CHAPTER 3

A SPHERICAL SYMMETRIC BODY EMMITTING CHARGED NULL FLUID IN EINSTEIN'S UNIVERSES.
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1. Introduction:

The familiar Schwarzschild's solution describes the gravitational field of a mass particle whose geometry can be described by the metric

$$ds^2 = 2dudr + (1 - 2m/r)u^2 - r^2(d\alpha^2 + \sin^2\alpha d\beta^2)$$  \hspace{1cm} (1.1)

where $m$ is a constant and measures the mass of the particle. Also the field of a charged mass particle is described by the well-known Nordström's metric

$$ds^2 = 2dudr + (1 - 2m/r + \frac{4\pi e^2}{r^2})u^2 - r^2(d\alpha^2 + \sin^2\alpha d\beta^2)$$  \hspace{1cm} (1.2)

where $e$ is the charge of the particle.

The generalization of (1.1) is obtained by Vaidya (1951) which describes the gravitational field of a radiating star with $m = m(u)$. Similarly the generalization of (1.2) is obtained by Bonner and Vaidya (1970) which describes the gravitational field of a radiating charged particle. In all these fields, the geometry at large distances from the particle reduces to that
of special relativity. The metric describing the gravitational field of an uncharged mass particle embedded in an expanding universe is given by McVittee (1933). The space round the mass particle is occupied by a spherically symmetric distribution of matter with non-zero density and isotropic pressure which at large distances from the particle go over smoothly to the cosmic density and pressure in expanding cosmological universe. McVittie's solution is generalized by Vaidya and Shah (1967) to include an electromagnetic field. Also Shah and Vaidya (1968) have obtained a solution of Einstein's field equations which describes the gravitational field of a charged particle embedded in a homogeneous universe.

The generalization of Desitter's and Fredman's cosmological solution with a mass point are available in the literature (McVittee 1933; Tolman 1934). Vaidya and Shah (1957) have discussed a radiating mass particle in an expanding universe. Patel and Shukla (1974) have discussed a radiating charged particle in an expanding universe. The solution given by Patel and Shukla (1974) includes the solution discussed by Vaidya and Shah (1957) as a particular case.
Tupper (1974) has obtained some solutions of the Einstein-Maxwell equations corresponding to electromagnetic fields plus the pure radiation fields. The geometry of these solutions is described by the metric

$$ds^2 = 2dudr + B du^2 - r^2 (d\alpha^2 + \sin^2 \alpha d\beta^2)$$  \hspace{1cm} (1.3)

where $B$ is of the form

$$B = 1 - 2m(u)/r + h(u)/r^2$$  \hspace{1cm} (1.4)

and $m, h$ being function of $u$.

In the absence of source (i.e. when $m = h = 0$), the metric (1.3) becomes flat. Thus the metric (1.3) is described under the flat background. It would be interesting to obtain the metric (1.3) in the cosmological background of Einstein's universe rather than the standard Minkowskian background. The object of the present chapter is to do just that.

Patel and Akabari (1979) have transformed the line-element representing the static Einstein’s universe

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{(xdx + ydy + zdz)^2}{b^2 - (x^2 + y^2 + z^2)}$$  \hspace{1cm} (1.5)

( where $b$ is a constant )
by the following transformation from \((x, y, z, t)\) to the coordinates \((r, \phi, \theta, u)\):

\[
\begin{align*}
    x &= b \sin \left( \frac{r}{b} \right) \sin \phi \cos \theta \\
    y &= b \sin \left( \frac{r}{b} \right) \sin \phi \sin \theta \\
    z &= b \sin \left( \frac{r}{b} \right) \cos \phi \\
    u &= t - r
\end{align*}
\]

Under the transformation (1.6) the metric (1.5) reduces to the form

\[
ds^2 = 2du \, dr + du^2 - b^2 \sin^2 \left( \frac{r}{b} \right) \left( d\phi^2 + \sin^2 \phi \, d\theta^2 \right) \quad (1.7)
\]

where \(b\) is constant.

2. Field equations, metric and the electromagnetic field:

We take the space surrounding the radiating charged particle to be occupied by a spherically symmetric matter distribution of density \(\rho\) and pressure \(p\). We shall use the following field equations.

\[
R^k_i - \frac{1}{2} \delta^k_i R = - 8\pi \left[ \frac{\mathcal{F}^k_i}{\mathcal{E}^k_i} + \sigma^k \xi^i + (p+\rho) \right] V^k_i V^i_k - \delta^k_i \quad (2.1)
\]

\[
+ \Lambda \delta^k_i
\]
where

\[ E^k_i = - F_{ia} F^k a + \frac{1}{2} \delta^k_i F_{ab} F^{ab} \]  \hspace{1cm} (2.2)

\[ F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \]  \hspace{1cm} (2.3)

\[ F^i_j = J^i \]  \hspace{1cm} (2.4)

\[ \xi^i \xi^i = 0, \quad V^i V^i = 1 \]  \hspace{1cm} (2.5)

Here \( \sigma, \lambda, F_{ik}, V^i \) and \( \xi^i \) are respectively the radiation density, the cosmological constant, the components of electromagnetic field tensor, the flow vector of the perfect fluid and the propagation null vector.

Now we take the Einstein's universe (1.7) as the background universe. For the description of our solution we consider the metric in the form

\[ ds^2 = 2dudr + 2L du^2 - b^2 \sin^2 r/b (d\phi^2 + \sin^2 \phi d\theta^2) \]  \hspace{1cm} (2.6)

Where \( L \) is a function of \( r \) and \( u \). The appropriate forms of \( \xi^i \) and \( V^i \) are (Patel and Akabarti 1979):

\[ \xi^i = \delta^i_1, \quad V^i = \left( \frac{1}{2L} \right)^{\frac{1}{2}} \delta^i_1 \]  \hspace{1cm} (2.7)
Here we have assumed that $2L$ is positive. We name the co-ordinates as

$$x^1 = r, \quad x^2 = \zeta, \quad x^3 = \beta, \quad x^4 = \psi.$$ 

The surviving components of the Ricci tensor for the metric (2.6) can be expressed as

$$R^1_1 = - \left[ L_{rr} + 2L_r/b \cot (r/b) - \frac{4L}{b^2} \right]$$

$$R^4_4 = - 2L_u \cot (r/b)/b$$

$$R^2_2 = R^3_3 = \left[ (1 - 2L) \cosec^2 (r/b)/b^2 + \frac{4L}{b^2} - 2L_r \cot (r/b)/b \right]$$

$$R^4_4 = - \left[ L_{rr} + 2L_r \cot (r/b)/b \right]$$

As in chapter 2, our convention is that a suffix denotes partial derivatives e.g. $L_r = \partial L/ \partial r$

$L_{rr} = \partial^2 L/ \partial r^2$ etc....
Following the arguments similar to those made by Tupper (1974) we have to consider the following cases only.

Case (1)

\[ F_{12} = F_{13} = F_{24} = F_{34} = 0 \quad \text{and at least one} \]

\[ \text{of } F_{14}, F_{23} \text{ non zero}. \]

Case (2)

\[ F_{12} = F_{13} = F_{14} = F_{23} = 0 \quad \text{and at least one} \]

\[ \text{of } F_{24}, F_{34} \text{ non zero}. \]

For the sake of brevity, we are not repeating here the arguments made by Tupper (1974).

3. **The solutions of the field equations**

   This section is devoted to give the solutions of the field equations. We shall discuss the solutions for the two cases mentioned earlier (Thaker and Patel, 1982).

Case I:

This case is essentially the same as that discussed by Patel and Akabari (1979). If both \( F_{14} \) and \( F_{23} \) are non-zero then their solution is modified by the addition of
a term representing magnetic monopole.

In this case the Maxwell equation (2.3) and (2.4) gives

\[ F_{1+} = \frac{e(u)}{b^2} \csc^2 (r/b), \quad F_{23} = k \sin \alpha \]  

(3.1)

and

\[ J^i = - \frac{e_u}{b^2} \csc^2 (r/b), \quad 0, 0, 0 \]  

(3.2)

where \( e(u) \) is an arbitrary function of \( u \) and \( k \) is a constant. It can be easily seen that \( J^i \) is a null vector.

Using (3.1) and (2.2), (2.5), (2.7) and (2.3) in the equation (2.1) we obtain

\[ \left( \frac{1}{b^2} - \frac{2L}{b^2} \right) \csc^2 (r/b) - \frac{8\pi}{b^4} (e^2 + k^2) \csc^2 (r/b) = 0 \]  

(3.3)

\[ 8\pi \rho = - 4\pi \frac{L}{b^4} (e^2 + k^2) \csc^4 (r/b) \]  

(3.4)

\[ 8\pi (\rho + \rho) = \frac{4L}{b^2} \]  

(3.5)

\[ 8\pi \sigma = \frac{2Lu}{b} \cot (r/b) \]  

(3.6)
It is painless to see that the solution of the differential equation (3.3) is

$$2L = 1 - \frac{2m}{b} \cot \left( \frac{r}{b} \right) + \frac{4\pi}{b^2} (e^2 + k^2) \left( \cot^2 \frac{r}{b} - 1 \right)$$

(3.7)

where \( m \) is an arbitrary function of \( u \). With this expression of \( 2L \) the expression for \( p, \rho \) and \( \sigma \) become

$$8\pi p = -\Lambda - \frac{2L}{b^2}$$

(3.8)

$$8\pi \rho = \Lambda + \frac{6L}{b^2}$$

(3.9)

$$8\pi \sigma = \frac{2m_0}{b} \cot \frac{r}{b} + \frac{8\pi e u}{b^2} \left( \cot^2 \frac{r}{b} - 1 \right)$$

(3.10)

The final form of the metric can be explicitly expressed as

$$ds^2 = 2dudr - b^2 \sin^2 \frac{r}{b} \left( d\alpha^2 + \sin^2 \alpha \, d\beta^2 \right)$$

$$+ \left[ 1 - \frac{2m}{b} \cot \frac{r}{b} + \frac{4\pi}{b^2} (e^2 + k^2) \left( \cot^2 \frac{r}{b} - 1 \right) \right] du^2$$

(3.11)
when \( k = 0 \), the metric (3.11) reduces to that discussed by Patel and Akabari (1979). When \( k = 0 \) and \( b \) tends to infinity, the metric (3.11) reduces to that discussed by Bonner and Vaidya (1970). Also when \( e \) and \( m \) are constants and \( k = 0 \) the metric (3.11) reduces to the Nordström metric in the cosmological background of Einstein's universe.

In the absence of the source (i.e. when \( m = e = k = 0 \)) the metric (3.11) reduces to the metric (1.7) of the Einstein's universe.

Case II:

In this case the Maxwell's equations (2.3) and (2.4) give

\[
\frac{\delta F_{24}}{\delta r} = \frac{\delta F_{34}}{\delta r} = 0 \quad (3.12)
\]

\[
\frac{\delta F_{24}}{\delta \epsilon} = \frac{\delta F_{34}}{\delta \epsilon} \quad (3.13)
\]

and

\[
J^2b^2\sin^2 r/b = F_{24} \cot \alpha + \frac{\delta F_{24}}{\delta \alpha} + \frac{\delta F_{34}}{\delta \epsilon} \cosec^2 \alpha \quad (3.14)
\]
Here also $J^1$ is a null vector. The differential equation for the function $2L$ is

$$L_{rr} + \frac{(1 - 2L)}{b^2} \csc^2 r/b = 0 \quad (3.15)$$

The solution of (3.15) can be easily seen to be

$$2L = 1 - \frac{2m}{b} \cot r/b \quad (3.16)$$

where $m$ is an arbitrary function of $\dot{u}$. In this case the expressions for $p, \rho$ and $\sigma$ become

$$8\pi p = -\Lambda - 2L/b^2 \quad (3.17)$$

$$8\pi \rho = \Lambda + 6L/b^2 \quad (3.18)$$

$$\sigma^2 = \left[ -\frac{m_{u}}{4\pi} \cos^2 r/b - F_{24} F_{24} - F_{34} F_{34} \csc^2 \alpha \right] \frac{\csc^2 r/b}{b^2} \quad (3.19)$$

suppose that $m_u$ is negative (i.e. the Swarzschild mass is decreasing). Putting $\alpha^2(u) = -\frac{m_{u}}{4\pi}$, Tupper (1974) has given some solutions of (3.12) and (3.13).
Solution (a) :

\[ F_{24} = \alpha(u) \cos \beta \cos \alpha \]  \hfill (3.20)

\[ F_{34} = -\alpha(u) \sin \beta \sin \alpha \]

For this solution \( J^1 \) and \( \sigma \) become

\[ J^1 = -\frac{2\alpha(u)}{b^2} \csc^2 \frac{r}{b} \sin \alpha \cos \beta \]  \hfill (3.21)

\[ \sigma = \frac{\alpha(u)}{b^2} \left[ -1 + \sin^2 \alpha \cos^2 \beta \csc^2 \frac{r}{b} \right] \]

Solution (b) :

\[ F_{24} = \beta(u) \]  \hfill (3.22)

\[ F_{34} = 0 \]

For this solution \( J^1 \) and \( \sigma \) are given by

\[ J^1 = \frac{\beta(u)}{b^2} \cot \alpha \csc^2 \frac{r}{b} \]  \hfill (3.23)

\[ \sigma = \frac{\alpha^2(u) - \beta^2(u)}{b^2} \cot^2 \frac{r}{b} - \frac{\beta^2(u)}{b^2} \]
Solution (c):

\[ F_{2+} = \alpha(u) \sin \alpha \]

\[ F_{3+} = 0 \]

For this, \( \sigma \) and \( \sigma' \) become

\[ J^1 = \frac{2\alpha(u)}{b^2} \cos \alpha \csc^2 \frac{r}{b} \]

\[ \sigma' = \frac{\alpha'(u)}{b^2} \left( \cot^2 \alpha \cot^2 \frac{r}{b} - \sin^2 \alpha \right) \]

When \( b \) tends to infinity in the solutions (a), (b) and (c) we recover the results obtained by Tupper (1974). The metric for case II can be expressed as

\[ ds^2 = 2dudr - b^2 \sin^2 \frac{r}{b} \left( d\alpha^2 + \sin^2 \alpha d\beta^2 \right) \]

\[ + \left( 1 - 2m/b \cot \frac{r}{b} \right) du^2 \]

When \( b \) tends to infinity (3.26) reduces to the metric

\[ ds^2 = 2dudr - r^2 (d\alpha^2 + \sin^2 \alpha d\beta^2) \]

\[ + \left( 1 - 2m/r \right) du^2 \]
It should be mentioned that the above metric is also discussed by Vaidya (1953) as a solution of Einstein-Maxwell equations (without null fluid).

In the absence of the source (i.e., $m = 0$) the metric (3.26) reduces to the (1.7) of the Einstein's universe.

4. Concluding remarks:

In the foregoing lines, we have obtained some exact solutions of Einstein-Maxwell equations. The energy-momentum tensor has been taken to be a combination of a perfect fluid distribution, electromagnetic fields and pure radiation fields. The solution discussed in case I is a simple generalization of the solution discussed by Patel and Akabari (1979). The solutions discussed in case II are Tupper's (1974) solutions described under the cosmological background of Einstein's universe.

The next chapter is intended mainly to investigate a Kerr-NUT metric in connection with the electromagnetic fields. We shall obtain some exact solutions of Einstein-Maxwell equations corresponding electromagnetic fields plus pure radiation fields.
REFERENCES