CHAPTER 6

A GENERALIZED NUT METRIC.
A generalized NUT metric

1. Introduction:

This chapter is about a metric which is a generalization of the metric discovered by Newman, Tamburino and Unti (1963). Their metric is generally abbreviated and called NUT metric. This metric is a stationary axially symmetric solution of Einstein's empty space equations.

\[ R_{ik} = 0 \]  \hspace{1cm} (1.1)

In chapter 4 we have expressed the NUT metric in the form

\[ ds^2 = 2( du - 2a \cos \varphi \, d\varphi )dr \]

\[ - \left( r^2 + a^2 \right) \left( d\varphi + \sin^2 \varphi \, d\varphi \right) \]

\[ + \left[ 1 - 2 \left( r^2 + a^2 \right)^{-1} \left( m + a^2 \right) \right] (du - 2a \cos \varphi \, d\varphi)^2 \]  \hspace{1cm} (1.2)

where \( m \) and \( a \) are constants. These constants are interpreted as mass and magnetic monopole-type mass respectively (Demianski, 1972). Misner (1963) introduced a periodic coordinate time and then showed that the NUT metric possesses the strange property that every observer at rest in the coordinate system has
closed time-like geodesic line. This is an unpalatable feature of the NUT solution.

Bonnor (1969) has interpreted the NUT metric as the field of (i) a mass around the origin of the co-ordinates and (ii) a semi-infinite massless source of angular momentum. If a = 0 the metric (1.2) reduces to the well-known Schwarzschild exterior metric for an isolated spherically symmetric mass m. It is also well-known that the NUT metric is of type D in Petrov classification.

The main purpose of the present investigation is obtain a type II empty-space generalization of NUT metric. For the derivation of such a metric, we shall use complex vectorial formalism, developed by Cahen, Debever and Defrise (1967). We have given an outline of the mathematical formulation of this formalism in section 2 of chapter 5.

In the present chapter we shall derive some exact solutions of the field equations

$$E_{pq} = 0, \quad R = 0$$

(1.3)

which are equivalent forms of the field equations (1.1).
The metric and the field equations:

We consider the metric (Patel and Thaker 1981) in the form

\[ ds^2 = 2 \left( du + g \sin \alpha \, dp \right) \, dr - M^2 \left( d\alpha^2 + \sin^2 \alpha \, d\beta^2 \right) - 2L \left( du + g \sin \alpha \, dp \right)^2 \]  

(2.1)

Here \( g \) is a function of \( \alpha \) and \( \beta \) while \( M \) and \( L \) are functions of \( r, \alpha \) and \( \beta \).

Introducing the basic 1 forms

\[ \theta^1 = du + g \sin \alpha \, dp, \quad \theta^4 = dr - L \theta^1 \]

\[ \sqrt{2} \theta^3 = M \left( d\alpha + i \sin \alpha \, dp \right) \]  

(2.2)

\[ \sqrt{2} \theta^5 = \sqrt{2} \theta^2 = M \left( d\alpha - i \sin \alpha \, dp \right) \]

we can express the metric (2.1) as

\[ ds^2 = 2 \left( \theta^1 \partial^1 - \theta^2 \theta^3 \right) \]  

(2.3)

Using (2.2) we can obtain \( d\theta^a \), which by using the defining expressions

\[ Z^1 = \theta^3 \theta^4, \quad Z^2 = \theta^1 \theta^2, \quad Z^3 = \frac{1}{2} \left( \theta^1 \theta^4 - \theta^2 \theta^3 \right) \]
for $Z^p$, will give us $dZ^p$. Using these expressions for $dZ^p$ in Cartan's first structure equation, we can determine the complex connection 1-forms $\sigma^-_p$. The explicit expressions for $\sigma^-_p$ are given by

\[
\sigma^-_1 = -2 \left[ \left( \frac{M_r}{M} - i \frac{f}{M^2} \right) \theta^2 \right]
\]

\[
\sigma^-_2 = - \left( \frac{\sqrt{2}}{M} \right) \left[ L_r - i L_\phi \cosec \alpha \right] \theta^1
\]

\[
+ 2L \left[ \left( \frac{M_r}{M} - i \frac{f}{M^2} \right) \theta^3 \right]
\]

\[
\sigma^-_3 = -2 \left( L_r + iL/M^2 \right) \theta^1 - \sqrt{2} (F - iE) \theta^4
\]

\[
+ \sqrt{2} (F + iE) \theta^3 + 2i \left( \frac{f}{M} \right) \theta^4
\]

