1. Introduction:

Two vacuum solutions of Einstein's field equations describing gravitational field of the rotating bodies are well-known in literature, they are Kerr (1963) and the NUT (1963) solutions. These two solutions have been proved to be of great interest in gravitational theory. Kerr metric in the cosmological background of Einstein's universe has been discussed by Vaidya (1977). It would be interesting to discuss NUT metric in the background of Einstein's static universe. We know that the geometry of Einstein's universe is described by line- element

\[ \text{ds}^2 = \text{dt}^2 - \text{dx}^2 - \text{dy}^2 - \text{dz}^2 - \frac{(\text{dx} + \text{ydx} + \text{zdz})^2}{\text{r}^2 - (x^2 + y^2 + z^2)} \quad (1.1) \]

where \( \text{r} \) is a constant.
We have seen in Chapter-3 that the metric (1.1) can be transformed to the form

\[ ds^2 = 2 \, du \, dr + du^2 - R^2 \sin^2 \left( \frac{r}{a} \right) \left( d\omega^2 + \sin^2 \alpha \, d\theta^2 \right) \tag{1.2} \]

The metric (1.2), \( (r \to \infty) \) is discussed by Kerr and Schild (1965). The NUT metric can be expressed in the following form,

\[ ds^2 = 2 \left( du - 2bcos\alpha \, d\phi \right) dr - (r^2 + b^2) \left( d\omega^2 + \sin^2 \alpha \, d\theta^2 \right) \]

\[ + \left[ 1 - \left( 2mr + 2b^2 \right) / (r^2 + b^2) \right] \left( du - 2bcos\alpha \, d\phi \right)^2 \tag{1.3} \]

where \( m \) and \( b \) are constants. These constants \( m \) and \( b \) are interpreted as mass and magnetic monopole type mass respectively (Demianski, 1973).

The NUT metric represents the exterior gravitational field of a Schwarzschild body of mass \( m \) together with a semi-infinite spike of pure angular momentum (Sackfield, 1969). Misner (1963) introduced a periodic coordinate time and then showed that the NUT metric possesses the strange property that every observer at rest in the coordinate system has a closed time-like world-line. Bonnor (1969) has given an interesting interpretation of the NUT metric according to which the metric describes the field
of a spherically symmetric mass $m$ together with a semi-infinite massless source of angular momentum $b$ along the axis of symmetry. Here it should be noted that when $b = 0$, the metric (1.3) reduces to the well-known Schwarzschild exterior metric. For the description of our solution we shall use the techniques of differential forms. A brief account of exterior calculus of differential forms is given in the next section.

2. Exterior calculus of differential forms

Physics as well as geometry look very simple when one tries to see it locally and this was the motivation behind the approach initiated by Cartan (1928). He tried to reduce the problems of Riemannian geometry to those of Euclidean geometry.

Consider a Riemannian space $\mathcal{M}$ with the metric $g$ given over $\mathcal{M}$. At each point $x \in \mathcal{M}$ we can have a tangent space $T_x$ (the space of contravariant vectors) and the corresponding dual (cotangent) space $T^*_x$ (the space of covariant vectors). The elements of $T^*_x$ are known as 1-forms defined on $\mathcal{M}$. The scalars are known as zero forms.
A tensor of rank $p$ with all of its indices covariant and which is totally anti-symmetric on all indices is call a differential $p$-form or a $p$-form. A totally antisymme- 
trized direct product of forms is denoted by a wedge $\wedge$. The wedge product have the following property. 
If $\omega_1$ is a $p$-form and $\omega_2$ a $q$-form then $\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1$. The exterior derivative of a differential form $\omega$ can be defined axiomatically by the following properties.

(i) If $\omega$ is a $p$-form $d\omega$ is a $(p+1)$ form.

(ii) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

(iii) For a zero-form $f$ and an arbitrary vector $\lambda$, $d f(\lambda) = \lambda(f)$ i.e. $d$ is an ordinary differential of $f$.

(iv) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{pq} \omega_1 \wedge d\omega_2$ where $\omega_1$ is a $p$-form.

(v) $d(d\omega) = 0$, for any $p$-form $\omega$.

The exterior calculus of differential forms has been used extensively in Mathematical physics. The detailed discussion of differential forms and their applications to
physical sciences is given by Westenholz (1978). Israel (1978) has given a lucid description of differential forms in general relativity. We shall use his notations.

Let $g^a_a$ be the basis for $T_x$. Throughout the remainder of this thesis, the Greek indices will be used to refer to the co-ordinates $x^\gamma$ and the tensor components with respect to them. First half of the Latin indices (e.g. $a, b, c, \ldots$) will refer to the frame $g^a_a$ and the frame components (tetrad components in four dimensions) of a tensor. Thus the frame components of a tensor say $T^\gamma_\alpha$, are

$$T_{\alpha \beta} = e^\gamma_\alpha e^\beta_\beta T^\gamma_\gamma$$

$e^\gamma_\alpha$ are contravariant components of the vector $g^a_a$.

We now introduce an inner product $g_{\alpha \beta}$ as

$$g_{\alpha \beta} = g^a_a g^b_b = e^\gamma_\alpha e^\gamma_\gamma g_{\alpha \beta}$$

Then, since $g^a_a$ form a basis for $T_x$ it follows that the inverse matrix $g^{ab}$ and consequently the dual basis $g^a_a g^b_b$ will exist. We find from (3.2) that
\( e^{(a)} e^{(b)} = \delta^a_b \)

Thus matrices \( e^{(a)}_\alpha \) and \( e^{\alpha}_{(a)} \) are inverse of each other. Introducing basis 1-forms

\[ e^\alpha = e^{(a)}_\alpha dx^\alpha \]  \hspace{1cm} (2.3)

the line element can be written as

\[ ds^2 = e_{\alpha\beta} dx^\alpha dx^\beta = e_{ab} e^a e^b \]  \hspace{1cm} (2.4)

A bivector space (which can be easily extended to a multilinear space or the space of \( r \)-forms) or a space of two forms denoted by \( \wedge^2 T_x \) can be formed which will have basis \( e^a \wedge e^b \). Here \( \wedge \) denotes the wedge product. Any bivector, say \( P \) can be expressed as

\[ P = (1/2!) P_{ab} e^a \wedge e^b \] \hspace{1cm} (2.5)

\( P_{ab} \) is an anti-symmetric tensor.

In the absence of torsion in the Riemannian space the affine connection 1-form \( \omega_{ab} \) and the curvature 2-form \( \Omega_{ab} \) are given by the following two equations known as Cartan's equations of structure:
\[ d\epsilon^a + \omega^a_b \wedge \epsilon^b = 0 \] (2.6)

\[ \Omega^a_{\ bc} - d \omega^a_{\ bc} = \omega^a_c \wedge \omega^c_b = 0 \] (2.7)

Here \( d \) denotes the exterior differentiation.

The connection 1-form \( \omega_{\ ab} \) and the curvature 2-forms \( \Omega_{\ ab} \) are related to Ricci-rotation coefficients \( \Gamma_{\ abc} \) and the curvature tensor \( R_{\ abcd} \) as follow:

\[ \omega_{\ ab} = \Gamma_{\ abc} \epsilon^c \] (3.8)

\[ \Omega_{\ ab} = (1/2) R_{\ abcd} \epsilon^c \wedge \epsilon^d \] (3.9)

The connection 1-form \( \omega_{\ ab} \) can be used to describe the covariant differentiation of a tensor field by the formula:

\[ d^{\cdot}_{\ ab\ldots} = d\epsilon^c_{\ b\ldots} + \omega^c_{\ b\ldots} \epsilon_{\ ab\ldots} \ldots \] (3.10)

\( \epsilon_{\ ab\ldots} \) being an arbitrary tensor. The domain of the definition can be extended from the tensorial fields to tensorial r-forms.
If we assume that 1-form $\omega_{\text{ab}}$ are torsion free (and this will be the case in this and the remaining chapters of the thesis) it follows by using (2.10) that

$$\Gamma g_{\text{ab}} = d g_{\text{ab}} - \omega_{\text{ab}} = 0 \quad (2.11)$$

Here and in what follows the round and the square brackets including indices will indicate symmetrization and anti-symmetrization respectively.

The tetrad components $R_{(ab)}$ of Ricci tensor can be obtained from $R_{abcd}$ by the formula

$$R_{(ab)} = R_{\text{c}}^{\text{c}}_{\text{abcd}} \quad (2.12)$$

In the next section we shall use the above technique to derive our solution.

3. The equations of structure:

We take the Einstein's universe described by the metric (1.2) as the background. For the description of the NUT metric in the background of the Einstein's universe we take metric in the form (Patel and Akabari, 1978)
\[ ds^2 = 2(du - 2b \cos \phi \, d\phi) \, dr - M^2(du^2 + \sin^2 \phi \, d\phi^2) + 2L(du - 2b \cos \phi \, d\phi)^2 \] 

(3.1)

where \( L \) is a function of \( r \) and \( M^2 \) is given by

\[ M^2 = (r^2 - b^2) \sin^2 (r/a) + b^2 \] 

(3.2)

Now we shall calculate tetrad components \( R_{(ab)} \) of the Ricci tensor for the metric (3.1) with the help of the formalism discussed in section 2.

Let us introduce the basis 1-forms

\[ \theta^1 = du - 2b \cos \phi \, d\phi ; \quad \theta^2 = M \, d\phi ; \]
\[ \theta^3 = M \sin \phi \, d\phi ; \quad \theta^4 = dr + L \, \theta^1 \] 

(3.3)

Then the metric (3.1) takes the form

\[ ds^2 = 2\theta^1 \theta^4 - (\theta^3)^2 - (\theta^4)^2 = \delta_{(ab)} \theta^a \theta^b \] 

(3.4)

Now using (3.3) with (2.7) the connection 1-forms \( \omega^a_b \) can be obtained as
\[
\begin{align*}
\omega^2_2 &= \omega^3_3 = \omega^1_4 = \omega^4_1 = 0; \\
\omega^1_1 &= -\omega^4_4 = -L_r e^1; \\
\omega^1_2 &= \omega^2_4 = (M/M) e^2 + (b/M^2) e^3; \\
\omega^1_3 &= \omega^3_4 = -(b/M^2) e^2 + (M/M) e^3; \\
\omega^2_1 &= \omega^4_2 = L(M/M) e^2 + L(b/M^2) e^3; \\
\omega^2_3 &= -\omega^3_2 = L(b/M^2) e^1 - \cot\theta e^3 + (b/M^2) e^6; \\
\omega^3_1 &= \omega^4_3 = -L(b/M^2) e^2 - L(M/M) e^3 
\end{align*}
\tag{3.5}
\]

where \( M^2 \) is given by (3.2). Here and in what follows a suffix denotes a partial derivatives e.g. \( L_r = \partial L/\partial r \) etc.

Now with the help of equation (3.5) and (2.7) one can obtain the curvature 2-form \( \Omega^b_a \) and from (2.9), (2.12) and \( \Omega^a_b \) we find the tetrad components \( R_{(ab)} \) of Ricci tensor. The curvature 2-forms \( \Omega^a_b \) are listed in the appendix-2 for ready reference.
The non-zero components of $R_{(ab)}$ are as follow.

\[ R_{(44)} = -2L^{-2} \; ; \quad R_{(11)} = L^2 R_{(44)} \; ; \]
\[ R_{(14)} = -L_{rr} - 2L_r(M_p/M) + 2L_{r^2} - (2Lb^2/M^4) \; ; \]
\[ R_{(22)} = R_{(22)} = -2L_{r^2} + 2L_r(M_p/M) - 4Lb^2/M^4 \]
\[ -(1/M^2) + 2L(M_p/M)^2 \quad (3.6) \]
where $M^2$ is given by (3.3).

4. The field equations and their solutions:

We now try to make the metric (2.1) compatible with the perfect fluid distribution. The field equations corresponding to perfect fluid distribution can be expressed in the tetrad form as

\[ R_{(ab)} = -3 \left[ (p+\varphi) \nu_{(a)} \nu_{(b)} -(l/2) g_{(ab)} (p - \varphi) \right] + \lambda \delta_{(ab)} \]
\[ \nu_{(a)} \cdot \nu_{(a)} = 1 \quad (4.1) \]
where \( \lambda , p , \varphi \) and \( V(\alpha) \) are respectively the cosmological constant, the pressure, the density and the tetrad components of the flow vector. We shall assume the following form for \( V(\alpha) \).

\[
V(\alpha) = \left( \frac{1}{2n}, 0, 0, n \right) \quad (4.2)
\]

where \( n \) is the parameter to be determined from the field equations. From (4.1) and (4.2) we get the following relations.

\[
R(22) = \bar{R}(33) \quad (4.3a)
\]

\[
\bar{R}(44) = 4n^2 R(11) \quad (4.3b)
\]

\[
R(14) + \Lambda (22) + 2n^2 R(11) = 0 \quad (4.3c)
\]

\[
\bar{R}(14) = -R(14) + \lambda \quad (4.4)
\]

\[
\bar{R}(44) = -2R(11) - R(22) + \lambda \quad (4.5)
\]

where \( R(\alpha \beta) \) are given by (3.6). From (4.3b) and (3.6) it is obvious that \( 2\Lambda = n^{-2} \). Here we have assumed that \( L \) is positive. The equation (4.3c) gives us the following differential equation for \( L \).
\[ L_{rr} + 2L \left( \frac{R^L}{R} \right)_r + 4L \left( \frac{b^2}{R^2} \right) + R^{-2} = 0 \]  
(4.6)

where \( R^2 \) is given by (3.2). It can be easily verified that (2.1) admits the solution

\[ 2L = 1 + b^2 (3R^2 - b^2) (R^2 - b^2)^{-3} \]

\[ R^{-2} \left[ 2 \sin (r/R) \cos (r/R) + 2b^4 (R^2 - b^2)^{-2} \right] \]  
(4.7)

where \( \lambda \) is a constant.

From (4.4) and (4.5), pressure and density are found to be

\[ 8 \pi p = \lambda - 2L/R^2 \]  
(4.8)

\[ 8 \pi \rho = - \lambda + 2L/R^2 \]  
(4.9)

where \( 2L \) is given by (4.7).

5. **Concluding remarks** :

If \( m \) and \( b \) are zero, the metric (3.1) along with (3.2) and (4.7) reduces to the metric (1.2) of
Einstein's universe, when \( R \to \infty \) the metric (3.1) along with (3.2) and (4.7) becomes NUT metric (1.3). Thus we have found a metric which in the vicinity of source reduces to the NUT metric and which in the absence of the source reduces to the metric of Einstein's static universe. We designate the metric (3.1) along with (3.2) and (4.7) as Einstein-NUT metric. If we put \( b = 0 \) in Einstein-NUT metric, we get the Schwarzschild exterior metric in the cosmological background of Einstein's universe.

Vaidya (1977) has discussed the Kerr metric in the background of Robertson-Walker universe. Following the scheme developed by Vaidya, we have tried to express NUT metric in the background of Robertson-Walker universe. But we have not been able to do so.

In the next Chapter we shall discuss a radiating-NUT metric.
References:


