1. Introduction:

Kerr and Schild (1965) have given exact vacuum solutions \( R_{ik} = 0 \) for a space-time where the metric tensor has the form

\[
\mathbf{E}_{ik} = \eta_{ik} + \xi_i \xi_k
\]  

(1.1)

where \( \eta_{ik} \) is the metric-tensor of the Minkowskian space-time in rectangular cartesian co-ordinates and \( \xi_i \) is a null vector field. Many investigators have taken keen interest in the Kerr-Schild type solutions of Einstein's field equations.

In the present investigation we are interested in the generalized Kerr-Schild metric describing gravitational
waves. The theory of plane gravitational waves in general relativity has been discussed by many researchers. Takeno (1961) has discussed the mathematical theory of the plane gravitational waves in details. Peres (1959) has studied the plane-wave like line element

$$\text{ds}^2 = -\text{dx}^2 - \text{dy}^2 - \text{dz}^2 + \text{dt}^2 - 2f(\text{dt} - \text{dz})^2$$  (1.2)

where $u = t - z$ and $f$ is a function of $x, y$ and $u$.

The line element (1.2) can be expressed in the form

$$\text{ds}^2 = -\text{dx}^2 - \text{dy}^2 + 2 \text{dudz} + (1 - 2f) \text{du}^2$$  (1.3)

Vaidya and Pandya (1960) have studied the metric (1.3) in connection with the gravitational and electromagnetic radiation. Patel and Vaidya (1971) have discussed the generalized plane-wave metric and have shown that the metric (1.3) is of Kerr-Schild type. Gursas and Gursey (1977) have studied a generalized form of the metric (1.3) in which $f$ is a function of $x, y, u$ and $v$, where $v = t - z$.

Pandya and Vaidya (1981) have studied a more general metric in connection with wave solutions in general relativity their metric can be expressed in the form

$$\text{ds}^2 = -3^2(\text{dx}^2 + \text{dy}^2) + 2 \text{dudz} + L \text{du}^2$$  (1.4)
where $B$ and $L$ are functions of co-ordinates $x$, $y$ and $u$.

In this Chapter we are going to study a more general metric than (1.4) in which $L$ is a function of $x$, $y$, $u$ and one new co-ordinate $v$.

2. The metric and the Ricci tensor:

In Minkowskian space consider an arbitrary smooth world line $L$ that is everywhere time-like. Let $\lambda_1$ be the unit time-like tangent vector at any point of $L$ with the co-ordinates $x^1$. Let $A^1$, $B^1$ and $C^1$ be three mutually orthogonal space-like unit vectors lying in the three space orthogonal to $\lambda_1$ at the point under consideration. Thus we have the following relations:

$$\chi^1 \lambda_1 = 1; \quad A^1 A_1 = B^1 B_1 = C^1 C_1 = -1 \tag{2.1}$$

$$\chi^1 A_1 = \chi^1 B_1 = \chi^1 C_1 = 0. \tag{2.2}$$

Here it should be noted that the raising and lowering of the vector indices of $\lambda^1$, $A^1$, $B^1$ and $C^1$ are carried out with respect to the Minkowskian metric

$$\eta_{ik} = \text{diag}(-1, -1, -1, 1) \tag{2.3}$$
Let us define the new co-ordinates $x$, $y$, $z$ and $t$ in terms of the co-ordinates $x^i$ by the following relations

$$x = x^i A_i \ ; \ y = x^i B_i \ ; \ z = x^i C_i \ ; \ t = x^i \lambda_i$$

(2.4)

then clearly

$$x_{,k} = A_k \ ; \ y_{,k} = B_k \ ; \ z_{,k} = C_k \ ; \ t_{,k} = \lambda_k .$$

(2.6)

Here and in what follows a comma followed by a lower index will imply partial differentiation with respect to the corresponding co-ordinate.

Now let us define

$$\xi_1 = \lambda_1 - C_1 \ ; \ \nu_1 = \lambda_1 + C_1$$

(2.6)

then

$$\xi_1, \xi_1 = 0 \ ; \ \nu_1, \nu_1 = 0 \ \text{and} \ \nu_1, \xi_1 = 2$$

(2.7)

$\xi_1$ and $\nu_1$ are null vectors with respect to Minkowskian metric. In this Chapter we shall confine ourselves to the case in which $\xi^i$, $A^i$, $B^i$ and $C^i$ are all constant vectors.
Consider a Riemannian 4-space whose metric is given by

$$ds^2 = g_{ik} dx^i dx^k$$  \hspace{1cm} (2.8)

where $g_{ik}$ is defined below.

$$g_{ik} = \eta_{ik} + \xi_i \xi_k + (1-\xi^2)(A_{ik} + B_{ik})$$  \hspace{1cm} (2.9)

Here $\eta_{ik}$ is the Minkowskian metric, $\xi$ is a function of co-ordinates $u, v, x$ and $y$, $\xi^2$ is a function of $u, x, y$ and $v$ with $u = t-z$ and $v = t+z$.

Then the contravariant components of the metric tensor (2.9) are given by

$$g^{ij} = \eta^{ij} - 3 \xi^i \xi^j + (1-\xi^2)(A^{ij} + B^{ij})$$  \hspace{1cm} (2.10)

and the determinant $g$ of $g_{ik}$ is

$$| g_{ik} | = g = -\xi^2$$  \hspace{1cm} (2.11)

The Christoffel symbols of the second kind for the metric (2.8) with (2.9) are given by
\[
\begin{align*}
\{ i j k \} &= H_{i(k} \xi_{j)} \xi^{i} - (1/2) H_{s} \xi_{j} \xi^{k} \eta_{s}^{L} \\
+ H_{v} \xi^{1} \xi_{j} \xi^{k} + (e^{-\gamma} - 1) \eta_{(k} \xi^{i)} \eta_{s}^{L} \\
+ (e^{-\gamma} - 1) \eta_{(k} \xi^{i)} 3^i + (1/2) (e^{-\gamma} - 1) \eta_{a} \eta_{s}^{18} (H_{j} a_{k} + B_{j} B_{k}) \\
+ (1/2) (e^{-\gamma} - 1) \eta_{a} (A_{i} A_{j} + B_{i} B_{j}) 3 \xi_{j} \xi_{k} \\
+ (1/2) (e^{-\gamma} - 1) (1 - e^{-\gamma}) \eta_{a} (A_{i} A_{j} + B_{i} B_{j}) (H_{j} A_{k} + B_{j} B_{k})
\end{align*}
\]

where round brackets denote symmetrization.

e.g., \( H_{i(k} A_{j)} = (1/2) \left( H_{k} A_{i} + H_{i} A_{k} \right) \).

The Ricci tensor \( R_{ik} \) takes the form

\[
R_{ik} = \left[ -2H_{v} + (1/2) e^{-\gamma} (H_{x} A_{y} + A_{x} H_{y}) - \gamma_{u} - (1/2) \gamma_{u}^{3} \right.

- \gamma_{u}^{i} \gamma_{u}^{j} \] \( \xi_{i} \xi_{k} \) 

\[-2H_{v} \xi^{(i} k) \\

+ (\gamma_{x} A_{y} + \gamma_{y} A_{x}) \left( A_{i} A_{k} + B_{i} B_{k} \right) \\

+ (\gamma_{u} A_{v} - 2H_{v} \xi^{(i} k) + (\gamma_{u} A_{v} - 2H_{v} \xi^{(i} k) (A_{i} A_{k} + B_{i} B_{k}) \right) \] 

\[(2.13)\]

Here and in what follows the suffix denotes the partial derivatives e.g., \( H_{v} = \partial H / \partial v \), \( H_{uv} = \partial^{2} H / \partial u \partial v \) etc.
From (2.9) and (2.13) it is easy to obtain the curvature scalar \( R \). It is given by

\[
\hat{\lambda} = g^{ij} R_{ij} = -4H_{yy} - \gamma_{xx} - \gamma_{yy} \tag{3.14}
\]

For gravitational waves we have

\[
R_{ij} = 0 \tag{2.15}
\]

Solving these equations we find

\[
H = f(x, y, u) v + g(x, y, u) \tag{2.16a}
\]

\[
\gamma_u = 2f(x, y, u) \quad \gamma_{xx} + \gamma_{yy} = 0 \tag{2.16b}
\]

where \( f \) and \( g \) are functions of co-ordinates \( x, y \) and \( u \), which satisfy

\[
f_{xx} + f_{yy} = 0 ;
\]

\[
(1/2) e^{-\gamma} (g_{xx} + g_{yy}) - 4f^2 - f_u = 0 \tag{2.16c}
\]

3. The Einstein-Maxwell equations and their solutions.

In this section we shall obtain a solution of Einstein-Maxwell field equations corresponding to the electromagnetic fields. The field equations are
\( F_{ij} = E_{i,j} - D_{j,i} \) 
\( p_{i,j} = 0 \)

\[ E_{ij} = (1/4) \varepsilon_{ij} F_{ab} F_{km} g^{kn} g^{bm} - F_{ik} F_{jm} g^{km} \quad (3.3) \]

and

\[ R_{ij} = -8\pi \varepsilon_{ij} + \Lambda \varepsilon_{ij} \quad (3.4) \]

where the symbols have their usual meanings.

We choose \( D_i \) as \( D_i = \varnothing \xi_i \), where \( \varnothing \) is a function of co-ordinates \( x, y, u \) and \( v \).

Using (3.1) and (3.3) we get

\[ \varnothing_{yy} = 0; \quad \varnothing_{xy} = 0; \quad \varnothing_{yv} = 0 \quad (3.5) \]

\[ \varnothing_{xx} + \varnothing_{yy} + 2\Lambda \nu \varphi \psi = 0 \quad (3.6) \]

and

\[ E_{ij} = -\varnothing_\nu^2 \eta_{ij} + \left[ \varepsilon \psi (\varnothing_x^2 + \varnothing_y^2) + 2\varepsilon \varnothing_\nu^2 \right] \xi_i \xi_j \]

\[ -2 \varnothing_\nu^2 (1 - \psi)(A_i A_j + B_i B_j) + 4\varnothing_\nu^2 v (1 \xi_j) \]

\[ + 4\varnothing_\nu \left[ \varnothing_x A_i (1 \xi_j) + \varnothing_y B_i (1 \xi_j) \right] \quad (3.7) \]
Substituting (2.9), (2.13) and (3.7) in (3.4), the following relations between \( \theta, \mathcal{H} \) and \( \mp \) can be obtained.

\[
\mp \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0 \tag{3.8}
\]
\[
\mp \psi_{xx} - 2 \mathcal{H}_{v} = -32 \psi_{v} \tag{3.9}
\]
\[
\mp \psi_{yy} - 2 \mathcal{H}_{v} = -32 \psi_{y} \tag{3.10}
\]
\[
\mathcal{H}_{vv} = 16 \psi_{v}^2 \tag{3.11}
\]
\[
16 \psi_{v}^2 + \wedge = 0 \tag{3.12}
\]

\[
-(1/2)e^{-\frac{1}{\mathcal{H}}} (\mathcal{H}_{xx} + \mathcal{H}_{yy}) + \mathcal{H}_{v} \mp \psi_{u} + \mp uu
\]
\[
+(1/2) \mp \psi_{u}^2 + \mathcal{G} \psi (\psi_{x}^2 + \psi_{y}^2) = 0 \tag{3.13}
\]

Now we shall solve all these equations for \( \theta, \mathcal{H} \) and \( \mp \) and find out their forms. They are

\[
\theta = h v + h(x, y, u) \; ; \quad \wedge = \sqrt{-(1/16 \psi) } \tag{3.14}
\]
\[
\mathcal{H} = -(\wedge/2) v^2 + f(x, y, u) + g(x, y, u) \tag{3.15}
\]
\[
\mp \psi_{u} = 2f - 32 \psi \wedge h \tag{3.16}
\]

with the following relations.
where \( f, g \) and \( h \) are functions of co-ordinates \( x, y \) and \( u \).

Now let us set \( \Lambda = 0 \) in the above equations, then they are reduced to the form

\[
\begin{align*}
\bar{\psi} &= h(x, y, u) ; \\
\bar{u} &= f(x, y, u)v + g(x, y, u) ; \\
\gamma' &= 2f(x, y, u) \quad (3.20)
\end{align*}
\]

with the conditions

\[
\begin{align*}
\gamma_{xx} + \gamma_{yy} &= 0 \\
-(1/2)e^{-\gamma} (\varepsilon_{xx} + \varepsilon_{yy}) + 2f_u + 4f^2 + 8\varepsilon e^{\gamma} (h_x^2 + h_y^2) &= 0 \quad (3.21)
\end{align*}
\]
Moreover taking $f = 0$, we see that $\mathcal{V}$ is a function of $x$ and $y$ only. In this case the above solution reduces to the solution obtained by Pandya and Vaidya (1961). Moreover if we put $\mathcal{V} = 0$ then we get the solution discussed by Vaidya and Pandya (1960) in connection with gravitation and electromagnetic radiation.

4. Concluding remarks:

In this Chapter we have discussed a generalized Kerr-Schild metric in connection with the Einstein and the Einstein-Maxwell field equations.

In the next Chapter we shall discuss the NUT metric in the cosmological background of Einstein's static universe.
References:


