5. PION CAPTURE IN $^3$He

In this chapter we shall consider pion capture on $^3$He nucleus. In this case the capture can take place by either a p-p pair or a p-n pair, so that the process is somewhat more complex than in the case of $^6$Li. We have discussed already in previous chapter the treatment of the problem by various authors, Diwakaran (D65a), Sakamato and Tohsaki (S67b), Figureau and Ericson (F69a) etc. In these treatments the proton-structure of the wave-function is explicitly exhibited. Our treatment differs from these authors in that we use the fully antisymmetrised wave function in three nucleons (in space, spin and isospin coordinates) obtained by solving the Schrödinger equation in momentum space with the non-local separable nucleon-nucleon potentials. It will thus be possible to exhibit explicitly the contributions of various symmetry terms in the wave functions. Further, we shall see that in such a three-body treatment, even the apparently uncorrelated or spectator proton can have large momentum components, and hence can capture a pion, whereas in other treatments the uncorrelated nucleon is incapable of capture. This happens since in the three-body treatment (in which we follow essentially the work of Mitra and collaborators (M66c, M68a), even the spectator
function has large dynamic effects built into it. This was already demonstrated in the case of $^6$Li. Our calculations are thus more complex and we believe more complete than those of other authors reported previously.

Pion capture in $^3$He leads to several possible final states.

$$\pi^- + ^3\text{He} \rightarrow n + d \ (d), \ ^3\text{H} + \gamma \ (\gamma)$$

$$n + n + p \ (p), \ ^3\text{H} + \pi^0 \ (CE)$$

$$n + n + p + \gamma \ (\gamma), \ n + d + \gamma \ (d\gamma) \ (5.1)$$

Our interest here is mainly in the deuteron (d) and proton (p) modes which are purely nuclear pair-absorption processes. The last two modes in which an additional $\gamma$-ray is also emitted have very small crosssections because of the small phase-space availability. The relative crosssection for the (d$\gamma$) mode is known to be only about $(3.6 \pm 1.2) \%$ (Z67a). We do not consider them in the following. The $\gamma$-mode and the charge-exchange scattering (CE) are single-nucleon processes and are usually calculated in the impulse approximation. Our pion-nucleon interaction is linear in pion-field so that the CE mode can occur only as a second-order process. We do not consider it either. The $\gamma$-mode is calculated here (for sake of comparison with d - or p-modes) somewhat crudely using the gauge-invariant
interaction \( P \rightarrow P - \frac{e}{c} A \).

5. A \( \pi - ^3\text{He} \) interaction.

For writing down the wave function of \(^3\text{He}\) we shall use the full group of permutation symmetries of three nucleons in coordinate, spin and ispin space. It will then be convenient to express pion-nucleon interaction also in a 3-body form rather than a two-body form as was done in equation (2.16) of chapter 2. The required algebra is given by Verde in (V57a).

For any three operators or functions \( \Theta_1, \Theta_2 \) and \( \Theta_3 \), one can construct a completely symmetric combination

\[
\Theta^s = \Theta_1 + \Theta_2 + \Theta_3
\]

and two combinations which can form the basis for \([2, 1]\) representation of the symmetry group,

\[
\Theta' = \frac{V_2}{2} (\Theta_3 - \Theta_2), \quad \Theta'' = \Theta_1 + \frac{1}{\sqrt{2}} (\Theta_2 + \Theta_3)
\]

The permutation operators \( P_{ij} \) leave \( \Theta^s \) invariant and have the following matrix representations in the space of \( \Theta', \Theta'' \).

\[
P_{23} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_{12,33} = \begin{pmatrix} 1/2 & \pm\sqrt{3}/2 \\ \pm\sqrt{3}/2 & -1/2 \end{pmatrix}
\]
We can now write the total pion-nucleon interaction for three nucleons as

\[
H_{\text{int}} = \left( -\frac{m}{M} G \right) \sum_{i=1}^{3} \left( \tau_{x} \cdot \Phi(x) \right) \left( \tau_{x} \cdot \Phi(x) \right) \\
= \left( -\frac{m}{M} G \right) \frac{2}{9} \left[ \Phi \cdot \{ \left( \tau_{x} \cdot \tau_{x} \right) \phi + \tau_{x} \cdot \tau_{x} + \tau_{x} \cdot \tau_{x} \phi \} \phi + \left( \tau_{x} \cdot \tau_{x} \phi + \tau_{x} \cdot \tau_{x} \phi \right) \phi \right] \\
(5.5)
\]

In the above derivation we note that for the pion wave function

\[
\Phi(\pi) = \Phi(\pi) = \Phi(\pi) = \Phi \\
\Phi^{s} = 3 \Phi \\
\Phi^{s} = \Phi^{s} = \Phi
\]

Further for S-wave pion capture which alone we consider, the requirement of parity change between initial and final state wave-functions prohibits contributions from terms involving \( p^{s} \) in the \( H_{\text{int}} \) and hence they have been dropped. With the above form it will be very simple to evaluate the matrix elements. For the negative pion capture we shall specialise to each \( \tau \rightarrow \sqrt{2} \tau \), as in the previous cases.
5.B Wave function for the three-nucleon system:

It is now well established that $^3\text{H} - ^3\text{He}$ constitute an isospin doublet ($T = 1/2$). The Coulomb energy evaluation for $^3\text{He}$ (A68a, G67b) confirms that nuclear forces are essentially charge-independent. However, the analysis of the charge form factor $F_{\text{ch}}$ ($^3\text{He}$) by Griffy (G64a) appears to indicate a small mixing of $T = 3/2$ state (about 2\% probability) in the ground state of $^3\text{He}$. In our treatment we only consider $T = 1/2$ state for the $^3\text{He}$ wave function. Also the magnetic moments of $^3\text{H}$ and $^3\text{He}$ indicate that the two wave-functions are almost symmetric in configuration space, the most important states thus being $L = 0$, $J = S$. The inclusion of tensor force will result in a small probability for $L \neq 0$ (mainly D-state) also being present in the wave function. Fortunately, the choice of a special form of the tensor force as in chapter 2 enables such a force to be included very easily. Thus the major component of the wave function is the $^2S_{1/2}$ state ($L = 0$, $S = 1/2$, $T = 1/2$). The space-part of the wave function is the symmetric S-state $\Psi_S$ with a very small admixture of the mixed symmetric S-states $\Psi'_S$ and $\Psi''_S$. The solution of the Schrödinger equation with various forces yields the probability
of mixed symmetric S-states to be about 1%.

Let $\Psi$, $\chi$ and $\mathcal{S}$ denote the space, spin and isospin components of the wave function. For the spin-isospin part we can construct ($I = 1/2$, $S = 1/2$) the various combinations (see V57a).

$$\Psi^S = \frac{1}{\sqrt{2}} (\chi' S + \chi'' S)$$  
$$\Psi^\alpha = \frac{1}{\sqrt{2}} (\chi' S' - \chi'' S)$$  
$$\Psi^\prime = \frac{1}{\sqrt{2}} (\chi' S'' + \chi'' S')$$  
$$\Psi'' = \frac{1}{\sqrt{2}} (\chi' S' - \chi'' S'')$$

$$\chi^S_{m_s = \frac{1}{2}} = \frac{1}{\sqrt{3}} \left\{ (++ ) + (-+ ) + (-+ ) \right\} \mathcal{S}$$

$$\chi^\prime_{m_s = \frac{1}{2}} = \frac{1}{\sqrt{2}} \left\{ (++ ) - (+- ) - (-+ ) \right\} \mathcal{S}$$

$$\chi''_{m_s = \frac{1}{2}} = \frac{1}{\sqrt{6}} \left\{ -2 (-+ ) + (++) + (+- ) \right\} \mathcal{S}$$

(5.7)

and then the complete antisymmetric wave function for the three-nucleon system can be written as

$$\mathcal{U} = \Psi^\alpha \Psi^S - \Psi^\prime \Psi^S + \Psi^\prime \Psi'' - \Psi'' \Psi'$$  (5.8)

Here the subscript 'a' indicates a completely antisymmetric state. An extensive treatment of the three-body system has been given in a series of papers by Mitra and collaborators (B63a, M63a, M66c),
and we shall mostly borrow their results for wave functions with different types of non-local separable interactions. We only consider the following forms of the potentials.

(i) A spin-independent average central S-wave force with $\beta_t = \beta_s = \beta = 1.4487 \text{ F}^{-1}$;
$$\bar{\lambda} = \frac{1}{2}(\lambda_t + \lambda_s) = 0.353 \text{ F}^{-3}.$$ This force is defined as $C^{\text{avr}}_Y$ in the following text.

(ii) A spin-dependent but central S-state force with parameters denoted by $C^\text{eff}_Y$ and $S^\text{eff}_Y$ given in table 1 of chapter 2.

(iii) A central + tensor force in the triplet even states, and a central singlet even S-state force, with parameters that are also defined in table 1 of chapter 2.

Since we only consider even-state potentials, we can immediately drop the $\psi^{S \neq S}_S$ term of equation (5.8). The space-parts of the wave function $\psi^S$ (major term) and $\psi'$, $\psi''$ (the mixed-symmetric $S'$ state) are obtained by solving the Schrödinger equations. For details we refer to (B65b, M66c). Here we give only the results for the structure of the space-parts of the
wave-functions.

(i) The structure of the Schrödinger equations for $\Psi^5$, $\Psi'$ and $\Psi''$ shows that in the absence of the spin-dependence of the N-N interaction, the equation for $\Psi^5$ is completely uncoupled from $\Psi'$ and $\Psi''$, and the ground state has no $\Psi'$ or $\Psi''$ components. We have for

$$\Psi^5 = D^{-1}_E \sum_{ijkl} g(q_{ij}) F(P_K)$$

where

$$D_E = q_{ij}^2 + P_k^2 + a_T^2$$

with $q_{ij} = (P_i - P_j)/2$, $a_T^2 = -M\varepsilon_b$

Numerical integration of the eigen-value equation for $\Psi^5$ yields $F(P_K)$, the spectator function for the $K^{th}$ nucleon, in a numerical form. When the three-body value of $\lambda$ is properly matched to the two-body value (0.353 $F^{-3}$), the binding energy for $^3$He turns out to be 12.43 MeV, which is rather large compared to the experimental value viz., 8.45 MeV (the Coulomb-corrected value). However, the correct binding energy is obtained for a value of $\lambda$ (three-body) = 0.328 $F^{-3}$ which is only about \(7\%\) decrease from the appropriate two-body value.
(ii) In this case, we obtain

\[ \chi^s = D_0^{-1} \sum_{ijk} \left[ g(q_{i,j}) f(P_k) + f(q_{i,j}) G(P_k) \right] \]

\[ \chi' = \sqrt{\frac{3}{2}} D_0^{-1} \left\{ \left[ g(q_{12}) f(P_3) - f(q_{12}) G(P_3) \right] - \left[ (12, 3) \rightarrow (3, 1, 2) \right] \right\} \]

\[ \chi'' = D_0^{-1} \left\{ \left[ g(q_{23}) f(P_1) - f(q_{23}) G(P_1) \right] + \frac{1}{2} \left[ (23, 1) \rightarrow (31, 2) \right] \right\} + \frac{1}{2} \left[ (23, 1) \rightarrow (12, 3) \right]^2 \]

(5.10)

Here \((ij, k) \rightarrow (jk, i)\) represents the similar term with the specified change.
Here $F$ and $G$ are the spectator wave-functions when the correlated pair is interacting via triplet and singlet interactions respectively. The probability for the mixed-symmetry states $P_s' = 2\langle \psi' | \psi' \rangle$ turns out to be $0.8\%$ (B65a). In figure 10, we have plotted the function for $F(P)$ for the case (i), as well as the functions $F(P) + G(P)$ and $F(P) - G(P)$ for case (ii).

The binding energy for this case obtained with Yamaguchi's effective central force defined in table 1 is still quite large, i.e. 12.9 MeV. However Naqvi has constructed a potential model which includes besides a spin-dependent central force, also tensor and spin-orbit forces. The parameters are given by Bhakar (B65a,b). We also use later in this chapter wavefunctions obtained with only the central force of Naqvi, for estimating the sensitivity of the capture rate to different parameters. With this truncated Naqvi force (central part only) the binding energy of $^3\text{He}$ is found to be 8.48 MeV (B65a).

(ii) The introduction of the tensor force brings about the additional complication of a quartet-$D$ ($^4D_{1/2}$) state component in the ground state wave-function. The wave function of equation (5.8) then gets modified to

$$\Psi = \frac{1}{\sqrt{2}} (A' \mathbf{J}' - A'' \mathbf{J}'')$$  \hspace{1cm} (5.11)
$F$ for $C_Y^{AVR}$

$(F + G)$ for $C_Y^{eff} + S_Y^{eff}$

$(F - G)$ for $C_Y^{eff} - S_Y^{eff}$
with \( A', A'' \) being mixed-symmetric space-spin composite wave functions, and are expressed as

\[
\begin{align*}
(A'') &= \mathcal{D}_E^{-1} \left( A_T + A_S \right) (\chi') \\
A_T &= \sum_{i, j, k} \mathcal{g}(q_{ij}) F_{ij}(P_k) P^+_{\sigma}(ij) \\
A_S &= \sum_{i, j, k} \mathcal{f}(q_{ij}) G_{\sigma}(P_k) P^-_{\sigma}(ij) \\
F_{ij}(P_k) &= F(P_k) + \frac{1}{\sqrt{3}} s_{ij}(\hat{P}_k) I(P_k)
\end{align*}
\]  

Here \( P^+_{\sigma}(ij) \) are projection operators for triplet and singlet spin states respectively. \( F, G \) and \( I \) are the spectator functions. Our notation here is somewhat different from that of (B65b, M66c) to make the later equations look simpler. The explicit forms of \( A' \) and \( A'' \) are given in Bhakar's work (B63a), and are not repeated here.

For future reference we also define \( \Psi_{\mu} \)

\[
(\Psi_s, \Psi_r, \Psi_\delta)
\]

where the subscript denotes the angular nature of the wave-functions.

\[
\Psi_{00}(i) = \Psi_s(i)
\]

\[
= c(q_{jk}) F(P_i) + T(q_{jk}) I C(P_i) P x(q_{jk}, P_i) + s(q_{jk}) G C(P_i)
\]
\[ \Psi_{1\mu}(i) = \{ \Psi_p(i) \}^3_{\mu} \]
\[ = i T(q_{jk}) I(p_i) p_1(q_{jk} \cdot p_i)(q_{jk} \times p_i)_{\mu} \]
\[ \Psi_{2\mu}(i) = \{ \Psi_d(i) \}^3_{\mu} \]
\[ = \left[ \sum c(q_{jk})^2 + \frac{1}{\sqrt{2}} T(q_{jk})^2 \right] I(p_i) W_{\mu}^2 (\hat{p}_i, \hat{q}_{jk}) \]
\[ + T(q_{jk}) \left( \frac{1}{\sqrt{2}} I(p_i) \right)^2 W_{\mu}^2 (\hat{q}_{jk}, \hat{q}_{jk}) \]
\[ - \frac{3}{\sqrt{2}} T(q_{jk}) I(p_i) p_1(q_{jk} \cdot p_i) W_{\mu}^2 (\hat{p}_i, \hat{q}_{jk}) \] (5.13)

where \( W_{\mu}^2 \) is defined in table 7. One could again define various symmetric combinations of such wavefunctions as done in (5.2 - 5.3). For the construction of \( \Psi_5^S \) (\( \Psi_5^S \) or \( \Psi_5^r \)) the plus (minus) sign preceding the last term in the expression for \( \Psi_5^S(i) \) is the needed one.

The final states for three nucleons for cases of our interest would be \( ^3\text{H} \) or \( n + d \) or \( n + n + p \). In the last two states, the total angular momentum \( J(=1/2) \) would be a good quantum number, but the total spin or isospin may not be a good quantum number. Thus the complete final state wave function can be written as
\[ \Psi_f = A \Psi (s_f = \frac{1}{2}, T_f = \frac{1}{2}, [3]) \]
\[ + B \Psi (s_f = \frac{3}{2}, T_f = \frac{1}{2}, [2, 1]) \]
\[ + C \Psi (s_f = \frac{3}{2}, T_f = \frac{3}{2}) \]
\[ + D \Psi (s_f = \frac{1}{2}, T_f = \frac{3}{2}) \]  
\hspace{1cm} (5.14)

where

\[ \Psi (s_f = \frac{1}{2}, T_f = \frac{1}{2}, [3]) = -\varphi_{\frac{5}{2}}^{\frac{5}{2}} \]
\[ \Psi (s_f = \frac{1}{2}, T_f = \frac{1}{2}, [2, 1]) = \frac{1}{\sqrt{2}} (\varphi_{\frac{3}{2}}^{\frac{3}{2}} - \varphi_{\frac{3}{2}}^{\frac{3}{2}}) \]
\[ \Psi (s_f = \frac{3}{2}, T_f = \frac{1}{2}) = \frac{1}{\sqrt{2}} (\varphi_{\frac{5}{2}}^{\frac{5}{2}} - \varphi_{\frac{5}{2}}^{\frac{5}{2}}) \chi^s \]
\[ \Psi (s_f = \frac{1}{2}, T_f = \frac{3}{2}) = \frac{1}{\sqrt{2}} (\varphi_{\frac{3}{2}}^{\frac{3}{2}} - \varphi_{\frac{3}{2}}^{\frac{3}{2}}) \]  
\hspace{1cm} (5.15)

Here \( \varphi \)'s represent space wave functions with

* In all subsequent equations the pion wavefunction appear only through \( \varphi = 3 \varphi = \varphi_1 + \varphi_2 + \varphi_3 \), and hence there should be no confusion between this, and the space functions of three nucleons being defined in equation (5.15).
different symmetries for the three nucleons. They assume different forms for different final states, and their explicit representations will be discussed later as we calculate each capture rate. The values of $A, B, C$ and $D$ would be decided by the statistical availability of different states.

5.6 Transition amplitude: $\mathcal{M}_{fi}$

We now calculate the matrix elements for $\pi^-$ capture from $1S$ orbit for each component of the final state given in equations (5.14-5.15) with the initial wave-function (5.11). The results for the special cases of $N-N$ forces (ii) and (i) are obtained with suitable approximations viz., for case (ii) $t \to 0$, $I(P^+_K) = 0$, and in addition for case (i) $F + G = F$, $F - G = 0$ etc. To evaluate the spin-Ispin part of the matrix element, we use the double-barred-matrix-element listed in Table 9. Only the non-zero matrix elements are shown there. We define

$$\mathcal{M}_{fi} = -\frac{m}{m} \frac{\alpha}{a_1} \frac{2\sqrt{2}}{2L+1} \times$$

$$\sum_{\mu, m_s} C_{s_i} L_{s_f} \mathcal{M}_{fi}$$

where

$$\mathcal{M}_{fi} = \frac{1}{(2S_{f}+1)(2S_{i}+1) - 1 (2L+1)^{-1}}$$

upon spin-averaging the
matrix element. We shall define $L$ a little later.

In the following we write only expression for $M^A_{fi}$.

Now we give the explicit expressions for the transition matrix element leading to each of the four components with coefficients $A, B, C$ and $D$ in the final state.

$$\begin{align*}
M^A_{fi} &= \frac{9\sqrt{3}}{2}\sum_{l=1}^{4} \left< \phi' | \left( p \otimes \psi' + p'' \otimes \psi'' \right)_{L=1}^{l} \right|_{\mu} \\
&- \frac{9\sqrt{3}}{2}\sum_{l=1}^{4} \left< \phi'' | \left( p \otimes \psi' - p'' \otimes \psi'' \right)_{L=1}^{l} \right|_{\mu}
\end{align*}$$

(5.17)

Clearly, this matrix element becomes zero in the absence of the tensor force.

$$\begin{align*}
M^B_{fi} &= \frac{9\sqrt{3}}{4}\sum_{l=1}^{4} \left< \phi' \left| \left( p \otimes \psi' + p'' \otimes \psi'' \right)_{L=1}^{l} \right|_{\mu} \\
&- \frac{9\sqrt{3}}{4}\sum_{l=1}^{4} \left< \phi'' \left| \left( p \otimes \psi' - p'' \otimes \psi'' \right)_{L=1}^{l} \right|_{\mu}
\end{align*}$$

(5.18)
\[ M_c^{\pm} = \left\{ \begin{array}{c}
\frac{3}{2} \left[ \langle \phi \mid \nu_1 \nu_2 \psi_{L=1} \rangle + \langle \phi'' \mid \nu_1 \nu_2 \psi_{L=1} \rangle \right] \\
+ \frac{\sqrt{3}}{2} \left[ \langle \phi \mid \nu_1 \nu_2 \psi' \rangle + \psi' \right] \\
+ \langle \phi'' \mid \nu_1 \nu_2 \psi' \rangle \\
- \frac{9 \sqrt{3}}{4 \sqrt{2}} \left[ \langle \phi \mid \nu_1 \nu_2 \psi_{L=1} \rangle + \langle \phi'' \mid \nu_1 \nu_2 \psi_{L=1} \rangle \right] \\
- \frac{9 \sqrt{3}}{2 \sqrt{2}} \left[ \langle \phi \mid \nu_1 \nu_2 \psi' \rangle + \psi' \right] \\
+ \langle \phi'' \mid \nu_1 \nu_2 \psi' \rangle \\
+ \frac{9 \sqrt{5}}{4 \sqrt{2}} \left[ \langle \phi \mid \nu_1 \nu_2 \psi_{L=2} \rangle + \langle \phi'' \mid \nu_1 \nu_2 \psi_{L=2} \rangle \right] \\
+ \frac{3 \sqrt{5}}{2 \sqrt{2}} \left[ \langle \phi \mid \nu_1 \nu_2 \psi' \rangle + \psi' \right] \\
+ \langle \phi'' \mid \nu_1 \nu_2 \psi' \rangle \\
- \frac{9 \sqrt{5}}{2 \sqrt{2}} \left[ \langle \phi \mid \nu_1 \nu_2 \psi_{L=2} \rangle + \langle \phi'' \mid \nu_1 \nu_2 \psi_{L=2} \rangle \right] \\
+ \langle \phi'' \mid \nu_1 \nu_2 \psi' \rangle + \psi' \right] \\
\right\} \\
\]
Table - 9

\[ \langle x' \parallel \sigma^0 \parallel x' \rangle = \sqrt{3}, \quad \langle x' \parallel \sigma^1 \parallel x' \rangle = -\sqrt{3}, \]
\[ \langle x'' \parallel \sigma^0 \parallel x'' \rangle = \sqrt{3}, \quad \langle x'' \parallel \sigma^1 \parallel x'' \rangle = -\sqrt{3}, \]
\[ \langle x'' \parallel \sigma_1 \cdot \sigma_3 \parallel x'' \rangle = -\sqrt{3}, \quad \langle x'' \parallel i(\sigma_1 \times \sigma_3) \parallel x'' \rangle = 2, \quad \langle x'' \parallel i(\sigma_1 \times \sigma_3) \parallel x' \rangle = 1, \]
\[ \langle x'' \parallel \sigma^0 \parallel x' \rangle = -\sqrt{3}, \quad \langle x'' \parallel \sigma^1 \parallel x' \rangle = -\sqrt{3}, \]
\[ \langle x'' \parallel X^2(\sigma_1, \sigma_3) \parallel x' \rangle = -\sqrt{5/2}. \]

For definition of \( x^2 \) and the double-barred matrix elements see the table 7.

The structure of the matrix element \( M_{\tilde{f}i}^D \) is the same as that of \( M_{\tilde{f}i}^C \), except that the numerical coefficients of terms involving \( \psi_s \) (\( \psi_s', \psi_s'' \)) are multiplied by \(-1/\sqrt{2}\) (\(+1/\sqrt{2}\)), of terms involving \( \psi_P \) (\( \psi_P', \psi_P'' \)) are multiplied by \( \sqrt{2} (-\sqrt{2}) \), and terms with \( L=2 \), as well as those involving \( \psi_P', \psi_P'' \) turn out to be zero. Table 10 contains various formulae needed for this reduction. \( \xi_L \otimes \psi^L \) is the vector product also defined in table 10.
\[ \{ p^\alpha \otimes \psi \}_{\mu} = \sum_{m=0}^{L} \frac{C}{m \mu} \psi_{\mu} \psi_{\mu} \]

\[ \{ a \otimes I \}_{\mu} = (a)_{\mu} \]

\[ \{ a \otimes t \}_{\mu} = \frac{i}{\sqrt{2}} (a \times b)_{\mu} \]

\[ \{ a \otimes q \}_{\mu} = W_{\mu}(a, t) \]

\[ \{ a \otimes W_{\mu}(t, r) \}_{\mu} = \left( -\frac{3}{2 \kappa^2} \right) \left[ (a \cdot t) r + (a \cdot r) t - \frac{2}{3} (t \cdot r) a \right] \]

\[ \{ a \otimes W_{\mu}(t, r) \}_{\mu} = \left( \frac{i}{\kappa} \right) \left[ W_{\mu}(a \times t, r) + W_{\mu}(a \times r, t) \right] \]

where \( a, t \) and \( r \) are arbitrary vectors.
We note a few selection rules here. The transition operator connects the major component $\psi^S_S$ of the initial state to only the $S_f = 3/2$, $T_f = 1/2$ and $S_f = 1/2$, $T_f = 3/2$ components of the final state. The pseudoscalar nature of the pion prohibits transition to the final state with $S_f = 1/2$, $T_f = 1/2$ from initial state $\psi^S_S$; this is analogous to the selection rule from equation (2.17) which requires that for $S$-wave pion capture either $S$ or $T$ or both must change. We also note that the mixed-symmetric $S$-component of the initial state cannot lead to the (partially) fully symmetric final state.

Before we proceed to actual calculation of the capture rates for different modes, we shall show how different terms occurring in $M_{fi}$ can be expressed in forms which are more useful for numerical evaluation, and also discuss the physical interpretation of such terms. We have normalised $\psi^S$, $\psi'$ and $\psi''$ according to the definitions of $\chi^S$, $\chi'$ and $\chi''$ (see equation (5.7)) rather than as for $\psi^S$, $\psi'$ and $\psi''$ as given in equations (5.10). Then the matrix elements contain terms of the following types which can written as,

$$
\langle \psi' | \sum_i \theta \otimes \psi^S_{i}^{L} \rangle + \langle \psi'' | \sum_i \theta \otimes \psi^S_{i}^{L} \rangle = \sqrt{\frac{3}{2}} \sum_{i,k} \langle \phi(i) | \sum_i \theta \otimes \psi^S_{i}^{L} \rangle (5.20)
$$
and similarly

\[
\left[ \langle \phi' | \hat{p} \otimes \psi'_e + \hat{p}'' \otimes \psi'_e \overrightarrow{\mathcal{f}}_{\mu} \right] \\
+ \langle \phi'' | \hat{p} \otimes \psi'_e - \hat{p}'' \otimes \psi''_e \overrightarrow{\mathcal{f}}_{\mu} \right]
= \sqrt{\frac{3}{2}} \sum_i \langle \phi(i) | \hat{p} \otimes \psi(i) \overrightarrow{\mathcal{f}}_{\mu} \right]
+ \hat{p} \otimes \psi(k) + \hat{p} \otimes \psi(j) \overrightarrow{\mathcal{f}}_{\mu} \right]^{L} \tag{5.21}
\]

\[
\langle \phi^s | \hat{p} \otimes \psi'_e + \hat{p}'' \otimes \psi''_e \overrightarrow{\mathcal{f}}_{\mu} \right]
= \sqrt{\frac{3}{2}} \sum_{ik} \langle \phi(i) | \hat{p} \otimes \psi(k) \overrightarrow{\mathcal{f}}_{\mu} \right] \tag{5.22}
\]

The major contribution to the capture rates comes from the equation (5.20) with \( l = 0, L = 1 \). Let us examine its structure in some detail. It reduces to

\[
\sum_i \langle \phi(i) | (\hat{P}_i)_{\mu} | \psi_s(i) + \psi_s(2) + \psi_s(3) \rangle
\]

The interaction operator operates upon the \( i \)th nucleon which captures the pion. In the sum we shall have following terms;
This may be interpreted as capture on the uncorrelated $i$-nucleon, which would emerge with a high-momentum, leaving the remaining $n-p$ pair bound as a deuteron or in a fragmented state. Physical arguments suggest that the contribution of such a term should be negligible. We shall see later that indeed the contribution is small, but not altogether negligible, since this spectator nucleon in our model does have some dynamical interaction effects in its wave-function.

In this case the nucleon belongs to an interacting pair, and so this term represents truly a pair-capture. However in the final state the pair may emerge either as a bound state (deuteron) or as free nucleons ($np$ or $nn$ pair) leaving the spectator nucleon $K$ ($n$ or $p$) with small momentum. Alternatively, one of the nucleons in the absorbing pair may attach itself to the $K$-nucleon emerging then as a fast deuteron. This of course requires a rather specific structure of the initial state and final states. Schematic representations of processes discussed above are shown below.
5.D Deuteron mode.

Kinematics dictate that in this mode the final state consists of a neutron and a deuteron flying away.
from each other in a relative P-state. This final state cannot contain the component \( T_f = 3/2 \), i.e. the term with coefficient \( D \) in equation (5.14).

The space-part of the wave function of the final state (with particle 'i' being a neutron), can be written as

\[
\varphi(i) = \left( \frac{2\pi}{L} \right)^{3/2} \delta^{3} (p_i - k) \times N_d \sum_{j,k} (q_{jk}) (q_{jk}^2 + \lambda_j^2)^{-1}
\]

(5.23)

where we ignore final state-interaction, \( N_d = N_o / \sqrt{4\pi} \) as in equation (2.30), \( K \) is the relative momentum of the n-d system, and energy conservation requires

\[
\frac{3}{4} K^2 - \lambda_d^2 = m_i M - \lambda_f^2
\]

which gives \( K \approx 2\pi^{-1} \) i.e. 400 MeV/c. Such a large relative momentum justifies neglect of the final state interaction.

For calculating the actual numbers for the capture rates \( \psi_d \) for deuteron mode and \( \psi_p \) for proton mode) we keep the terms \( \psi_s^S \) as well as \( \psi_s^' \) and \( \psi_s^" \) in the initial wave function, and also include terms of the order \( \psi(P)/T(P) \) as well as \( T/C \) (compared with CF term) (see p.114, and
further discussion in (G67a) in the transition amplitude. The numerical spectator functions are given in Bhakar's thesis (B65b), and analytical forms fitted to these wave functions are given in (G67a) for cases (ii) and (iii). For case (i) we have calculated the relevant spectator function by numerical integration, and then fitted it to a suitable analytical form as was also done in the previous chapter for $^6$Li. All the angular integrations are facilitated by such analytical representations of the spectator functions, and carried out analytically. The integration over the magnitudes of the vectors are then done numerically.

It should be noted that when the tensor force is included in the initial state as in case (iii), for consistency it should also be included in the final deuteron state. This brings in some additional terms in the transition amplitude. Some of the important additional terms within our approximation scheme are given below. Writing

$$\psi(i) = \psi_5(i) + \psi_3(i)$$

we get additional terms.
\[ M_{fi} (\text{C-part}) = \]
\[ \frac{9\sqrt{3}}{2} \left[ \left< \Phi_3' \otimes P' + \Phi_3'' \otimes P'' \right|_{\mu}^{L=1} \left| \psi_s^+ \right> \right] \]
\[ + \frac{9}{\sqrt{2}} \left[ \left< \Phi_3' \otimes P' + \Phi_3'' \otimes P'' \right|_{\mu}^{L=2} \left| \psi_s^+ \right> \right]^2 \]
\[ (5.24) \]

Now we write down explicitly the integrals which have to be evaluated in the momentum space with \( \Phi(i) \) as given in (5.23). Apart from the normalisation factor, these integrals have the following forms.

\[ \left< \Phi'_w \mu \left| \psi_s^+ \right> + \left< \Phi''_w \mu \left| \psi_s^+ \right> \right. \]
\[ = \frac{3\sqrt{3}}{\sqrt{2}} (k_{\mu}) \left[ \int d^3x \frac{\left( \alpha \right)}{(\alpha^2 + \lambda^2)} \left. \cdot \frac{\xi (\infty) F(k) + f(\infty) \alpha(k)^2}{(\alpha^2 + \frac{3}{4} \lambda^2 + \kappa^2)} \right] \]
\[ + 2 \int d^3x \frac{\left( \alpha + \frac{\lambda}{2} \right)}{\sqrt{\left( \alpha + \frac{\lambda}{2} \right)^2 + \lambda^2}} \left. \cdot \frac{C (\alpha + \frac{\lambda}{2}) F(\alpha) + f(\alpha + \frac{\lambda}{2}) \alpha(\alpha)^2}{(\alpha^2 + \frac{3}{4} \lambda^2 + \kappa^2)} \right] \]
\[ \times \left( \alpha^2 + \frac{\lambda^2}{2} \right) \left( \alpha \cdot k + \lambda \right)^{\frac{1}{2}} \]
\[ = \frac{3\sqrt{3}}{\sqrt{2}} (k_{\mu}) \left[ I_{1+} + 2 I_{2+} \right] \]
\[ (5.25) \]
\[\left[ \langle \Phi | \{ p'_\mu | \Psi \rangle \right]^2 + \langle \Phi | \{ p'_{\mu} | \Psi \rangle - | p'_{\mu} | \Psi \rangle^2 \right] \]

\[= \frac{3 \sqrt{3}}{\sqrt{2}} \left[ K \right] \mu \int d^3 x \frac{e^{i x \cdot k}}{\left( x^2 + \alpha^2 \right)} \frac{x \left( x + k/2 \right) \left( 1 + x \cdot k/k^2 \right)}{\left( x^2 + k^2 + \alpha^2 \right)^2} \frac{e^{i x \cdot k} - f(x + k/2) G(x)^{\frac{1}{2}}}{x \left( x + k/2 \right) \left( 1 + x \cdot k/k^2 \right)} \]

\[= \frac{3 \sqrt{3}}{\sqrt{2}} \left[ K \right] \mu \int d^3 x \frac{e^{i x \cdot k}}{\left( x^2 + \alpha^2 \right)} \frac{e^{i x \cdot k} - f(x + k/2) G(x)^{\frac{1}{2}}}{x \left( x + k/2 \right) \left( 1 + x \cdot k/k^2 \right)} \]

We also write down the general structure of the integral arising from $\Psi_D$ as well as $\Psi_D$. Taking them all together as $I_{T+}$, we have

\[I_{T+} = \frac{3 \sqrt{3}}{\sqrt{2}} \left[ K \right] \mu \int d^3 x \frac{e^{i x \cdot k}}{\left( x^2 + \alpha^2 \right)} \frac{e^{i x \cdot k} - f(x + k/2) G(x)^{\frac{1}{2}}}{x \left( x + k/2 \right) \left( 1 + x \cdot k/k^2 \right)} \]

\[= \frac{3 \sqrt{3}}{\sqrt{2}} \left[ K \right] \mu \int d^3 x \frac{e^{i x \cdot k}}{\left( x^2 + \alpha^2 \right)} \frac{e^{i x \cdot k} - f(x + k/2) G(x)^{\frac{1}{2}}}{x \left( x + k/2 \right) \left( 1 + x \cdot k/k^2 \right)} \]

(5.26)
Here \( r, s \) are numerical factors, and \( n_i \)'s are simple functions of their arguments.

It is useful to remark at this stage that the integral \( I_{1+} \) has its origin in the matrix element of the type \( \langle \phi(i) | P_i | \Psi_S(k) \rangle \), i.e. the contribution from the capture on the uncorrelated nucleon, whereas \( I_{2+} \) has its origin from an element like \( \langle \phi(j) | P_i | \Psi_S(k) \rangle \) \( (i \neq k) \), i.e. from capture on a correlated pair. In view of the comments made at the end of the previous section, we should thus expect \( I_{2+} \) to be large, and \( I_{1+} \) small.

The final values for the capture rate \( W_d \) is obtained using Fermi's Golden rule. Table 11 gives the results for case (i) for two different values of \( \tilde{\lambda} \), but the effects of correlation are exhibited...
Table - 11.

<table>
<thead>
<tr>
<th>$E_b$(MeV)</th>
<th>$\lambda$(3-body)$^r$</th>
<th>$I_{1+}$</th>
<th>$I_{2+}$</th>
<th>$\mathcal{W}_d(10^{16}\text{sec}^{-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.43</td>
<td>0.352</td>
<td>0.1745</td>
<td>0.4642</td>
<td>1.89</td>
</tr>
<tr>
<td>8.29</td>
<td>0.328</td>
<td>0.1470</td>
<td>0.4291</td>
<td>1.21</td>
</tr>
</tbody>
</table>

Table - 12

<table>
<thead>
<tr>
<th>Potential</th>
<th>$E_b$(MeV)</th>
<th>$I(S)$</th>
<th>$I(S')$</th>
<th>$I(T)$</th>
<th>$\mathcal{W}_d(10^{16}\text{sec}^{-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{avr}$</td>
<td>12.43</td>
<td>1.103</td>
<td>.</td>
<td>.</td>
<td>1.89</td>
</tr>
<tr>
<td>$C_Y^{\text{eff}}+S_Y^{\text{eff}}$</td>
<td>12.9</td>
<td>1.507</td>
<td>-.044</td>
<td>.</td>
<td>1.35</td>
</tr>
<tr>
<td>$C_N+S_N$</td>
<td>7.04</td>
<td>1.037</td>
<td>-.0253</td>
<td>.</td>
<td>0.78</td>
</tr>
<tr>
<td>$(C+T)_Y+S_Y^{\text{eff}}$</td>
<td>10.40</td>
<td>1.270</td>
<td>-.0274</td>
<td>-.134</td>
<td>0.795</td>
</tr>
<tr>
<td>$(C+T)_N+S_N$</td>
<td>8.85</td>
<td>1.194</td>
<td>-.0237</td>
<td>-.082</td>
<td>0.66</td>
</tr>
</tbody>
</table>

explicitely. We note that although $I_{1+}$ is smaller than $I_{2+}$, it is by no means negligibly so. This is once again an illustration of the statement made earlier that in our model even the spectator nucleon has some effects of N-N interactions built into it. Table 12 contains the results for a variety of different NLS potentials.
We would like to point out some essential features of the results contained in tables 11 and 12. The capture rates calculated here are rather large compared to value obtained by Diwakaran which is \((1.40 \pm 0.51) \times 10^{15} \text{ sec}^{-1}\). However, Figureau and Ericson obtain 0.454, 1.93 and \(1.06 \times 10^{16} \text{ sec}^{-1}\) for gausssian (G) Irving-Gunn (I-G) and modified Irving-Gunn wavefunctions respectively. As we pointed out earlier Beckstein-Diwakaran version of pion-nucleon interaction contains implicitly the effects of nuclear correlations or short-range nuclear forces. It is obvious that the NLS interaction used here must also introduce quite sizable correlations in the nuclear wavefunction although it does not contain an explicit short-range repulsion term.

The effect of including the \(S^{'}\)-states and the D-state (Via tensor force) is to reduce the capture rates as seen from table 12. For Yamaguchi potentials the capture rate is reduced by about 12 \% for inclusion of \(S^{'}\) state and by about 50 \% again for inclusion of tensor force. The Naqvi version of potentials (besides giving better binding energies) gives somewhat smaller capture rates, and the effect of including tensor force is not so severe. We note here that the
full Naqvi potential contains also an additional L.S potential term which we ignore here and use only a truncated potential. The major contribution to the capture rates comes in all cases from the $I(S)$ term.

Before proceeding ahead, a comment may be made regarding pion capture from atomic $P$-state. A simple argument shows that this capture rate is of little importance. Crudely speaking,

$$\frac{W(P\text{-wave})}{W(S\text{-wave})} = \left| \frac{p_\pi \rho_p(\pi)}{\rho_s(\rho)} \frac{m}{M} \right|^2$$

$$\sim \left| \frac{\alpha}{m} \frac{M}{K} \frac{1}{m} \right|^2 \sim 10^{-3} - 10^{-4}$$

Thus $P$-wave pion capture rate is expected to be of the order of $10^{12} - 10^{13}$ sec$^{-1}$. This however is comparable to the radiative $2P \rightarrow 1S$ transition rate which is (M52b) known to be $3 \times 10^{12}$ sec$^{-1}$. One obtains the total $P$-wave capture rate more accurately. To do this we use only the $\psi^5_2$ component of the initial state, take only $p_\pi$ term in the $N\pi$ interaction (equation (2.3)) and then express the interaction in three-body permutation symmetry form, one gets
The overlap integral is proportional to

\[ M_{fi} (P\text{-wave}) = G \frac{2 \sqrt{2}}{q} \lambda \left( -\frac{q \sqrt{3}}{2} \right) \]

\[ \times C_{\frac{1}{2}}^{1/2} \frac{1}{2} \left\langle \Phi^S_{\frac{1}{2}} \Big| \Psi^S_{\frac{1}{2}} \right\rangle \]  

(5.33)

\[(I_{1}^+ + 2I_{2}^+)\), and \(N_{\lambda}\) is the normalisation factor for the pion wave function. The integrals \(I_{1}^+\) and \(I_{2}^+\) are the same as defined in equation (5.25). We then obtain

\[ W_d(P\text{-wave}) = 2.1 \times 10^{12} \text{ sec}^{-1} \]

which is quite comparable to the \(2P \rightarrow 1S\) radiative transition rate. The effect of this factor should be to reduce the absolute value of the pion capture rate from 1S state, since only about half the pion going into atomic orbit reach the 1S state. It seems to us however that since the pion capture rate from 2P-state is so small, the relative ratios for various capture modes from 1S-capture should remain practically unaffected by consideration of P-wave capture. On the other hand, Figureau and Ericson argue that pions captured from 2P-orbit give practically no radiative capture, and hence reduce by 19% \(\left( = \frac{W(P\text{-wave})}{W(2P-1S)} \right)\) the ratios of observed capture rates for comparison to calculated values. We do not really understand their argument.
5.E Proton Mode

For the calculation of the proton capture mode we follow the same treatment as outlined in sections 5C and 5D. In this case \( f(i) \) is defined as

\[
f(i) = \left( \frac{2\pi}{L} \right)^3 \delta \left( \mathbf{p}_i - \mathbf{K} \right) \delta \left( \mathbf{q}_{jk} - \mathbf{l}_{jk} \right)
\]

where \( \mathbf{l}_{jk} \) is the relative momentum of the \((jk)\) pair of nucleons and \( \mathbf{K} \) is the momentum of the \('i'\) nucleon relative to the c.m. of the \((jk)\) pair.

Energy conservation requires

\[
\mathbf{k}^2 + \frac{3}{4} \mathbf{k}^2 + \alpha_T^2 = m M
\]

We do not include here the final-state-interaction for simplicity. Diwakaran has shown (with the closure approximation) that the final-state-interaction has little effect on the capture rate.

Diwakaran has pointed out that using the closure approximation with three particles in the final state taken as plane wave fields the total capture rate, so that the actual proton capture rate is to be obtained by subtracting from this total rate the deuteron mode capture rate \( W_d \). This does not appear to be quite valid to us, since the overlap integral of the \( n-d \) state with the \( n-n-p \) state as defined in the above
equation (5.34') turns out to be zero. Thus the use of equation (5.34) for the final state should yield only the proton capture rate. We have thus taken the results of the capture rate obtained for three final nucleons in plane wave states as directly the proton capture rate in the work of Diwakaran and Ericson and Figureau also. This should be kept in mind in comparison of our results with those of the above authors.

The proton capture rate is calculated following the standard procedure as was done earlier say for $^6\text{Li}$.

Table 13 shows the capture rates for different versions of the non-local separable potentials as was done in table 12 for the deuteron mode. The effects of including the $S'$ state and the tensor force (D-state) are also shown explicitly. Again our results are quite large compared to those of Diwakaran, but comparable to those of Ericson and Figureau. The effect of $Q'$ state and D-state is also again to reduce the capture rates. The famous 3-D cancellation is only about 50%. Cheon's calculation (as was mentioned in the introduction), uses an incorrect value for the size parameter of the $^3\text{He}$ nucleus, his value for the capture rate however is $1.54 \times 10^{17}$ sec$^{-1}$ in the absence of correlation.
Table - 13

<table>
<thead>
<tr>
<th>Potential</th>
<th>$E_b$(MeV)</th>
<th>Capture rate for proton mode ($10^{16}$sec$^{-1}$)</th>
<th>$W_p(S)$</th>
<th>$W_p(S+S')$</th>
<th>$W_p(S+S'+D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cyr</td>
<td>12.43</td>
<td>4.56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cyr + Sy</td>
<td>12.90</td>
<td>4.39 4.19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N + Sn</td>
<td>7.04</td>
<td>2.44 2.37</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(O+T)$\gamma$ + Sy</td>
<td>10.40 3.15 3.02</td>
<td>2.05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(O+T)$\gamma$ + Sn</td>
<td>8.85 2.15 2.39</td>
<td>1.79</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Diwakaran: $W_p = 0.785$ (G)

Figureau: $W_p = 1.30$ (G)

Ericson: $W_p = 3.01$ (I-G)

Diwakaran: $W_p = 1.99$ (Modified I-G)
The experiments on pion capture in $^3\text{He}$ generally measure the momentum spectrum of the outgoing proton. In our formulation of the problem full $I$-spin symmetry of the wavefunctions is used. Thus it is not obvious to identify the outgoing proton. One can obtain the momentum distribution for the $i$th nucleon, but this would be a coherent superposition of the proton as well as the neutron components of the wavefunction. To obtain the momentum-spectrum for the proton, we use the following method. The complete wavefunction given in equation (5.14) is rewritten in terms of the full three-particle antisymmetrisation operator operating on a function in which a proton is separated out. We use the notation where 'u' denotes a proton state and 'n' a neutron state. Then (5.14) is rewritten as

$$
\Psi_f = \frac{1}{3\sqrt{2}} \mathcal{A} \Psi_f^{(1)}
$$

$$
= \frac{1}{3\sqrt{2}} \mathcal{A} \left[ A + B + D \left( \varphi(1) + \varphi(2) + \varphi(3) \right)^2 \right.
$$

$$
+ 3 C \varphi(3) \frac{1}{\sqrt{3}} \chi^5 + \quad
$$
where \( \psi_f^{(1)} \) indicates that the particle number 1 is the proton.

\[
\mathcal{H} = 1 - P_{12} - P_{13} - P_{23} + P_{123} + P_{132}
\]

where the P's are well-known permutation operators. Note that \( \mathcal{H} \) commutes with an operator fully symmetric in all particles (viz., \( H_{\text{int}} \)), and

\[
\mathcal{H} \Psi_i = 3! \Psi_i ,
\]

we get

\[
\mathcal{M}_{f,i} (\mathcal{H}) = \left< \frac{1}{3 \sqrt{2}} \mathcal{H} \psi_f^{(1)} \right| \leq \tau_i \cdot \phi_i (n) \sigma_i \cdot \sigma_i \cdot P_i | \Psi_i \right> \]

\[
= \sqrt{2} \left< \psi_f^{(1)} \right| \leq \tau_i \cdot \phi_i (n) \sigma_i \cdot P_i | \Psi_i \right> \]

and then with the same conventions and approximations...
as in sections C and D, one can obtain for the
transition amplitude

\[
M_{fi}(\mathcal{H}) = \left[ \frac{2}{3}(A+B+D) \left\{ \phi(1)+\phi(2)+\phi(3) \right\} \right] + 3\phi(3)^3 \\
\times \sqrt{\frac{2}{3}} \left\{ \phi \otimes \psi_5 \right\}^{L=1}_\mu + \frac{2\sqrt{3}}{3} \left\{ \phi'' \otimes \psi''_5 \right\}^{L=1}_\mu \\
+ \frac{3\sqrt{5}}{2} \left\{ \phi' \otimes \psi'_5 \right\}^{L=1}_\mu + \frac{\sqrt{15}}{2} \left\{ \phi'' \otimes \psi''_5 \right\}^{L=1}_\mu \\
+ \frac{9\sqrt{5}}{2} \left\{ \phi' \otimes \psi'_5 \right\}^{L=2}_\mu + \frac{3\sqrt{15}}{2} \left\{ \phi'' \otimes \psi''_5 \right\}^{L=2}_\mu \right] \hat{S}_{5,1/2} \\
+ \frac{3}{2} - \frac{3}{2} \left\{ \phi(1)+\phi(2)+\phi(3) \right\}^2 \\
\times \left\{ \frac{1}{3\sqrt{3}} \phi \otimes \psi_5 - \phi'' \otimes \psi''_5 \right\}^{L=1}_\mu \\
+ \left\{ \phi' \otimes \psi'_5 - \phi'' \otimes \psi''_5 - \sqrt{3}\phi' \otimes \psi''_5 \right\}^{L=1}_\mu \\
- \frac{1}{3\sqrt{3}} \phi'' \otimes \psi''_5 \right\}^{L=1}_\mu \\
+ \sqrt{15} \phi'' \left\{ \phi'' \otimes \psi''_5 \right\}^{L=1}_\mu + \sqrt{15} \phi' \otimes \psi'_5 + \phi' \otimes \psi'' \\
+ \frac{1}{3} \phi' \otimes \psi'_5 \right\}^{L=1}_\mu \right] \hat{S}_{5,1/2}
\]

(5.38)
where all the terms which turn out to be zero upon momentum space integration are already dropped.

The proton momentum spectrum can be calculated in a straightforward manner, and the result is exhibited in fig. 11, for wavefunctions obtained with Yamaguchi's non-local interactions $C_{av}^r, C_{eff} + S_{eff}^r$ and $(C+T)+S_{eff}^r$, which in turn show the effect of ignoring both the $S'$ and the $D$-states, including $S'$ states only, and including both $S'$ as well as $D$-states, respectively.

The most striking feature of the momentum spectra shown in Fig. 11 is the absence of a prominent low-momentum peak, which one should expect to arise when the capture takes place on an $n-p$ correlated pair, with the proton a mere spectator. Such a low-momentum peak is indeed observed in experiments (Z67a).

The calculations do give a rather small peak in the low-momentum region, but the correlations dominate to give a much more prominent peak at high momentum, $\sim 320$ MeV/C. The effect of including the mixed-symmetric $S'$ state appears to be to reduce the high-momentum side of the spectrum somewhat, whereas the inclusion of $D$-state (tensor force) results in a pronounced reduction of the low-momentum part of the spectrum. In figure 12 we plot the energy-distribution $(dW/dE_p)$ of the outgoing proton, and the results
are similar to those of Fig. 11. The experimental results of the Russian group are shown in Fig. 13 along with the calculations of Diwakaran (D65a) and Figureau and Ericson (E69a). Clearly the results with uncorrelated wavefunctions and the phenomenological interaction of Eckstein agree much better with experiments than do our calculations. In this connection a brief comment may be made also on the results of Cheon (C66a) and revised calculations of Sakamoto and Totsuki (S67b). Here again the $^3$He wavefunctions are Gaussian wavefunctions, with adhoc introduction of correlations via Jastrow factors, and the pion-nucleon forces used are simple $ps-pv$ first order interactions. One sees from Figs. 2 and 3 of reference (C66a) that the momentum distribution shows only one peak in the large momentum region, in the absence of correlations. Even with a suitable value of the correlation length parameter, only a small peak in the low-momentum region (about 1/3 of the peak on high-momentum side) (see the Fig. 2 in the ref. (S67b)) is obtained. Perhaps this shows the inadequacy of the simple $ps-pv$ interaction.

5.F $\gamma$ -mode.

In the previous sections we considered the capture processes in which a pair of nucleons was involved. On
the other hand, the photocapture and photoproduction of pions by complex nuclei are generally related to single-nucleon processes e.g. \( \pi P \rightarrow n \gamma \) and \( \gamma P \rightarrow nn^+ \) in the impulse approximation; using the CGLN amplitude for the fundamental reactions. Irving and Gunn (G51a) used this method to test various wavefunctions for the tri-nucleon systems. A more sophisticated approach has recently been used by Ericson and Figureau (E67a). Here the tri-nucleon system is treated as a single entity with suitable form factors and the capture rate is calculated using the PCAC theory. In the following work we use a rather simple approach to calculate the capture rate in \( \gamma \)-mode. Our primary purpose is to obtain some estimate of the kind of results one can get for single-nucleon processes with non-local-separable potentials, and this test the appropriateness of the wavefunctions obtained for \( ^3\text{He} \).

Gauge-invariance is invoked to obtain the interaction appropriate for the radiative capture mode. Thus we replace \( P \) by \( P - eA \) in the expressions for the interaction (equation 2.3), and keeping only the terms relevant to this mode, we obtain
\[ H_{\pi N} = \mathcal{G}_K \sum_i \left\{ \sigma_i \cdot (-eA_{i}(\pi)) \gamma_i \cdot \mathcal{P}(\pi) \right\} \]
\[ -\frac{m}{\omega} \Phi(i) \left( \frac{1}{2} \right) \left[ \gamma_i \cdot \frac{1+\gamma_3(i)}{2} \right] \sigma_i \cdot (-eA_{i})^2 \]

\[ (5.39) \]

where \([ , ]_+\) is an anti-commutator, and \(\gamma_3\) is the Z-component of the I-spin. Since \(\gamma_3\) does not commute with \(\gamma\), the expression has been symmetrised.

Further simplifying the expression, we get,
\[ H_{\pi N} = (-e\mathcal{G}_K) \sum_i \left\{ \sigma_i \cdot A_{i}(\pi) \gamma_i \cdot \mathcal{P}(\pi) \right\} \]
\[ -\frac{m}{\omega} \gamma_3 \sigma_i \cdot \Phi(i) \sigma_i \cdot A_{i}^2 \]

\[ (5.40) \]

where
\[ A_{i}(\rho_i,\lambda) = \left( \frac{2\pi}{L} \right)^{3/2} \frac{1}{\sqrt{\omega}} \int (\rho_i - K) \xi_\lambda(K) \]

\[ (5.41) \]

In the above expressions \(K\) and \(W\) are the momentum and energy of the photon and \(\xi_\lambda(K)\) denotes its polarisation vector. Including the recoil of the \(^3\text{H}\) nucleus, we have
\[ \frac{K^2}{2m(^3\text{H})} + K = m, \text{ which gives,} \]
\[ K \approx 0.65 \text{ F}^{-1} = 130 \text{ MeV/c} \]

\[ (5.42) \]

It will be noted that the interaction \(H_{\pi N}\) is simply the dipole term in the photoproduction process,
and in the lowest order theory higher multipole terms will not contribute. Further, since the photon momentum value is far from the resonance region, and (pion being at rest) there is no pionic current, the terms known as 'shaking-off' and 'photo-electric' in the usual field-theory jargon are not important in the non-relativistic region under consideration. We are only including in $H_{\pi\gamma}$ here the wellknown 'catastrophic' term.

The method of calculating the transition amplitude is already described in section 50. We directly write the matrix element here. For simplicity, we include only the symmetric $S$ and $S'$ terms in the initial and final states ($^3\text{He}$ and $^3\text{H}$), and ignore the complications of the tensor force. Then,

$$M_{\gamma} = \left\langle \Psi_f \mid H_{\pi\gamma} \mid \Psi_i \right\rangle$$

$$= \left\{ -e G_F \left( 1 - \frac{m}{2m} \right) \Phi_i^2 \sum \sum \frac{1}{m_s - \mu} \frac{1}{m'} \right\} \times \left[ -\frac{3\sqrt{3}}{2} \langle \Psi^5_s \mid A^5_{\mu'} \mid \Psi^5_i \rangle + \sqrt{3} \langle \Psi^s_s \mid A^5_{\mu'} \mid \Psi^s_i \rangle \right.$$

$$- \sqrt{3} \langle \Psi^s_s \mid A_{\mu}^5 \mid \Psi^5_i \rangle \right\} \frac{2}{f} \tag{5.43}$$

The capture rate $W_{\gamma}$ is calculated using Fermi's golden rule. If the contribution of the fully symmetric $S$-state alone is taken into account,

$$W_{\gamma} = 0.58 \times 10^{16} \text{ sec}^{-1}$$
The contribution of the mixed symmetric states reduces this value to
\[ W_Y = 0.54 \times 10^{16} \text{ sec}^{-1} \]

These values may be compared with \( 0.35 \times 10^{16} \text{ sec}^{-1} \) obtained by Ericson and Figureau (E67a) using PCAC, \( 0.10 \times 10^{16} \text{ sec}^{-1} \) obtained by Diwakaran and \( 0.83 \times 10^{16} \text{ sec}^{-1} \) of Fujii and Hall (F62a) using the CGLN amplitudes.

5.6 Results with Variational Wavefunctions:

The calculations reported in the previous sections have all been carried out in momentum space and with fully antisymmetrised wavefunctions. Thus it was not possible to compare our results at any intermediate steps with those of other authors. In this section we present calculations of capture rates with variational wavefunctions used by other authors, but with different pion-nucleon interaction forms. For example, Figureau and Ericson calculate capture rates with Irving-Gunn (I\_G) or modified Irving-Gunn etc. wavefunctions, but use the Bockstein form of \( \pi NN \) interaction. It should be interesting to see what results one would obtain with the first order \( ps-pv \) \( \pi N \) interaction such as the one we use.

The simplest variational wavefunction viz., a
Gaussian form, can be fitted to the gross properties of $^3$He, but fails to give good agreement with the form factors obtained from electron scattering etc. Irving (I51a) and later Irving and Gunn (G51a) have suggested other forms for the wavefunction which fit better the data, especially for photoproduction of pions. I-G wavefunction gives better agreement with data on electron scattering and radiative neutron capture by deuteron compared to the Irving wavefunction, whereas the latter gives better agreement with muon capture data on $^3$He. Since the I-G wavefunction contains a singularity for small inter-nucleon distances, it fits poorly the form factors for momentum transfers larger than $2\pi^{-1}$. To remedy this Gibson (G67c) has modified the I-G wavefunction at short distances to remove the singularity. Figureau and Ericson (F69a) also consider a somewhat different version of the modified I-G wavefunction.

In view of our result that the inclusion of the mixed-symmetric and D-states in the wavefunctions of $^3$He reduces the capture rates, but gives practically unchanged values of $\frac{\psi_p}{\psi_n}$, and since our interest is only in comparing the results of different variational wavefunctions with our results, we take for simplicity only the S-state in the variational $^3$He wavefunctions.
Since all such wavefunctions are expressed as some functions of 
\[ \sum \chi_{ij}^2 = (r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_1)^2 \]
or equivalently functions of \( (\vec{r} \cdot \vec{R})^2 + \frac{3}{2} \vec{r}^2 \) with \( \vec{r} = r_2 - r_3 \) and \( \vec{R} = r_1 - \frac{1}{2} (r_2 + r_3) \). They become in momentum-space description, simple algebraic functions of \( (p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 = \frac{9}{8} \vec{p}^2 + 3/2 q_{jk}^2 \).

Different wavefunctions used in this section are given below both in coordinate as well as momentum-space representation:

Gaussian

\[
\psi^S_s(x_1, x_2, x_3) = \mathcal{N} \exp \left\{ -\frac{\lambda}{2} (\sum x_{ij}^2) \right\}
\]

\[
\psi^S_s(p_1, p_2, p_3) = \mathcal{N} \exp \left\{ -\frac{\lambda}{2} (p_{23}^2 + \frac{3}{4} p_1^2) \right\}
\]

with \( \lambda = 0.147 \text{ F}^{-2} \)

Irving

\[
\psi^S_s(x_1, x_2, x_3) = \mathcal{N} \exp \left\{ -\frac{\lambda}{2} (\sum x_{ij}^2) \right\}
\]

\[
\psi^S_s(p_1, p_2, p_3) = \mathcal{N} \exp \left\{ -\frac{\lambda}{2} (p_{23}^2 + \frac{3}{4} p_1^2 + \frac{3}{8} \vec{p}^2) \right\}
\]

with \( \lambda = 1.27 \text{ F}^{-1} \)
Irving-Gunn

\[ \psi_s^5 (x_1, x_2, x_3) = \mathcal{N} (x_i^2)^{-1/2} \exp \left\{ -\frac{1}{2} (x_i^2) \right\} \]

\[ \psi_s^5 (p_1, q_{23}) = \mathcal{N} \left[ q_{23}^2 + \frac{3}{4} p_1^2 + \frac{3 \lambda^2}{8} \right]^{-5/2} \]

with: \( \lambda = 0.771 \, \text{F}^{-1} \)

modified Irving-Gunn (F69a)

\[ \psi_s^5 (x_1, x_2, x_3) = \mathcal{N} \left( x_i^2 \right)^{-1/2} \times \]

\[ \left[ \exp \left\{ -\frac{1}{2} (x_i^2)^{1/2} \right\} - \exp \left\{ -\frac{\sqrt{5}}{2} (x_i^2)^{1/2} \right\} \right] \]

\[ \psi_s^5 (p_1, q_{23}) = \mathcal{N}_{\text{modified}} \left[ q_{23}^2 + \frac{3}{4} p_1^2 + \frac{3 \lambda^2}{8} \right]^{-5/2} \]

with: \( \sqrt{5} = 2 \lambda = 1.874 \, \text{F}^{-1} \)

Now we follow the approach of section 5.0,

with the final wavefunctions given by equations (5.23)
or (5.34), and the interaction as given in equation (5.5). We obtain for the transition amplitude $M_{fi}$ (initial state $\psi_s$):

$$M_{fi} = \frac{3\sqrt{3}}{2} \left[ \langle \varphi' | p' | \psi_s \rangle + \langle \varphi'' | p'' | \psi_s \rangle \right]$$

(5.44)

and the proton momentum spectrum always turns out to be

$$\frac{dW}{dk_1} \sim (8/k_{23}^2 + 1/k_{12}^2) k_1^2 k_{23}$$

(5.45)

where $k_1$ is the proton-momentum.

For each of the above wavefunctions, the capture rates $W_d$ and $W_p$ are evaluated analytically, and the results are summarised in table 14. The Gaussian wavefunction gives not only very low results for the capture rates, but also a rather large ratio for $W_p/W_d$. This may be due to the lack of high momentum components in this wavefunction. Other wavefunctions also yield rather small values of capture rates compared to the results for non-local separable wavefunctions or Figureau-Ericson calculations. It is obviously necessary to build into these wavefunctions more correlations as in our approach or to modify the $\pi N$ interaction as in Figureau-Ericson calculation.

The proton momentum distribution is found to
be identical for all the wavefunctions since they are all functions of \((\frac{3}{4} P_1^2 + q_{23}^2)\) which turns out to be constant as a result of energy conservation. The spectrum is plotted in figure 11, and shows only one peak in the high momentum region.

Table - 14

<table>
<thead>
<tr>
<th>Wave function</th>
<th>(W_d \text{ (sec)})</th>
<th>(W_p \text{ (sec)})</th>
<th>(\frac{W_p}{W_d})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>2.11 x 10^{13}</td>
<td>2.72 x 10^{14}</td>
<td>12.9</td>
</tr>
<tr>
<td>Irving</td>
<td>2.25 x 10^{15}</td>
<td>7.24 x 10^{15}</td>
<td>3.22</td>
</tr>
<tr>
<td>I-G</td>
<td>5.48 x 10^{15}</td>
<td>1.31 x 10^{16}</td>
<td>2.39</td>
</tr>
<tr>
<td>modified I-G</td>
<td>2.56 x 10^{15}</td>
<td>7.83 x 10^{15}</td>
<td>.3.06</td>
</tr>
<tr>
<td>NLS</td>
<td>1.89 x 10^{16}</td>
<td>4.57 x 10^{16}</td>
<td>2.41</td>
</tr>
<tr>
<td>experimental</td>
<td></td>
<td></td>
<td>3.6 \pm 0.6</td>
</tr>
</tbody>
</table>
5. H Summary

We summarise in table 15 the results for different capture modes and their ratios obtained with different wavefunctions (using Yamaguchi's spin-independent and spin-dependent central forces and the central + tensor force model), as well as theoretical results of some other authors and the experimental results. It should be noted again that what we call the "proton capture rate $W_p" is not the same as that quoted in Diwakaran or Ericson-Figureau papers (see comments on p. 167).

We see that although the introduction of spin-dependence and tensor terms in NN interactions reduces the various capture rates, the relative ratios remain practically the same. The results with different potentials and methods of calculations appear to be reasonably well in agreement with each other as well as with experimental results. However, none of them gives a really outstanding agreement with experiments.
<table>
<thead>
<tr>
<th>Specification and details of the work</th>
<th>different capture rates ((10^{16} \text{ sec}^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(W_d)</td>
</tr>
<tr>
<td>NLS Wave function. (S) state, (E_b = 12.43) Mev.</td>
<td>1.89</td>
</tr>
<tr>
<td></td>
<td>(S+S') state, (E_b = 12.9) Mev.</td>
</tr>
<tr>
<td></td>
<td>(S+S'+D) state, (E_b = 10.4) Mev.</td>
</tr>
<tr>
<td>Diwakaran - Gaussian wave fn. (g_0^2 = 0.32), (g_1^2 = 0.29)</td>
<td>.140</td>
</tr>
<tr>
<td>Figureau-Ericson Irving-Gunn (g_0^2 = .64), (g_1^2 = .15) Modified</td>
<td>1.93</td>
</tr>
<tr>
<td></td>
<td>(g_0 g_1 &gt; 0) I-G</td>
</tr>
<tr>
<td>Experimental</td>
<td>3.6±0.6</td>
</tr>
</tbody>
</table>
Our purpose in this chapter (as also in the previous chapter) was to check what kind of results for pion capture may be obtained with 'exact' wavefunctions for $^3\text{He}$ obtained with realistic potentials in the non-local separable form. We find that the wavefunctions obtained with such potentials do contain sufficient correlations to give reasonably good values for capture rates even with the first order ps-pv interactions. We have also explicitly shown the relative effects of mixed-symmetric terms in the wavefunctions and of tensor forces in NN interaction. One result is that the effect of tensor forces (via S-D cancellation) on capture rates in $^3\text{He}$ is by no means so alarmingly severe as in the Deuteron. The need for including second-order pion-scattering terms is thus not so obvious. On the other hand, the pion momentum spectrum obtained in our calculations appears defective in the sense that the experimentally observed peak for low proton momenta is rather weak, and is further reduced by inclusion of tensor forces. Also the strong interference between the high and low momentum regions does not allow a clear view of the low momentum peak. We have seen in the last section that even with variational wavefunctions for $^3\text{He}$ which are currently used, there appears only one peak
(at high proton momenta) in the proton momentum distribution, when only first order \textit{ps-pv} interaction is used. On the other hand, Diwakaran and Ericson and Figureau who use the phenomenological pion-nucleon-pair interaction (which presumably already contains the pion-scattering-before-absorption effects) do obtain the low-momentum peak. It is possible that the inclusion of pion-scattering terms is really essential to give the low momentum peak in proton spectrum.