Here $2f = e_\alpha + gcot \alpha$, $M^2F = M_\phi + M \cot \alpha$

$M^2E = M_\phi \cosec \alpha$ and the suffixes denote partial derivatives viz

$L_r = \frac{\delta L}{\delta r}, \quad L_{r_2} = \frac{\delta L}{\delta \phi \delta z}$

e.t.c.

The absence of the terms involving $\theta^3$ and $\theta^4$ in $\sigma^-_1$ indicates that the congruence $k^\alpha$ of null tangents is geodesic as well as shear-free.
One can now use \( \sigma_p \) given by (2.4) and Cartan's second structure equation to obtain the complex curvature 2-forms \( \Sigma_p \). The expression for \( \Sigma_p \) are lengthy and recorded in the appendix (3) for ready reference. These expressions for \( \Sigma_p \), the result

\[- \frac{1}{2} R \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = \Sigma_p \wedge Z^p\]

and the result

\[\Sigma_p = C_{pq} Z^q \cdot \frac{1}{6} R \gamma_{pq} Z^q + E_{pq} - q \text{ will} \]

then determine \( E_{pq} \), \( R \) and \( C_{pq} \). The expression for \( R \) is

\[R = 2L_{rr} + 8L_r (M_r/M) + 4L \left[ (M_r/M)^2 + (r/M^2)^2 \right] \]

\[\quad - (2/M^2) \left[ (M_{\xi}/M)^2 + (M_{\eta}/M)^2 \cosec^2 \xi + (M_{\eta}/M) \cot \xi - 1 \right] \]

\[+ 4LE_{11} \]

Also the expressions for \( E_{pq} \) are given by

\[E_{11} = 2 \left[ (M_r/M)^2 + (M_r/M)^2 - (r/M^2)^2 \right] \]

\[E_{12} = 0 \]
\[ E_{13} = \frac{1}{M} \left[ (M_r/M)\alpha + (f/M^2)\beta \csc \alpha \right] \]

\[ E_{22} = \frac{1}{M^2} \left[ L\alpha + L\cot \alpha + L\beta \csc^2 \alpha \right] + L^2 E_{11} \]

\[ E_{23} = \frac{\sqrt{2}}{M} \left[ L\alpha + L(M_r/M)\alpha - \alpha \left( L(f/M^2)\beta + 2(f/M^2)\beta \right) \csc \alpha \right] \]

\[ - i \left\{ L\alpha + L(M_r/M)\beta \right\} \csc \alpha \]

\[ - i \left\{ 2L\alpha(f/M^2) + L(f/M^2)\beta \right\} \]

\[ E_{33} = \frac{2L}{rr} - 4L(M_r/M)^2 + \text{i} j 2L(f/M^2)^2 \]

\[ + (2/M^2) \left[ (M_k/M)\alpha + (M_p/M)\beta \csc^2 \alpha + \mu(M/M)\alpha \csc \alpha \right. \]

\[ \left. + 1 \right] \]

We shall now try to make the metric (2.1) compatible with the field equations

\[ E_{\mu \nu} = 0 \quad \text{and} \quad R = 0 \]
The equations $E_{1l} = 0$ and $E_{13} = 0$ involve only one unknown function $M$. They can be solved to obtain the form of $M$ as

$$M^2 = (f/Y) (X^2 + Y^2) \quad (2.6)$$

where $X = X (r, \alpha, \beta)$, $Y = Y (\alpha, \beta)$ with

$$X_{\alpha} = -Y_{\beta} \cosec \alpha, \quad Y_{\alpha} = X_{\beta} \cosec \alpha, \quad X_\beta = -1 \quad (2.7)$$

We set $R = 0$ and use $M^2$ given by (2.6) and (2.7) to determine the following form of the function $2L$:

$$2L = 2S + \left(2E^* X + 2F^* Y\right) (X^2 + Y^2)^{-2} \quad (2.8)$$

where $S$, $E^*$ and $F^*$ are function of $\alpha$ and $\beta$ and $E^*$ and $S$ are related by

$$E^* + 2 SY = 0 \quad (2.9)$$
Next we consider the equation $E_{33} = 0$. Using the above relations in this equation we find that

$$2S = \left(\frac{Y}{f}\right) \frac{1}{2} \left[ \left(\frac{Y}{f}\right) \sqrt{(f/Y) \csc^2 \alpha - 1} - \left(\frac{Y}{f}\right)^2 \left( (f/Y)^2 + (f/Y)^2 \frac{\partial}{\partial \xi} \csc^2 \alpha \right) \right]$$  \hspace{1cm} (2.10)

where $z = \log \tan \alpha/2$ and $\Delta^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2}$

It then follows from the equation $E_{23} = 0$ that

$$E_\alpha = F_\phi \csc \alpha, \quad E_\phi \csc \alpha = -F_\alpha$$  \hspace{1cm} (2.11)

Finally using the results of this section in the equation $E_{22} = 0$ we find that

$$S_\alpha Y_\alpha + S_\phi Y_\phi \csc^2 \alpha = 0$$  \hspace{1cm} (2.12)

Now $E^*$ is determined from (2.11) and $2S$ from (2.10).

These expression for $E^*$ and $2S$ must satisfy the equations (2.9) and (2.12). We have only one unknown function $f$ (i.e., $g_\alpha + \cot \alpha$) at our disposal. Thus we have one additional equation. Therefore the question of compatibility arises.
However in the case \( f = Y \) we have \( 2S = -1 \) and (2.12) is satisfied identically. Thus for \( f = Y \) the compatibility condition is automatically satisfied.

We shall now take the case \( f = Y \).

3. The case \( f = Y \).

In this case we have from (2.6)

\[
M^2 = X^2 + Y^2
\]  

(3.1)

where \( X \) and \( Y \) satisfy (2.7). The solution of (2.7) can be taken as

\[
X = -r + A \sin \psi \csc \epsilon
\]

(3.2)

\[
Y = -A \cos \psi \cot \epsilon + a
\]

where \( a, A \) are constants of integration. It is apparent from (2.10) that \( 2S = -1 \) and therefore from the result (2.9) we have \( E^* = Y \). Consequently the function \( E^* \) is given by

\[
E^* = X + r - m
\]  

(3.3)

where \( m \) is a constant of integration. Hence on substituting these values \( E^* = Y \), \( 2S = -1 \) and \( F^* \) given by (3.3) in (2.8) we get the form of function \( 2L \) for this case.
It is given by

\[ 2L = -1 + \left[ 2Y^2 + 2 (X + r - m) \right] \left( X^2 + Y^2 \right)^{-1} \quad (3.4) \]

From the result \( f = Y \) we obtain, with the help of the relation

\[ g \cot \alpha + gcot \alpha = 2f = 2Y \]

the value of \( g \). It is given by

\[ g \sin \alpha = -2 \cos \alpha - 2 \sin \alpha \cos \beta \quad (3.5) \]

Thus the final form of the line-element describing the empty space is

\[ ds^2 = -2 (du + g \sin \alpha df) \, dr 
- \left( \frac{2}{X + Y} \right) \left( d\alpha^2 + \sin^2 \alpha d\beta^2 \right) \quad (3.6) \]

\[ + \left[ 1 - \left( 2Y^2 + 2 (X + r - m) \right) X \right] \left( X^2 + Y^2 \right)^{-1} \]

\[ X (du + g \sin \alpha df)^2 \]

where \( X, Y \) are given by (3.2), \( F^* \) by (3.3) and \( g \sin \alpha \) by (3.4).
In the case $A = 0$, the $\beta$-dependence of $g$, $M$, and $L$ disappears and we are left with an axially symmetric case. Thus when $A = 0$, it is painless to verify that the metric (3.6) reduces to the well-known NUT metric (1.2). We designate the metric (3.6) as the generalized NUT metric.

We know that the coefficients $C_{pq}$ occuring in

$$F_p = C_{pq} \gamma^q - \frac{1}{6} R_{pq} \gamma^q + E_{pq} \gamma^q \gamma^q$$

are related to the five Newman Penrose components $\Psi_A$ in terms of which the Petrov classification can be made. The relationship between $C_{pq}$ and $\Psi_A$ is given by the equation (2.19) of chapter 5.

We have obtained the expressions of $\Psi_A$ for the metric (3.6). These expressions are very lengthy and therefore are not given here. These expressions are listed in the appendix (4) for reference. We have verified that

$$\Psi_0 = \Psi_1 = 0$$

$$2 \Psi_3 - 3 \Psi_2 \Psi_4$$

(3.7)
Hence we conclude that this empty space-time is of type II and not of type I (Carmeli and Kaye 1977, Akabari 1980).

However if $A = 0$ then we have seen that

$$2\psi_3^2 = -3\psi_2 \psi_4$$

and the metric becomes of type $D$. Thus the generalized NUT metric is of type II in Petrov classification.

Also we have obtained the Newman-Penrose spin coefficients (1962) by the help of $dZ^P$ and $\sigma_P$ for the case $f = Y$ where $X$ and $Y$ are given by (3.2). They are given by

$$\psi = \sigma = k = \lambda = \tau = 0$$

$$\rho = -\frac{A(\sin\phi \csc\alpha - \cos\phi \cot\alpha)}{r^2 + a^2 - 2A (\sin\phi \csc\alpha + \cos\phi \cot\alpha)} - (r - ia)$$

$$+ A^2 (\sin^2 \phi \csc^2 \alpha + \cos^2 \phi \cot^2 \alpha)$$

$$\nu = -\frac{1}{\sqrt{2}} \left[ (r^2 + a^2) - 2A (\sin\phi \csc\alpha + \cos\phi \cot\alpha) 
+ A^2 (\sin^2 \phi \csc^2 \alpha + \cos^2 \phi \cot^2 \alpha) \right]^{-\frac{3}{2}}$$
\[
x \left[ 2 \left( \cos \phi \sin \alpha (a - \Lambda \cos \beta \cot \alpha) - \text{is} \sin \phi \cos \alpha \cosec^2 \alpha \right) \right.
\]
\[
x \left( a - \Lambda \cos \beta \cot \alpha \right) \right] \left[ 2 \right.
\]
\[
+ \alpha \left( (r+m) \cosec^2 \alpha (\sin \phi \cos \alpha + \cos \beta) \right)
\]
\[
- 2\Lambda \cosec^3 \alpha (\sin^2 \phi \cos \alpha + \cos \beta)
\]
\[
- 2 \left(r^2 + a^2 - 2\Lambda \left( \text{rs} \sin \phi \cosec \alpha + \cos \beta \cot \alpha \right) \right)
\]
\[
+ \left( a \right)^2 (\sin^2 \phi \cosec^2 \alpha + \cos^2 \beta \cot^2 \alpha) \right] \left[ -1 \right.
\]
\[
x \left( mr + a^2 \right) - \Lambda \left( m + r \right) \sin \phi \cosec \alpha
\]
\[
- 2\Lambda \cos \phi \cot \alpha + \left( \cosec^2 \alpha + \cos^2 \beta (\cot^2 \alpha - \cosec^2 \alpha) \right)
\]
\[
x \left( r + ia \right) \sin \phi \cos \alpha \cosec^2 \alpha + (a + ir) \cos \beta \cosec^2 \alpha
\]
\[
- \Lambda \cos \phi \sin \phi \cosec^3 \alpha (\sin \phi + \cos \alpha \cos \beta)
\]
\[
- \Lambda \cosec^2 \alpha \left( \cos^2 \phi \cot \alpha + \text{is} \sin \phi \cosec \alpha \right)
\]
\[
\mathcal{M} = \left\{ r^2 + a^2 - 2\Lambda \left( \text{rs} \sin \phi \cosec \alpha + \cos \beta \cot \alpha \right) \right.
\]
\[
+ \left( a \right)^2 (\sin^2 \phi \cosec^2 \alpha + \cos^2 \beta \cot^2 \alpha) \right] \left[ -1 \right.
\]
\[
\begin{align*}
\gamma &= \frac{1}{2} \left[ r^2 + a^2 - 2\Lambda \left\{ r \sin \beta \csc \alpha + a \cos \beta \cot \alpha \right\}^2 \\
&\quad + \frac{\Lambda^2}{2} \left\{ \cot^2 \alpha + \sin^2 \beta \left( \cot^2 \alpha - \csc^2 \alpha \right) \right\} \right] \\
x &= \sqrt{\frac{a^2 - r^2}{2}} + m - \Lambda \left( m \sin \beta \csc \alpha + a \cos \beta \cot \alpha \right) \\
&\quad + \frac{\Lambda^2}{2} \left\{ \cot^2 \alpha + \sin^2 \beta \left( \cot^2 \alpha - \csc^2 \alpha \right) \right\} \\
&\quad + \frac{1}{2} \left\{ \sin^2 \beta \csc^2 \alpha + \cos^2 \beta \cot^2 \alpha \right\} \\
&\quad \left( r - a \right) + 2\Lambda \left( a \cos \beta \cot \alpha - r \sin \beta \csc \alpha \right) \\
&\quad + \Lambda^3 \left( \sin \beta \csc \alpha + \cos \beta \cot \alpha - \cot^2 \alpha \right) \right\} \right] \\
&\quad \left( m + \frac{1}{2} \cos \beta \cot \alpha \right) - \left( m + \frac{a}{2} \right) \right\} \right]
\end{align*}
\]
\[ \alpha = -\frac{A}{2\sqrt{2}} \left[ r^2 + a^2 - 2A \left( r \sin \beta \csc \alpha + a \cos \beta \cot \alpha \right) \right]^{\frac{1}{2}} + A^2 \left( \sin^2 \beta \csc^2 \alpha + \cos^2 \beta \cot^2 \alpha \right) \]

\[ \beta = \frac{A}{2\sqrt{2}} \left[ r^2 + a^2 - 2A \left( r \sin \beta \csc \alpha + a \cos \beta \cot \alpha \right) \right]^{\frac{1}{2}} + A^2 \left( \sin^2 \beta \csc^2 \alpha + \cos^2 \beta \cot^2 \alpha \right) \]
\[
\left\{ r + ia - \lambda \left( \sin \phi \csc \alpha + i \cos \phi \cot \alpha \right) \sin \phi \cos \alpha \right\\
+ \left\{ a - ir - \lambda \left( \cos \phi \cot \alpha - i \sin \phi \csc \alpha \right) \cos \phi \right\} \csc \alpha \right\\
\]

\[
x \left( r^2 + a^2 - 2 \lambda \left( r \sin \phi \csc \alpha + a \cos \phi \cot \alpha \right) \right) \right]^{-1} \right\\
+ \cot \alpha \right]
\]

\[
\zeta = -\frac{i}{2} \left( -\lambda \cos \phi \cot \alpha + a \right) \right\\
\]

\[
x \left( r^2 + a^2 - 2 \lambda \left( r \sin \phi \csc \alpha + a \cos \phi \cot \alpha \right) \right) \right]^{-1} \right\\
+ \lambda^2 \left( \sin^2 \phi \csc^2 \alpha + \cos^2 \phi \cot^2 \alpha \right) \right]^{-1} \right]
\]

5. Concluding remarks:

In the analysis of the previous sections, we have obtained a stationary NUT-like empty space-time without any symmetry. This space-time is such that when a certain parameter (i.e., \( \lambda \)) is zero, it reduces to the
well-known NUT metric. The generalized NUT solution obtained by us is of type II in Petrov classification.

In the next chapter we shall consider an extension of the arguments of the present chapter to a situation in which there is a source-free electromagnetic field. We shall obtain a stationary NUT like electrovac space-time as the outcome of this extension.
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