1. Introduction:

In the general theory of relativity much attention has been paid to the problem of obtaining two types of exact solutions of Einstein or Einstein-Maxwell field equations which are algebraically special and algebraically general. Despite the fact that an exact gravitational wave solution radiating from a finite source must be algebraically general (Sach, 1961), many investigators have taken keen interest in obtaining algebraically special solutions of Einstein's field equations. There are several reasons for it. One reason is that the Schwarzschild solution, Kerr solution (1963), the NUT solution (Newman, Tamburino and Unti (1963)), Demianski solution (1972) and the vacuum solutions of the Klinkersley (1969) are familiar members of this class.
Many investigators have discussed the non-static generalizations of some of the above mentioned algebraically special vacuum solutions. Vaidya (1961) has obtained a non-static generalization of Schwarzschild solution which represents the gravitational field of a spherically symmetric source emitting null fluid. The non-static generalizations of Kerr and NUT solutions have been treated extensively [see e.g. Vaidya and Patel, 1973; Vaidya 1974; Vaidya, Patel and Bhatt, 1974]. Patel (1978) has obtained the radiating version of Demianski's solution. The object of the present investigation is to obtain radiating version of some of the vacuum metrics discussed by Kinnersley (1961) which are algebraically special.

The field equations corresponding to pure radiation fields are

\[
R_{\alpha\beta} = -8\pi \sigma^b k^{\alpha} k^{\beta}, \quad k^{\alpha} k^{\alpha} = 0
\]  

(1.1)

where \( \sigma^b \) is the density of the flowing radiation.

The formalism which we are going to use for the derivation of our solution is the complex vectorial formalism developed by Cahen, Debever and Defrise (1967). A detailed account of this formalism has been given by Israel (1970) and we shall use his notations.
In this Chapter a radiating space-time will be a simply connected differentiable 4-dimensional manifold with a metric tensor field $g$ of signature $+, -, -, -, -$, that satisfy the field equation (1.1).

In the next section we shall give a brief account of the complex vectorial formalism.

3. Complex vectorial formalism:

Consider a 4-dimensional pseudo Riemannian space-time manifold $\mathcal{M}$, with the metric $g$ given over $\mathcal{M}$. At each point $x \in \mathcal{M}$, we can have a tangent space $T_x$ called the space of contravariant vectors and the corresponding cotangent space $T^*_x$ called the space of covariant vectors. Let $\mathfrak{g}(\mathfrak{a})$ be a basis for $T_x$. If we introduce a tetrad of null vectors $\{k, m, \bar{m}, n\}$ as a dual basis $\mathfrak{g}^\dagger(\mathfrak{a})$ then the computation becomes simple, $k$ and $n$ are real and future pointing whereas $\bar{m}$ is a complex. They satisfy the relations,

$$k \cdot n = 1; \quad m \cdot \bar{m} = -1$$  \hspace{1cm} (2.1)

all other innerproducts vanish. The bar indicates complex conjugation.
They are such that the metric of $V_4$ has the form

$$g_{\alpha\beta} = 2k(\alpha_p) - 2m(\alpha^m_p)$$  \hspace{1cm} (2.2)

The greek indices and the first half of the Latin indices run from 1 to 4 and second half of the Latin indices run from 1 to 3, throughout the remaining part of the thesis.

Let us introduce the basic 1-forms

$$e^1 = k_{\alpha} dx^\alpha ; \quad e^2 = m_{\alpha} dx^\alpha ; \quad e^3 = e^2 ; \quad e^4 = n_{\alpha} dx^\alpha$$  \hspace{1cm} (2.3)

Here $x^\alpha$ are the local co-ordinates in $V_4$.

It follows from (2.1) and (2.2) that

$$g_{14} = g_{41} = 1 = -g_{23} = -g_{32}$$  \hspace{1cm} (2.4)

The remaining $g_{\alpha\beta}$ are zero and consequently we get the metric in the form

$$ds^2 = e^1 e^4 - e^2 e^3$$  \hspace{1cm} (2.5)

Given a bivector $p_{ab}$ we define a dual bivector $p^{ab}$ as
\[ P_{ab}^* = (1/2) \epsilon_{abcd} P^{cd} \]  (2.6)

where \( \epsilon_{abcd} = i \epsilon_{abcd} \), \( i = \sqrt{-1} \) and \( \epsilon_{abcd} \) is the Levi-Civita's permutation symbol. A bivector \( P_{ab} \) will be called self-dual or anti-self-dual according as

\[ P_{ab}^* = i P_{ab} \quad \text{or} \quad P_{ab}^* = -i P_{ab} \]  (2.7)

respectively.

It has been shown that (Debever et. al., 1967; Israel, 1970) that the six dimensional space \( \wedge^2 \mathbb{T}^*_x \) of bivectors is isomorphic to the complex 3-dimensional space \( \mathbb{C}^3 \) of self-dual 2-forms. Therefore to the basis \( \epsilon^a \wedge \epsilon^b \) of \( \wedge^2 \mathbb{T}^*_x \), we can associate a basis of \( \mathbb{C}^3 \) given as

\[ Z^1 = \epsilon^3 \wedge \epsilon^4; \quad Z^2 = \epsilon^1 \wedge \epsilon^2; \quad Z^3 = (1/2)(\epsilon^1 \wedge \epsilon^4 - \epsilon^2 \wedge \epsilon^3) \]  (2.8)

Any 2-form, say \( P \), can be expressed as

\[ P = (1/2) P_{ab} \epsilon^a \wedge \epsilon^b = P_m Z^m + \bar{P}_m \bar{Z}^m \]  (2.9)

with
\[ p_1 = p_{34} ; \quad p_2 = p_{12} ; \quad p_3 = p_{14} - p_{23} \]
\[ \overline{p}_1 = p_{24} ; \quad \overline{p}_2 = p_{13} ; \quad \overline{p}_3 = p_{14} + p_{23} \]  \hspace{1cm} (2.10)

The metric \( \gamma_{pq} \) for the space \( C^3 \) is given by

\[ \gamma_{pq} = -2 \varepsilon^{(p \bar{q})} q - (1/2) \varepsilon^{p \bar{q} q} \]  \hspace{1cm} (2.11)

In the absence of torsion in the Riemannian space, the complex valued connection 1-forms \( \sigma_m \) and the complex valued curvature 2-forms \( \Xi_p \) are determined by the following equations known as Cartan's equations of structure.

\[ dz^p = (1/2) \varepsilon^{p mn} \sigma_m \wedge \sigma_n \]  \hspace{1cm} (2.12)

\[ \Xi_p = d\sigma_p - (1/2) \varepsilon^{p mn} \sigma^m \wedge \sigma^n \]  \hspace{1cm} (2.13)

where \( d \) and \( \wedge \) denote respectively the exterior differentiation and exterior product. \( \sigma_m \) are complex valued 1-forms which serve as six connection 1-form \( \omega_{ab} \).

\( \sigma_m \) are related to \( \omega_{ab} \) as follows:

\[ -\omega_1 = \omega_4 = (1/4)(\sigma_3 + \overline{\sigma}_3) \]

\[ -\omega_2 = \omega_3 = (1/4)(\sigma_3 - \overline{\sigma}_3) \]
\[
\begin{align*}
\omega^1_3 &= \omega^2_3 = -(1/2) \sigma^1_1; \\
\omega^1_2 &= -\omega^3_3 = -(1/2) \sigma^3_1; \\
\omega^4_3 &= \omega^3_1 = (1/2) \sigma^3_2; \\
\omega^4_3 &= \omega^2_1 = (1/2) \sigma^2_2. \\
\end{align*}
\]

The tetrad components \( \sigma^p_{pe} \), defined by \( \sigma^p = \sigma^p_{pe} e^e \),

are related to Newman-Penrose spin coefficients as follow:

\[
\sigma^p_{pe} = 2 \begin{pmatrix}
z & -\rho & -\sigma & \kappa \\
-\rho & -\lambda & -\mu & \pi \\
-2\nu & 2\alpha & 2\tau & -2\rho
\end{pmatrix}
\]  
(2.15)

\( \nu_p \) are three complex 2-forms which are related to curvature 2-forms \( \Omega_{ab} \) in exactly the same manner as \( \sigma^p \) are related with \( \omega_{ab} \).

A complex 2-form \( \Sigma_p \) can be expressed in terms of \( z^p \) and \( \bar{z}^p \) as follows:

\[
\Sigma_p = C_{pq} z^q - (1/6) R_{pq} z^q + z^p \bar{z}^q
\]  
(2.16)

Here \( C_{pq} \) is a complex valued trace-free symmetric
tensor which corresponds to the Weyl tensor, \( C_{\rho\sigma} \) is a Hermitian tensor, corresponding to the trace-free part of the Ricci tensor, \( R \) is the scalar curvature. \( \gamma_4 \) are related to the five Newman-Penrose components \( \gamma_5 \) in terms of which Petrov classification can be made.

\[
C_{\rho\sigma} = 2 \begin{pmatrix}
\gamma_0 & -\gamma_2 & \gamma_1 \\
-\gamma_2 & -\gamma_4 & 2\gamma_3 \\
\gamma_1 & 2\gamma_3 & -4\gamma_2
\end{pmatrix}
\]  

(2.17)

The Einstein-Maxwell field equations for the source-free electromagnetic field can be expressed as

\[
R_{\alpha\beta} - (1/2) R g_{\alpha\beta} = f_{\alpha\rho} f^\rho_{\beta} - (1/4) g_{\alpha\beta} f^\rho_{\rho} f_{\gamma\delta} - \sigma \epsilon_{\alpha\beta} k^\gamma k^\delta;
\]

\[
k^\alpha k_{\alpha} = 0; \quad f^\alpha_{\beta} f_{\alpha} = 0
\]  

(2.18)

where \( R_{\alpha\beta} \) are the components of Ricci tensor, \( R \) is the scalar curvature, \( g_{\alpha\beta} \) is metric tensor, \( f_{\alpha\rho} \) are the components of the electromagnetic field tensor, \( \sigma \) is the density of the flowing radiation and \( k_{\alpha} \) is the null vector.
From the results discussed above in this section we can derive

\[ \text{in}^2_{p-q} = \Sigma_p \wedge \Sigma_q ; \quad (1/2) \text{in}^2 = \Sigma_p \wedge \Sigma_p \quad (2.19) \]

where \( n = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \) represents the volume of the 4-cell. Using the fact that

\[ \Omega_{ab} - i \Omega^{ab} = \Sigma_p^2 \Sigma_{ab} \quad (3.20) \]

where

\[ \Omega_{ab} = (1/2) R_{abcd} e^c \wedge e^d \quad (3.21) \]

and (2.16) and the cyclic identity for \( R_{abcd} \) we find after contraction that

\[ R_{ab} = (1/4) R e_{ab} + \chi^p_{ab} \Sigma_p^q \quad (2.22) \]

where

\[ \chi^p_{ab} = g^{cd} z^p_{ac} z^q_{dp} \quad (2.23) \]

\( R_{ab} \) are the components of Ricci tensor, \( z^p_{ab} \) are the
tetrad components of $\mathbf{z}^P$, which are as follow:

$$
\begin{align*}
\mathbf{z}_{ab}^1 &= \mathbf{e}_{\delta^4}^3 \mathbf{e}_{\delta^4}^2 \mathbf{e}_{\delta^4}^b \\
\mathbf{z}_{ab}^2 &= \mathbf{e}_{\delta^4}^1 \mathbf{e}_{\delta^4}^3 \mathbf{e}_{\delta^4}^b \\
\mathbf{z}_{ab}^3 &= \mathbf{e}_{\delta^4}^2 \mathbf{e}_{\delta^4}^3 \mathbf{e}_{\delta^4}^b \\
\mathbf{z}_{ab}^4 &= \mathbf{e}_{\delta^4}^1 \mathbf{e}_{\delta^4}^2 \mathbf{e}_{\delta^4}^b.
\end{align*}
$$

(2.24)

With the help of equations (2.23) to (2.24), $\mathbf{R}^a_b$ can be written in terms of $\mathbf{e}_{PQ}$ and $R$ as under.

$$
\begin{align*}
\mathbf{R}^1_{11} &= \mathbf{z}_{22}^2 \\
\mathbf{R}^1_{12} &= (1/2) \mathbf{z}_{23}^3 \\
\mathbf{R}^1_{13} &= (1/2) \mathbf{z}_{32}^3 \\
\mathbf{R}^1_{14} &= (1/4) (\mathbf{R} + \mathbf{z}_{33}^3) \\
\mathbf{R}^2_{22} &= \mathbf{z}_{21}^2 \\
\mathbf{R}^2_{23} &= (1/4) (\mathbf{z}_{33}^3 - R) \\
\mathbf{R}^2_{24} &= (1/3) \mathbf{z}_{31}^3 \\
\mathbf{R}^3_{33} &= \mathbf{z}_{12}^1 \\
\mathbf{R}^3_{34} &= (1/2) \mathbf{z}_{11}^3 \\
\mathbf{R}^4_{44} &= \mathbf{z}_{11}^3.
\end{align*}
$$

(2.25)

Since $\mathbf{R}^a_b$, the electromagnetic field tensor is a bivector which can be written as

$$
\mathbf{F} = (1/2) \mathbf{R}^a_b \mathbf{e}^a \wedge \mathbf{e}^b = \mathbf{e}^P \mathbf{z}^P + \mathbf{e}^P \mathbf{z}^P
$$

(3.26)

So the electromagnetic energy tensor
\[ \xi_{ab} = -\varepsilon^{cd} F_{ac} F_{bd} + (1/4) F_{cd} F^{cd} g_{ab} \]

can be written in terms of \( F_p \) and \( \bar{F}_p \) as

\[ \xi_{11} = -F_2 F_2 ; \quad \xi_{12} = F_2 F_3 ; \quad \xi_{13} = F_3 F_2 ; \]

\[ \xi_{14} = (1/2) F_3 F_3 ; \quad \xi_{22} = F_3 F_1 ; \quad \xi_{23} = (1/2) F_3 F_3 ; \]

\[ \xi_{24} = F_3 F_1 ; \quad \xi_{33} = 2 F_1 F_2 ; \]

\[ \xi_{34} = F_1 F_3 ; \quad \xi_{44} = 2 F_1 F_1 . \]  

(2.27)

Using equations (2.25), (2.27) and the null vector

\( k_p = (1, 0, 0, 0) \) the equation (2.19) can be written as

\[ \bar{\varepsilon}_{pq} = -2 F_p \bar{F}_q - \sigma^{\kappa \ell} \sigma_{\kappa \ell} F_p \bar{F}_q \]  

(2.28)

Finally introducing a self-dual bivector

\[ F^{(+)}_{ab} = F_{ab} - i F^\kappa_{ab} ; \quad i F^\kappa_{ab} \beta^\kappa = 0 \]

can be replaced by the equation

\[ d F^{(+)}_{ab} = 0 \]  

(2.29)

In this Chapter we shall consider only the case of pure radiation fields. The case of source-free
electromagnetic field plus pure radiation fields will be taken up in the next Chapter. Therefore in the present Chapter we shall derive some exact solutions of the field equations

$$\delta_{pq} = -\sigma^{-2} \phi^2 \delta_{pq}$$  \hspace{1cm} (2.30)

Here $\sigma^{-2}$ is the density of the flowing radiation.

3. The metric and the field equations:

For the description of our solutions we consider the metric in the form

$$ds^2 = 2(du + gGde)(dr + hGde) - 2L(du + gGde)^2 \left( dy^2/\sigma^2 + G^2de^2 \right)$$  \hspace{1cm} (3.1)

Here $L$ and $M$ are functions of $u, r$ and $y$ and $h, G$ and $g$ are functions of $y$ only.

Introducing the basic 1-forms

$$\theta^1 = du + gGde ; \quad \sqrt{2} \theta^3 = M(dy/\sigma + iGde) ;$$

$$\theta^4 = dr - L\theta^1 + hGde ; \quad \theta^3 = \phi^2$$  \hspace{1cm} (3.2)
Therefore (3.1) can be expressed as

\[ ds^2 = 2(e^1 e^4 - e^2 e^3) \]  \hspace{1cm} (3.3)

Using (2.7), (2.9) and (3.2) we can obtain the connection 1-forms \( \sigma_p \). They are given by

\[ \sigma_1 = -2 \left[ \left( \frac{K_r}{M} \right) - \frac{1}{2} \right] e^2 \;
\]

\[ \sigma_2 = - \sqrt{2} \left[ \frac{1}{2} \left( \frac{2 L_y + 2^n - 2}{\ell^2} \right) + \frac{2 L_y}{M} \right] e^1 \;
\]

\[ + 2 \left[ \frac{2 (M_x + M_y)}{M} + \frac{1}{2} \left( (g_0)_{yy} - (g_0)_{yy} \right) \right] e^2 \;
\]

\[ \sigma_3 = -2 \left[ L_x + 1 \left( \frac{(g_0)_{yy} - (g_0)_{yy}}{M} \right) \right] e^1 \;
\]

\[ - \sqrt{2} \left[ \frac{1}{2} \left( \frac{2 L_y + 2^n - 2}{\ell^2} \right) + \frac{2 L_y}{M} \right] e^2 \;
\]

\[ - \sqrt{2} \left[ \frac{2 (M_x + M_y)}{M} + \frac{1}{2} \left( (g_0)_{yy} - (g_0)_{yy} \right) \right] e^2 \;
\]

\[ + 1 \left[ \frac{(g_0)_{yy}}{M^2} \right] e^4 \]  \hspace{1cm} (3.4)

where the suffix denotes the partial derivatives.

From (2.12) and (3.4) the spin coefficients can be obtained. They are given by
\[ \zeta = \sigma = \kappa = \lambda = \eta = 0 \] ;
\[ \psi = \left( M_{r}/M \right) - 1(gG)_{y}/M^{2} \] ;
\[ \gamma = -\left( 1/\sqrt{2} \right) \left[ 1(hL_{r}+gL_{u})/M + GL_{y}/M \right] \] ;
\[ \mu = -\left( M_{u} + h_{r} \right)/M - 1\left[ (hG)_{y} - (gG)_{y} \right] / M^{2} \] ;
\[ \nu = \left( 1/2 \right) \left[ L_{r} + 1\left\{ L(gG)_{y} - (hG)_{y} \right\} / M^{2} \right] \] ;
\[ \alpha = -\psi = -\left( 1/\sqrt{2} \right) \left[ (hG)_{y} M^{2} + 2(h_{u} + h_{r})/M^{2} \right] \] ;
\[ \xi = -1(gG)_{y} / M^{2} . \quad (3.5) \]

The facts \( \sigma = 0 \) and \( \kappa = 0 \) indicate that the null congruence \( k^{\alpha} \) is geodesic as well as shear-free.

One can now use \( \sigma_{p} \) given by (3.4) and the Cartan's second equation of structure (2.13) to determine the curvature 2-forms \( \Sigma_{p} \). The expressions for \( \Sigma_{p} \) are very lengthy and therefore are not given here. These expressions for \( \Sigma_{p} \) are given in the Appendix - 5 (c) for ready reference. These expressions for \( \Sigma_{p} \) will give us \( \Sigma_{p q} \) and \( R \). They are given by
\[ \mathcal{L}_{12} = \frac{1}{2} \left[ M_{\nu} - \frac{1}{2} \frac{(g \mathcal{G})}{y} \right]^2 / \mathcal{M}^2 ] \] ;

\[ \mathcal{E}_{21} = \mathcal{E}_{31} = 0 \] ;

\[ \mathcal{E}_{33} = \mathcal{E}_{33} = \sqrt{2} \left[ L_{\nu} - \sqrt{2} \mathcal{M} \right] \left[ \mathcal{G}_{\nu} + \mathcal{G}^2 \right] \]

\[ \mathcal{E}_{33} = 2 \mathcal{L}_{\nu} - \left( g^2 \mathcal{G}^2 + g^2 \mathcal{G}^2 + g^2 \mathcal{G}^2 + g^2 \mathcal{G}^2 \right) / \mathcal{M}^2 \]

\[ + \left( 2 \left[ \frac{1}{2} \frac{(g \mathcal{G})}{y} \right]^2 / \mathcal{M}^2 + \left( g^2 \mathcal{G}^2 + \mathcal{G}^2 \right) / \mathcal{M}^2 \right] + \left( 2 \mathcal{M} \right) \left[ M_{\nu} \frac{1}{y} + \frac{1}{y} \right] (g \mathcal{G}) y \]

\[ + \mathcal{M}^3 \left( g \mathcal{G}^2 / \mathcal{M}^2 \right)_u \] ;

(3.6)
\( R = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} + 2\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} + 4\left(\frac{\partial t}{\partial x}\right)^2 \)

Using \( \Sigma_p \), (2.14) and (2.15) we can obtain Newman-Penrose components \( \Phi_A \). They are given in the Appendix 5 (b) for ready reference.

The field equation (2.30) imply that \( R = 0 \) and all \( \Sigma_{pq} \) except \( \Sigma_{22} \) are zero. The non-vanishing component \( \Sigma_{22} \) gives us the value of the radiation density \( \sigma^{-}\).

The equation \( \Sigma_{11} = 0 \) and \( \Sigma_{13} = 0 \) involve only one unknown function \( M \). They can be solved to get

\[
M^2 = F^2(x^2 + y^2) ; \quad F^2 = f/x ; \quad 2f = (gG)_y
\]

where

\[
x_u = -(y-z)_u ; \quad x_0 = (y-z)_u ; \quad x_r = -1 ; \quad y_r = 0
\]

Here and in what follows \( u \) and \( z \) are defined by the differential relations

\[
(g/G)dy = d\theta ; \quad (h/G)dy = dz
\]
We set \( R = 0 \) and use \( M^2 \) given by (3.9), (3.9) and (3.10) to determine the following form of the function \( 2L \) :

\[
2L = 2q^2 + 2J + 2(3^xY + F^x)/(x^2 + Y^2)
\]

(3.11)

where

\[
2q = -\frac{Y}{x}
\]

(3.12)

and \( J, S^z \) and \( F^z \) are functions of \( u \) and \( y \) subject to the condition

\[
2L^2 + 4JY - 2xuY - Y(h\phi)_x/f = 0
\]

(3.13)

Considering \( E_{23} = 0 \) and using the above relation we find that

\[
2J = 2x^u - Y - (g/f)^2 \left\{ (Y/u - 2y)_u + (Y/y - 2y)_y \right\}
\]

\[
- (f^u/3f)_y - f^u/e^2 - (g\phi)_{x} / g\phi
\]

(3.14)

It then follows from \( E_{23} = 0 \) that

\[
E^z = -f^u; \quad E^x = F^x
\]

(3.15)

Using \( E_{23} = -\sigma^r \), the radiation density \( \sigma^r \) can be calculated. The expression for \( \sigma^r \) in this general case is lengthy and complicated therefore it is not given here.
We have so far worked with the general metric (3.3). A case in which \( f = Y \) has been treated by Patel (1978) in connection with the radiating Demianski space-times. In the next section we shall consider one more case which seems to be of physical interest.

4. The case \( f \neq Y, Y = Y(y) \); \( G = \sin \omega \).

In this section the variable \( y \) is replaced by the variable \( \omega \) through the relation \( y = \cos \omega \). Taking the function \( G \) as \( G(y) = \sin \omega \), since \( Y = Y(y) \) it then follows from (3.9) that

\[
X = au - r; \quad Y - z = a\theta + b.
\]  

(4.1)

where \( a \) and \( b \) are constants of integration. No additional constant is added in \( X \) because such a constant can always be incorporated in \( r \) - co-ordinate.

Also the equation (3.13), (3.14) and (3.15) imply that

\[
E^2 = k\theta + \omega; \quad F^2 = -ku + m.
\]  

(4.2)

\( k, \omega \) and \( m \) being constants of integration. We now introduce a variable \( \gamma \) and a function \( h^*(\gamma) \) as follow:
$C = \sin \alpha \ ; \ (f/Y)^{1/2} \delta \ = \ dY \ ;$

\begin{equation}
(f/Y)^{1/2} \sin \alpha = h^\varphi (\varphi) \tag{4.3}
\end{equation}

Then we find from (3.13) that

$2J = 2\alpha + h^\varphi / h^\varphi ; \ \mathbb{E}^\varphi = -h^\varphi / (\alpha + (h^\varphi / h^\varphi)) \ Y \tag{4.4}$

with $2h^\varphi = (h \sin \alpha)_\varphi / h^\varphi \tag{4.5}$

Now we consider the case in which $h^\varphi / h^\varphi$ is a constant say $\xi$, where $\xi = 1, 0, -1$. The case $\xi = -1$ have been discussed by Patel (1978). Therefore we shall restrict our discussion to the case $\xi = 1$ and $\xi = 0$. Therefore we shall discuss the following cases.

(a) $h^\varphi = e^\varphi$
(b) $h^\varphi = \sinh^\varphi$
(c) $h^\varphi = k^\varphi + 3$

where $k$ and $3$ are constants. Here it should be noted that in the above three cases the radiation density $\sigma^\varphi$ is given by

$$\sigma^\varphi = -2k / (x^2 + y^2) \tag{4.6a}$$

The Newman-Penrose components are given by
Now we shall discuss the detail of these three cases.
Case (a) : \( n^* = e^T \)

In this case the result (4.4) reduces to the form

\[
3J = 2s + 1 ; \quad E^* = N^* - (s+1)Y .
\]  

(4.7)

\( N^* \) is given by (4.5).

The results (4.1), (4.2), (4.3) and (4.7) show that the functions \( Y \) and \( N^* \) satisfy the following differential equations.

\[
H^\phi_\psi + H_\phi^\psi = 2(s+1)N^* + \left[ 2a(s+1) + 2k \right] Y
\]

(4.8)

\[
Y_\phi^\psi + Y_\phi^\psi = -2N^* - 2aY
\]

(4.9)

If we set \( p^2 = 1+4k \), then it can be easily seen that the differential equations (4.8) and (4.9) are equivalent to the equations

\[
Z_\phi^\psi + Z_\phi^\psi = (1-p)Z
\]

(4.10)

\[
Z_{\phi^\psi}^* + Z_{\phi^\psi}^* = (1+p)Z^*
\]

(4.11)

with

\[
Z = N^* + (1/2) (p+2s+1)Y ;
\]

\[
Z^* = N^* + (1/2) (3s+1-p)Y .
\]

(4.12)
The solutions of (4.10) and (4.11) are given by

\[ z = c_1 e^{(1/2) \gamma (-1 + \sqrt{5 - 4p})} + c_2 e^{(1/2) \gamma (-1 - \sqrt{5 - 4p})} \]

\[ z^* = c_3 e^{-(1/2) \gamma (1 - \sqrt{5 - 4p})} + c_4 e^{(1/2) \gamma (-1 - \sqrt{5 - 4p})} \quad (4.13) \]

where \( c_1, c_2, c_3 \) and \( c_4 \) are constants of integration.

Knowing \( z \) and \( z^* \) from (4.13) the results (4.12) gives us \( Y \) and \( N^* \) as

\[ pY = z^* - z \quad 2pN^* = (1 + 2\alpha - p) \hat{z} + (1 + 2\alpha + p) \hat{z}^* \quad (4.14) \]

Hence we have

\[ h \sin \alpha = \int 2 \hat{z}^* e^\gamma d\tau \quad g \sin \alpha = -\int 2xe^\gamma d\tau \]

\[ E^* = -N^* - (a + 1)Y \quad (4.15) \]

One can therefore obtain the line-element in the final form as

\[ ds^2 = 2(du + g \sin \alpha \, d\phi)(d\tau + h \sin \alpha \, d\phi) - 2L(du + g \sin \alpha \, d\phi)^2 \]

\[ - (x^2 + y^2)(d\tau^2 + e^2 \gamma \, d\phi^2) \quad (4.16) \]

where \( x = a \, u - r \); \( Y, h, g \) are given by (4.13), (4.14) and (4.15) and
\[ 2L = 2a + 1 + 2 \left\{ Z^2 \mathcal{Y} + (c u - r)(c - km) \right\} / \left\{ (c u - r)^2 + Z^2 \right\} \quad (4.17) \]

where \( Z^2 \) is given by \( (4.15) \).

Now, if we take \( k = 0 \), the radiation density \( \sigma^0 \) vanishes and we get empty space-time. We have verified that this empty space-time is the transform of the type D vacuum metric (Case II D) of Kinnersley (1969). Thus the metric (4.16) is the radiative extension of Kinnersley's vacuum metric (Case II D). We have also verified that the metric (4.16) is algebraically special.

Case (b) : \( h^0 = \sinh^+ \) .

In this case the differential equations to be satisfied by \( X \) and \( N^0 \) are

\[
(1 - q^2) X_{qq} - 2q X_q = 2 N^0 + 2a Y \quad (4.18)
\]

\[
(1 - q^2) N^0_{qq} - 2q N^0_q = 2 \left[ -k - a(s + 1) \right] Y - 2(s + 1) N^0 \quad (4.19)
\]

where \( q = \cosh^+ \).

If we set \( p^2 = 1 + 4k \) and \( 0 < p \leq 5/4 \), then it can be easily seen that the differential equations (4.18) and (4.19) are equivalent to the equations
(1-q^2)Z_{qq} - 2qZ_q + n(n+1)Z = 0 \quad (4.20)

(1-q^2)Z^*_{qq} - 2qZ^*_q + k(k+1)Z^* = 0 \quad (4.21)

with \ l-p = n(n+1) ; \ l+p = k(k+1) ;

Z = N^* + (l/2) (2a+1+1)p^* \quad ;

Z^* = N^* + (l/2)(2a+1+l-p)^* \quad . \quad (4.22)

We need those solutions of (4.20) and (4.21) which will give us the Kinnersley vacuum metric (Case II 3) for \ a = k = 0 \ as a particular case. The solutions of the equations (4.20) and (4.21) can be seen from any standard text book like Coddington (1961). They are

\begin{equation}
Z = a_1 c_n(q) ; \ Z^* = b_0 q + b_1 c^*_n(q) \quad (4.23)
\end{equation}

where \ c_n(q) \ is the Legendre function of the second kind and its series expansion is given by

\begin{equation}
c_n(q) = 1 - \frac{n(n+1)}{2!} q^2 + \frac{(n+3)(n+1)(n-3)n}{4!} q^4 - \ldots \quad (4.24)
\end{equation}

Knowing \ Z \ and \ Z^* \ from (4.23) the result (4.23) will give \ Y \ and \ N^* \ as
The functions $g$ and $h$ now can be determined as

$$h \sin \omega = \int 2^m \sinh \psi \, d\psi ; \quad g \sin \omega = -\int 2^m \sinh \psi \, d\psi \quad (4.26)$$

Therefore the metric of the solution reduces to the form

$$ds^2 = 2 \left[ du - \int 2^m \sinh \psi \, d\psi \right] \left[ dt \right]$$

$$+ \left( \int 2^m \sinh \psi \, d\psi \right) \left[ du - \int 2^m \sinh \psi \, d\psi \right]$$

$$- (X^2 + Y^2) \left( d\varphi^2 + \sinh^2 \psi \, ds^2 \right) \quad (4.27)$$

where $X = e_{u-r}$, $N^* = -N^0 - (a+1)Y$;

$$2L = 2a+1 - 2 \left\{ X^0 X^1 \left( -k u + m \right) \right\} / \left( X^0 + Y^0 \right). \quad (4.28)$$

The functions $X$ and $N^*$ are given by (4.26). If we put $k = 0$, we get $p = 1$ and consequently $n = 0$ and $k = 1$. In this case we get an empty space-time for which $Y$ and $N^*$ are given by

$$Y = b_0 q + b_1 c_1(q) ;$$

$$N^* = -a_1 + (a+1) \left[ b_0 q + b_1 c_1(q) \right]. \quad (4.29)$$
From the equation (4.6b) it is obvious that for this vacuum solution the Newman-Penrose spinors $\chi_\alpha$ do not satisfy the equation

$$2 \psi_3^2 = -3 \psi_2 \psi_4$$  \hspace{1cm} (4.30)

Hence we conclude that this empty space-time is of type II and not of type D (Carmeli and Kaye, 1977). However if $b_1 = 0$ then (4.30) is satisfied and the metric becomes of type 0. It is painless to verify that this metric is the transform of the Kinnerley's metric (case II B). Thus the metric (4.27) represents a radiating Kinnerley's (case II B) metric.

**Case (c) :** $h^* = A^\gamma + B^\delta$.

In this case we have obtained the following two differential equations for the functions $Y$ and $N^*$

$$\nabla^3 \Delta_1 \Delta_2 + \nabla^2 \Delta_1 \Delta_2 + \nabla^2 Z = 0$$ \hspace{1cm} (4.31)

$$\nabla^2 \Delta_1 \Delta_2 + \nabla^2 \Delta_1 \Delta_2 - \nabla^2 Z^* = 0$$ \hspace{1cm} (4.32)

with $Z = N^*(a + \sqrt{k})Y$; $Z^* = N^*(a - \sqrt{k})Y$;

$$\nabla = 2\sqrt{k}(A^\gamma + B^\delta) / h$$ \hspace{1cm} (4.33)
The solutions of (5.31) and (5.32) are given by

\[ Z = C_0 J_0 (\tau) + C_2 K_0 (\tau) ; \]

\[ Z^* = d_1 J_0 (-\tau) + d_2 K_0 (-\tau) , \]  \hspace{1cm} (4.34)

where \( C_0, C_2, d_1 \) and \( d_2 \) are constants and \( J_0 \) and \( K_0 \) are zero-order Bessel's functions of the first and second kind respectively. In this case we have

\[ 2 \sqrt{k} Y = Z - Z^* ; \quad 2 \sqrt{k} N^* = (\pi + \sqrt{k}) Z^* - (\pi - \sqrt{k}) Z ; \]

\[ g \sin \alpha = - \int 2Y(\alpha^2 + B) \, d\tau ; \]

\[ h \sin \omega = - \int 2N^* (\alpha^2 + B) \, d\tau . \]  \hspace{1cm} (4.35)

where \( Z \) and \( Z^* \) are given by (4.34)

Here we have assumed that \( k \) is positive. The explicit form of the metric in this case can be expressed as

\[ ds^2 = 2 \left[ \, du + \left\{ J_2 Y (\alpha^2 + B) \, d\tau \right\} \, d\phi \right] \]

\[ + \left[ \, dr + \left\{ J_2 N^* (\alpha^2 + B) \, d\tau \right\} \, d\phi \right] \]

\[ - 2L \left[ \, du - \left\{ J_2 Y (\alpha^2 + B) \, d\tau \right\} \, d\phi \right]^2 \]

\[ - (x^2 + y^2) \left[ \, d\phi^2 + (\alpha^2 + B)^2 \, d\phi^2 \right] . \]

\hspace{1cm} (4.36)

where
\[ x = au - r ; \quad E^* = -N^* - sY ; \]
\[ zL = 2a + 2 \left\{ \frac{M^* Y + 2M(-ku+m)}{x^2 + y^2} \right\} / (x^2 + y^2) \]  \hspace{1cm} (4.37)

The functions \( Y \) and \( N^* \) are given by (4.35).

Since \( Y \) is a singular function for \( k = 0 \), it follows that the metric (4.37) is also singular for \( k=0 \).

5. Concluding remarks

In the present Chapter we have studied Demianski-type solutions of the Einstein's field equations corresponding to pure radiation fields. With the aid of complex vectorial formalism a general solution of this field equation is obtained. The solution is algebraically special. A particular case of the solution has been considered which includes many known solutions. Among them are the radiating versions of some of the Kinnerley solutions.

The next Chapter will be devoted to the study of some exact solutions of Einstein-Maxwell's field equations corresponding to source-free electromagnetic fields plus pure radiation fields.
References:

   for Advanced Studies, Ser A (19).

