Chapter - 3

A RADIATING CHARGED PARTICLE IN EINSTEIN'S UNIVERSE

1. Introduction:

The gravitational field of a mass particle is described by the well-known Schwarzschild solution. The geometry of this solution can be expressed by the metric

$$ds^2 = 2\, du\, dr + (1 - \frac{2m}{r})\, du^2 - r^2(du^2 + \sin^2\theta\, d\varphi^2) \quad (1.1)$$

where $m$ is the mass of the particle. The corresponding field of a charged mass particle is described by the Nordstrom's metric which can be expressed as

$$ds^2 = 2\, du\, dr + (1 - \frac{2m}{r} + \frac{4\, e^2}{r^3})\, du^2 - r^2(du^2 + \sin^2\theta\, d\varphi^2) \quad (1.2)$$

where $e$ is the charge of the particle.
Vaidya (1981) has obtained a generalization of (1.1) which describes the gravitational field of a radiating star. Vaidya metric can be described by (1.1) with \( m = m(u) \). Bonnor and Vaidya (1970) have obtained a generalization of (1.2) which describes the gravitational field of a radiating charged particle. In all these fields the space round the particle is empty and the geometry at large distances from the particle reduces to that of special relativity.

McVittie (1933) gave the line-element describing the gravitational field of an uncharged mass particle embedded in an expanding universe. The space around the mass particle is occupied by a spherically symmetric distribution of matter with non-zero density and isotropic pressure which at large distances from the particle go over smoothly to the cosmic density and pressure in expanding cosmological models. Vaidya and Shah (1967) have generalized McVittie's solution to the case in which there is an electromagnetic field also. Shah and Vaidya (1963) have obtained a solution of the Einstein's field equations which describes the gravitational field of a charged particle embedded in a homogeneous universe.

The generalization of DeSitter's and Friedman's cosmological solutions with a point mass is well-known in literature (McVittie, 1933; Tolman, 1934). Vaidya and Shah (1967) have discussed a radiating mass particle in an
expanding universe while Patel and Shukla (1974) have discussed a radiating charged particle in an expanding universe. However, a solution describing a radiating charged particle in an Einstein's universe has not yet been found. The purpose of this Chapter is to fill in this gap.

Einstein's model of the universe is the simplest geometrical model for an isotropic and homogeneous universe which is static. The geometry of this universe is described by the metric

\[ ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - \frac{(dx+xdx+ydy+zdz)^2}{r^2(x^2+y^2+z^2)} \]  

(1.3)

where \( r \) is a constant.

We carry out the following transformation from \( (x, y, z, t) \) to the co-ordinates \( (r, \alpha, \beta, u) \):

\[ x = R \sin(r/R) \sin\alpha \cos\beta; \]
\[ y = R \sin(r/R) \sin\alpha \sin\beta; \]
\[ z = R \sin(r/R) \cos\alpha; \]
\[ u = t - R. \]  

(1.4)

Under the transformation (1.4) the metric (1.3) reduces to the form
\[ ds^2 = 2 \, du \, dr + du^2 - a^2 \sin^2(r/R) \left( ds^2 + \sin^2 \omega \, d\theta^2 \right) \] (1.5)

Here it should be noted that the metric (1.2) with \( e \) and \( n \) being functions of \( u \), was given without physical interpretation by Plebanski and Stachel (1967). If \( m = e = 0 \), the metric (1.2) becomes flat. Thus the metric (1.2) is described in the flat background. It would be interesting to obtain the metric (1.2) in the cosmological background of Einstein's static universe rather than in the standard Minkowskian background and the object of the present investigation is to do just that.

2. Field equations, metric and the electromagnetic field:

We take the space surrounding the radiating charged particle to be occupied by a spherically symmetric matter distribution of density \( \rho \) and pressure \( p \). The field equations are

\[ R_{ik} - \left( 1/2 \right) g_{ik,\,rr} = -8\pi \left[ \left( \rho + \pi \right) V_{1,\,ik} - \rho \, g_{ik,\,r} + \sigma \rho \, g_{ik,\,\rho} \right] \]

\[ + \wedge g_{ik} \] (2.1)

\[ \Phi_{ik} = -\,\varepsilon^{mk} \Phi_{ik} - \left( 1/4 \right) \varepsilon_{ik} \varepsilon^{mn} \varepsilon_{nm} \] (2.2)

\[ \Phi_{ik} = \varepsilon_{ik} - \varepsilon_{ik} \] (2.3)

\[ \Phi_{ik} = J^1_{ik} \] (2.4)
where $R_{ik}$ are the components of the Ricci tensor, $\Lambda$ is the cosmological constant, $\sigma$ is the radiation density, $\text{E}_1$ and $\text{F}_{ik}$ are the electromagnetic vector potential and the field tensor respectively. $\text{V}_1$ is the flow vector of the perfect fluid and $\xi_1$ is the null vector satisfying

$$\varepsilon^{ik} \text{V}_i \xi_k = 1; \quad \varepsilon^{ik} \xi_i \xi_k = 0$$

$\varepsilon^{ik}$ being the metric tensor.

Now we take the Einstein's universe (1.8) as the background universe. Consider the metric with signature $(- - - +)$, (Patel and Akbari, 1979)

$$ds^2 = 2dudr + 2d\rho^2 - R^2 \sin^2(r/R) \left( 3\omega^2 + \sin^2 \alpha \varphi^2 \right)$$

where $L$ is a function of $r$ and $\rho$. The Ricci tensor $R_{ik}$ for the metric (3.6) can be expressed as

$$R_{11} = 2L^2 r_r r/M - 2L u u_r r/M ;$$

$$R_{14} = -2L r_r - 4L r_r r/M ;$$
\[ R_{33} = \frac{R^2}{M^2} \sin^2 \alpha \]

\[ = \frac{-2L/R^3}{R^2} - \csc^2 \left( \frac{r}{R} \right) / R^2 \]

\[ + \frac{2L \cot^2(r/R)}{R^2} + \frac{2L_r \cot(r/R)}{R} \]

\[ \hat{a}_{44} = -\frac{2L \cot(r/R)}{R} - \frac{2L_{rr}}{L_r} - \frac{2L \cot(r/R)}{R} \]  \hspace{1cm} (2.7)

with \( M^2 = R^2 \sin^2 (r/R) \). Here and in what follows the suffix denotes partial differentiation e.g. \( L_r = \frac{\partial L}{\partial r} \), \( L_{rr} = \frac{\partial^2 L}{\partial r^2} \) ... etc.

To obtain energy momentum tensor \( \varepsilon_{ik} \) let us consider the 4-potential \( \phi_4 \) with the form

\[ \phi_4 = \left(\frac{e}{r}\right) \cot \left(\frac{r}{R}\right) \delta_4 \]  \hspace{1cm} (2.8)

where \( e \) is an arbitrary function of \( u \).

Using (2.3) and \( \phi_4 \) given by (2.8), the electromagnetic field tensor takes the form

\[ F_{44} = -F_{41} = e(M_r/\beta) \]  \hspace{1cm} (2.9)

and the remaining components of \( F_{ab} \) vanish. Here

\[ M = R \sin (r/R) \]
With the help of (2.4) and (2.9) we get

$$J^i = \left( \frac{e_0}{R^2} \right) \cosec^2(r/R) \delta_i^1$$  \hspace{1cm} (2.10)

It is clear from (2.10) that the current is radial and null, i.e. \( \delta_{ik} J^i J^k = 0 \)

