Chapter - 6

**A RADIATING-NUT METRIC**

1. Introduction:

The Kerr (1963) metric describes the exterior gravitational field of a rotating body and is therefore of particular interest in the study of the gravitational collapse of rotating stars. Many investigators have tried to give interpretation of NUT (Newman, Tamburino and Unti; 1963) metric. According to the interpretation given by Bonnor (1966) and Sackfield (1971), NUT metric represents the exterior gravitational field of a static Schwarzschild body together with a rotating semi-infinite spike of pure angular momentum.

Carmeli and Kays (1975) have given a non-static generalization of Kerr metric which describes the gravitational field of a radiating rotating body. For the
derivation of this generalization they have used the Newman–Penrose (1962, 1963) null tetrad formalism. This null tetrad formalism not only simplify the computation involved but is also most convenient for analysing the physical consequences. Vaidya, Patel and Bhatt (1976) have shown that the usual Kerr metric can be expressed in a form similar to NUT metric. They have given a unified treatment of Kerr (1963) and NUT (1963) metrics. They have also obtained a one-parameter non-static solution of Einstein's equations corresponding to the field of flowing radiation. The new metric is algebraically special and a certain constant parameter put equal to zero it goes over to NUT metric. Therefore it would be interesting to obtain a new non-static generalization of NUT metric which can be interpreted as the metric describing the gravitational field of the radiating Schwarzschild body together with a semi-infinite spike of pure angular momentum. The object of the present Chapter is to do just that. We shall follow the approach adopted by Carmeli and Kaye (1977).

2. Radiating NUT metric

Let us consider the metric in the form

\[ ds^2 = \frac{2}{(2m+2b^2)} (du-2bcos\omega dr)^2 + \frac{2}{(2m+2b^2)} (d\omega + \frac{2m}{r^2+b^2} dr)^2 \]

\[ + \frac{1-(2m+2b^2)/r^2}{(r^2+b^2)} (du-2bcos\omega dr)^2 \]  \hspace{1cm} (2.1)
where \( b \) is a constant and \( m \) is an arbitrary function of \( u \). Here \( m \) is the mass of the body. When \( m \) is a constant, (2.1) reduces to the NUT metric. Clearly when \( b = 0 \), the metric (2.1) reduces to the Veidya's shining star metric. Thus the new metric bears the same relation to the NUT metric as does Veidya's metric to the Schwarzschild metric.

For the metric (2.1) let us introduce the following basic 1-forms.

\[
\begin{align*}
\delta^1 &= du - 2b \cos \alpha \, d\theta; \\
\delta^2 &= \Lambda \, d\alpha; \\
\delta^3 &= \Lambda \, \sin \alpha \, d\theta; \\
\delta^4 &= dr - L \delta^1.
\end{align*}
\]

(2.2)

where

\[
\Lambda = (r^2 + b^2)^{1/2}; \quad L = 1 - 2(r \alpha + b^2)/(r^2 + b^2)
\]

(2.3)

Then the metric (2.1) takes the form

\[
ds^2 = 2e^1 \delta^4 - (\delta^2)^2 - (\delta^3)^2
\]

(2.4)

Using the Cartan's first equation of structure (Chapter 3, (2.17), (2.18)) the non-zero \( \omega^a_b \) are obtained as follow:
\[ \omega_1^1 = -\omega_4^4 = L_x e_1^1 ; \]
\[ \omega_2^1 = \omega_4^2 = (A_x/l) e_2^2 + (b/l^2) e_3^3 ; \]
\[ \omega_3^1 = \omega_4^3 = -(b/l^2) e_2^2 + (A_x/l) e_3^3 ; \]
\[ \omega_1^2 = \omega_2^4 = L(A_x/l) e_2^2 - b(b/l^2) e_3^3 ; \]
\[ \omega_3^2 = -\omega_2^3 = -L(b/l^2) e_1^1 - (\cot\alpha/l) e_3^3 + (b/l^2) e_4^4 ; \]
\[ \omega_3^3 = \omega_4^3 = -L_0(2b\cot\alpha/l) e_1^1 + L(b/l^2) e_2^2 + L(A_x/l) e_3^3 ; \]

Here and in what follows a suffix denotes a partial derivative e.g. \( L_x = \partial L / \partial x \).

Using the Cartan's second equation of structure we can obtain \( \Omega^a_b \) and from \( \Omega^b_b \) we can find \( R_{(ab)} \). The expressions for \( \Omega^a_b \) are listed in the Appendix-3 for ready reference. The non-zero components of \( R_{(ab)} \) are

\[ R_{(12)} = 2r(r^2 + b^2)^{-3/2} (2m'' \cot^2 \alpha + rm') ; \]
\[ R_{(22)} = 4rb^2(r^2 + b^2)^{-5/2} \cot \alpha m' ; \]
\[ R_{(13)} = 2b\cot \alpha (b^2 - r^2)(r^2 + b^2)^{-5/2} m'' . \]
The overhead dash denotes the differentiation with respect to the co-ordinate $u$.

Now we shall solve the field equations

$$ R_{(ab)} - (1/2) \varepsilon_{(ab)} R = T_{(ab)} \quad (2.7) $$

where $R$ is the scalar curvature which is zero in the present case. $T_{(ab)}$ is the matter tensor with $T_{(ab)} = T_{(1)}^{(ab)} + T_{(2)}^{(ab)}$. Here $T_{(1)}^{(ab)}$ corresponds to pure radiation fields and $T_{(2)}^{(ab)}$ corresponds to non-radiation fields. They are defined in the tetrad form as

$$ T_{(1)}^{(ab)} = \delta \xi(a) \xi(b) \quad . $$

$$ T_{(2)}^{(ab)} = \xi(a) \xi(b) + 3( \xi(a) P(b) + \xi(b) P(a) ) $$

$$ + C( \xi(a) \sigma(b) + \xi(b) \sigma(a) ) \quad (2.9) $$

with

$$ \xi(a) = (1, 0, 0, 0) \quad ; $$

$$ P(a) = (0, 1, 0, 0) \quad ; $$

$$ \sigma(a) = (0, 0, 1, 0) \quad ; $$
\[ \sigma \text{ is the radiation density and} \]

\[ A = 4rb(r^2+b^2)^{-3} \cot^2 \alpha \cot m' \quad ; \]

\[ B = -4rb^2(r^2+b^2)^{-5/2} \cot \alpha \cot m' \quad ; \]

\[ C = 2b(b^2-r^2)(r^2+b^2)^{-5/2} \cot \alpha \cot m' \quad ; \]

\[ \sigma = 2m'(r^2+b^2)^{-2}. \quad (2.10) \]

Thus the metric (2.1) satisfies the field equations (2.7) where \( R_{(ab)} \) are given by (2.6) and \( T_{(ab)} = T^{(1)}_{(ab)} + T^{(2)}_{(ab)} \) are given by (2.8), (2.9) and (2.10).

Clearly the energy momentum tensor is divided into two parts, the first \( T^{(1)}_{(ab)} \) is of geometric optics form and the second \( T^{(2)}_{(ab)} \) is a residual contribution.

Asymptotically as \( (b/r) \to 0 \) , one sees that the residual part (2.9) tends to zero. On the other hand the geometrical optics term (2.8) consists of two terms. One of them goes to zero as \( (b/r) \) tends to zero. Hence due to the presence of residual term (2.9) in the energy momentum tensor, only asymptotically the non-stationary generalization of NUT metric behaves like the Vaidya solution.
3. The details of the solution:

In this section we shall give some physical features of the metric (2.1) which are analyzed with the aid of Newman-Penrose formalism.

Let us define the standard tetrad $(\xi_1, m_1, \bar{m}_1, n_1)$ for the metric (2.1) as

\[
\xi_i = \delta_1^4 - 2b \cos \alpha \delta_1^3
\]

\[
m_k = -(1/2) \sqrt{r^2 + b^2} (\delta_2^k - i \sin \alpha \delta_3^k)
\]

\[
n_1 = \delta_1^1 - \left\{ (1/2) - (m(u) r + b^2 (r^2 + b^2)^{-1}) \right\} \xi_1
\]

(3.1)

and $\bar{m}_k$ is the complex conjugate of $m_k$.

To obtain spin coefficients from (3.1) we have used the formalism which is given in the Appendix-4. The spin coefficients are obtained as follows.

\[
k = 0; \quad \lambda = \mu = \sigma = j = 0
\]

\[
c = -ib/2(r^2 + b^2)
\]

\[
p = -\alpha = -\cot \alpha / 2 \sqrt{2} (r^2 + b^2)^{1/2}
\]
\[ \varphi = \frac{(r-ib)}{(r^2+b^2)} ; \]

\[ \mu = (r-ib) \left[ \frac{(rm+b^2)}{(r^2+b^2)-(1/2)} \right] \left( \frac{r^3+b^2}{r^2+b^2} \right)^{-1} ; \]

\[ \gamma = \sqrt{2} \, i \, b \, r (r^2+b^2)^{-3/2} \cot \theta' \]

\[ \gamma = (1/2) \left[ m' (b^2-r^2) - (ib/2) (2rm+b^2-r^2) \right] (r^2+b^2)^{-1} \quad (3.2) \]

The optical scalars \( \psi, \sigma, \omega \) for the metric (1.2) are given by

\[ \psi = r (r^2+b^2)^{-1} ; \quad \sigma = 0 ; \quad \omega = b (r^2+b^2)^{-1} \quad (3.3) \]

The result (3.3) shows that the metric (2.1) admits a shear-free, twisting and diverging null congruence identical to that of NUT metric.

The components of Weyl spinors for the metric (2.1) are given below. The formalism for obtaining these components is given in the Appendix-4. They are

\[ \psi_0 = \psi_1 = 0 ; \]

\[ \psi_2 = \left[ b^2 (b^2+r^2) + rm (3b^2-r^2) - ibm (b^2-3r^2) \right. \]

\[ \left. - 3irb^3 \right] (r^2+b^2)^{-3} \; ; \]
To obtain the Petrov classification of the metric (3.1) we proceed as follows.

Corresponding to the null tetrad \( l^i, n^i, m^i, \bar{m}^i \) at each point in space-time there corresponds a tangent spin space with basis spinors \( k_\mu, n_\gamma \) (spinor indices ranging over the values 0, 1 will be denoted by upper case, Greek letters \( \mu, \gamma, \ldots \), with the normalization \( k_\mu n^\mu = -1 \)). This basis induces the basis

\[
\begin{align*}
\xi_0 \mu \gamma \sigma \delta &= n_\mu n_\gamma n_\sigma n_\delta; \\
\xi_1 \mu \gamma \sigma \delta &= -\epsilon \left( k_\mu k_\gamma n_\sigma n_\delta \right); \\
\xi_2 \mu \gamma \sigma \delta &= \sigma \left( k_\mu k_\gamma n_\sigma n_\delta \right); \\
\xi_3 \mu \gamma \sigma \delta &= -\epsilon \left( k_\mu k_\gamma k_\sigma n_\delta \right); \\
\xi_4 \mu \gamma \sigma \delta &= k_\mu k_\gamma k_\sigma k_\delta. 
\end{align*}
\]

in the 5-dimensional complex space \( \mathbb{C}^5 \) of completely
symmetric 4-spinors. Hence the Weyl spinors \( \gamma_{\mu \nu} \sigma_\delta \)
(which is the spinor equivalent to the Weyl tensor
\( C_{\epsilon \delta \mu \nu} \)) can now be written in terms of basis (3.5) as

\[
\gamma_{\mu \nu} \sigma_\delta = \sum_{m=0}^{3} \gamma_m \xi_m \mu \nu \sigma_\delta
\]

(3.6)

where \( \gamma_m \) are the tetrad components of the Weyl spinor.
However in the present case \( \gamma_0 = \gamma_1 = 0 \). So on
contracting (3.6) with \( k^\sigma k^\delta \) we get

\[
\gamma_{\mu \nu} \sigma_\delta k^\sigma k^\delta = k^\mu k^\nu
\]

Hence \( \gamma_{\mu \nu} \sigma_\delta k^\mu k^\nu k^\sigma = 0 \) and therefore \( \gamma_{\mu \nu} \sigma_\delta \)
must be of the form

\[
\gamma_{\mu \nu} \sigma_\delta = \varepsilon(\mu^\sigma \nu^\delta) \gamma^\sigma \gamma^\delta
\]

(3.7)

with \( \varepsilon_{\mu} \) proportional to \( k^\mu \). Then the solution will
be Petrov type II. (Isaacson, 1963) if \( \varepsilon_{\sigma} \neq \gamma_{\sigma} \)
or of type D (Misner, Lindquist and Schwarz; 1963) if
\( \varepsilon_{\sigma} = \gamma_{\sigma} \). We shall now show that \( \varepsilon_{\sigma} \neq \gamma_{\sigma} \). Writing
the spinors \( \varepsilon_{\sigma} \) and \( \gamma_{\sigma} \) explicitly we have
\[ \xi_{\mu} = p \xi_{\mu} + q n_{\mu} \]
\[ \eta_{\mu} = r \xi_{\mu} + s n_{\mu} \]

where \( p, q, r \) and \( s \) are arbitrary complex functions. \( \xi_{\mu} \) and \( \eta_{\mu} \) will be proportional if \( \xi_{\mu} \gamma^\nu \xi_{\mu} = 0 \), i.e. if \( p s = q r \), that this is not the case can easily be seen as follows.

From (3.6) and (3.7) we have

\[ \alpha(\mu^a \phi^b \sigma^\gamma \phi^c) = \sigma \gamma_2(\mu^a \phi^b \sigma^\gamma \phi^c) - 4 \gamma_3(\mu^a \phi^b \sigma^\gamma \phi^c) \]

+ \[ \gamma_4 \mu^a \phi^b \sigma^\gamma \phi^c \]

Contracting with \( n J n^\gamma n^\sigma n^\delta \) we get

\[ pr = \gamma_4 \quad (3.8) \]

contracting with \( n J n^\gamma n^\sigma \kappa^0 \) we get

\[ (ps + qr) = -4 \gamma_3 \quad (3.9) \]

Finally contracting with \( n J n^\gamma \kappa^0 \kappa^0 \) we get

\[ qs = \sigma \gamma_2 \quad (3.10) \]
Combining equations (3.8), (3.9) and (3.10) we find that the identity \( ps = qr \) is equivalent to \( 3 \gamma_2 \gamma_4 = -3 \gamma_3^2 \), however this equality does not hold, hence \( ps \neq qr \) and the solution is of Petrov type -II.

From (3.4) it is clear that \( 3 \gamma_2 \gamma_4 = 2 \gamma_3^2 \). Hence the metric (3.1) is algebraically special and of Petrov-type-II.

Asymptotically as \((b/r) \to 0\), it can be seen that the dominant term of the energy momentum tensor is \( T^{(1)}_{(ab)} \). Moreover when \((b/r) = 0\), we have verified that \( T^{(1)}_{(ab)} \) satisfies the Rainich, Misner, Wheeler conditions. Asymptotically the radiative field becomes purely electromagnetic.

Here it should be noted that although the \( w^{11} \) term in \( T^{(2)}_{(ab)} \) looks like a null radiation term, it has an asymptotic behaviour of \( \mathcal{O}(r^{-3}) \) which is not the characteristic property of electromagnetic radiation. This is the reason why it is included in the residual term \( T^{(2)}_{(ab)} \). Due to the non-linearity of the Einstein's field equations it seems difficult to give any physical interpretation of the residual term \( T^{(2)}_{(ab)} \) in the energy-momentum tensor.
In this Chapter we have obtained a non-stationary solution of the Einstein's field equations corresponding to the field of a radiating rotating body. The solution is algebraically special and of Petrov type II with twisting, shear-free null rays identical to those of the NUT metric. Asymptotically our solution behave like Vaidya solution. Our metric bears the same relation to the NUT metric as does Vaidya's metric to the Schwarzschild metric in the sense that in both the cases the radiating solution is generated from the non-radiating one by replacing the mass parameter by an arbitrary function of a retarded time co-ordinate.

In the next Chapter we shall discuss the radiating Demianski-type metric and derive many solutions describing pure radiation fields.
References:


