_e y = \log_e a + b x \]

Similarly, the equation

\[ y = a x^b \]

\[ \log_e y = \log_e a + b \log_e x \]
II.6 e) $\chi^2$ - test for goodness of fit:

A test for estimating the confidence level in similarity between a histogram derived from experimentally observed values and another hypothesized one, is described below. The method involves comparison of frequencies in every interval in the histogram with corresponding frequencies expected from the hypothesized frequency distribution. A quantity $(f_{oj} - f_{ej})^2 / f_{ej}$ is calculated for every interval where $f_{oj}$ is the observed frequency in the $j^{th}$ interval and $f_{ej}$ is the expected frequency in the same interval from the hypothesized distribution. The quantity $\chi^2$ is then defined as

$$\chi^2 = \sum_{j=1}^{n} \frac{(f_{oj} - f_{ej})^2}{f_{ej}} \quad \ldots (2.63)$$

where $n$ is the total number of intervals. A value of $\chi^2 = 0$, corresponds to the exact (theoretical) agreement between the observed and the hypothesized distribution, whereas the increasing values of $\chi^2$ indicate less and less agreement between the two. The probability $\gamma_{\chi^2}$ that the value of $\chi^2$ will lie between $\chi^2$ and $\chi^2 + d \chi^2$, on the basis of normal distribution, is given by

$$\gamma_{\chi^2} = \exp \left[ -\frac{\chi^2}{2} \right] \cdot \left( \chi^2 \right)^{\frac{n-2}{2}} / \sqrt{n/2} \cdot \left[ \frac{n-2}{2} \right]! \quad \ldots (2.64)$$

The probability that the $\chi^2$ value will be equal to or
greater than a calculated value of $\chi^2$ for (n-3) degrees of freedom (Smith and Duncan, 1944) is listed in $\chi^2$ - tables mentioned in standard statistical tables. Normally, if the observed value of $\chi^2$ is less than the $\chi^2$ value corresponding to 5% rejection level mentioned in the table, the agreement between the observed and hypothesized frequency distributions is taken to be satisfactory.

It should be mentioned that the $\chi^2$ -test does not indicate as to whether a single value in the histogram is responsible for the difference in the observed and hypothesized frequency distributions or it is due to an over all difference in the two distributions. Further, if the observed and the hypothesized distributions differ in a negative way on one side of the central point and in a positive way on the other side by small magnitudes, the $\chi^2$ -test does not bring out this difference, since the signs of the deviations from the expected values do not affect the value of $\chi^2$. 