Substituting the value of \( F_{ab} \) from (2.9), the nonzero components of \( F_{ik} \) can be obtained as

\[
\begin{align*}
E_23/R^2 \sin^2(r/R) &= E_33/R^2 \sin^2(r/R) \sin^2(r/R) = E_{44}/\lambda \\
&= 2 \left( e^2/2\lambda^4 \right) \cosec^4(r/R) \\
&= \frac{1}{14} = \frac{2}{e^2/2\lambda^4} \cosec^4(r/R) \hspace{1cm} (3.11)
\end{align*}
\]

3. The solutions of the field equations:

To obtain the solution of Einstein-Maxwell field equations (2.1) we take \( V_i \) and \( \xi_i \) as follow

\[
V_i = \left( \frac{1}{2\pi n}, 0, 0, \frac{L}{2\pi n} \right) ; \quad \xi_i = (0, 0, 0, 1) \hspace{1cm} (3.1)
\]

where \( n \) is the parameter to be determined from the field equations (2.1). The field equations (2.1) lead to

$$\frac{1}{n^2} = \lambda \hspace{1cm} (3.2)$$
Here we have assumed that $2L$ is positive) and

$$L_{rr}^+ (1-2L) \cosec^2(r/R)/R^2 - 3e^2 \cosec^4(r/R)/R^4 = 0 \quad (3.3)$$

The differential equation (3.3) is integrated to arrive at the following solution.

$$2L = 1 - 2m \cot(r/R)/R + 4re^2 \cosec^4(r/R)/R^4 \left\{ -4 \cosec^2(r/R) - 1 \right\} \quad (3.4)$$

where $m$ is an arbitrary function of $u$.

With the help of (3.4) we can find the explicit expressions for pressure, density and radiation density as

$$\Delta p = \Lambda - \frac{2L}{R^2} \quad (3.5)$$

$$\Delta \varphi = -\Lambda + \frac{6L}{R^2} \quad (3.6)$$

$$\Delta \delta = -3m_u \cosec^2(r/R)/R^2$$

$$+ \frac{8re u}{R^3} \cot(r/R) \left[ \cosec^2(r/R) - 1 \right] \quad (3.7)$$

From results (3.5) and (3.6) it is clear that

$$24p + \Delta \varphi = 2\Lambda \quad \text{i.e.} \quad 3p + \varphi = \text{constant.}$$
The metric of our solution can be expressed in the final form as

\[ ds^2 = 2 \, \text{d}u \, \text{d}r - R^2 \sin^2(r/H)(\text{d}e^2 + \sin^2 \alpha \, \text{d}\theta^2) + \left[ 1 - 2m \cot(r/H/R) + 4\pi e^2 R^{-2} \left( \cot^2(r/H) - 1 \right) \right] \text{d}u^2 \]  

(3.8)

4. Concluding remarks:

When \( m = e = 0 \), the metric (3.3) reduces to the metric (1.5) of Einstein's universe. When \( \beta \to \infty \), the metric (3.3) reduces to the Bonnor and Vaidya (1970) metric. (1.3) describes the field of a radiating charged particle. Thus we have found a metric which in the vicinity of the source reduces to the metric describing the field of a radiating charged particle and which in the absence of the source reduces to the metric of Einstein's universe.

When \( e = 0 \), the metric (3.3) becomes the radiating star metric of Vaidya (1963) in the background of Einstein's universe. Also when \( e \) and \( m \) are constants the metric (3.3) reduces to the Nordstrom metric in the cosmological background of Einstein's universe.

Here it should be noted that our uncharged non-radiating metric is closely related to the solution obtained
by Whittaker (1968). Be assumed the equation of state
3p + 9 = constant.

It is well-known that the Einstein's universe
given by the metric (1.6) is conformally flat. Therefore
one may be tempted to believe that our results are confor-
mally equivalent to those of Bonnor and Vaidya. But this
is not true. We have verified that our metric (3.8) is
not conformal to the metric (1.2) of Bonnor and Vaidya

In the next Chapter we shall investigate a more
general metric than that discussed by Pandya and Vaidya
(1961) in connection with the electromagnetic field.
References:


