Chapter 3

ANALYSIS OF COMPLETE DEMAND SYSTEMS
Introduction: The main focus in the early stages of development of applied demand analysis was either on Engel curve analysis or study of partial demand relations. The pioneering works of Allen and Bowley (1935), Wold and Jureen (1953), Prais and Houthakker (1955) have made the once forgotten work of Engel a scientific discipline. Schultz (1938), Wold and Jureen (1953) and Stone (1954) have highlighted the importance of price effects in partial demand functions and given a new dimension to applied demand analysis. However, these studies could not fully utilise the developments in the theory of consumer behaviour. More so, all the individual partial demand relations put together will not satisfy the consistency conditions and cannot be integrated with general equilibrium models. These limitations coupled with the recent developments in econometric methods have led to the development of Complete Demand Systems (CDS) in recent years.

There are two broad approaches for analysing CDS. In the first one, quantity demanded of each commodity is expressed as a function of all prices and income (total expenditure). There are again two ways of doing this. One way is by postulating a well-behaved utility function and then using the theory of consumer behaviour to arrive at the
derived demand system' which will automatically satisfy the theoretical properties. The other way is to postulate a set of demand relations apriori and impose the theoretical properties as restrictions in the estimation procedure. The majority of demand systems that are in empirical use belong to either of these two categories. Linear Expenditure System (LES), Direct Addilog System (DAS), Indirect Addilog System (IAS), Direct Translog System (DTS) and Indirect Translog System (ITS) belong to the first category. The second category consists of Frisch's Complete Scheme (FCS), Rotterdam Demand System (RDS), Constant Elasticity System (CES) and numerous other models. The other approach to CDS is the estimation of indifference surfaces. Quadratic Utility Function (QUF) is popular among them.

Some of the above models are mainly meant for describing consumer's behaviour while others are aimed at testing the empirical validity of postulates implied by theory. Each of the above models implies certain assumptions, both explicit as well as implicit, about the behaviour of the individual consumer. These assumptions in conjunction with the type of data used, will determine the quality of results that emerge. Often, it would be difficult to segregate the influence of these two factors and accordingly, comparisons between models based on numerical results become difficult. In what follows, we
shall discuss for each model its formulation, properties and limitations, reserving the methods of estimation and associated issues to the following chapter.

3.1 Linear Expenditure System: LES, proposed by Stone (1954) is the most popular and frequently used model. In this model, expenditure on each commodity is a linear function of all prices and income. The imposition of the properties adding-up, homogeneity and symmetry will reduce the model to its well known form:

\[ p_i q_i = p_i c_i + b_i \left( m - \sum_{j=1}^{n} p_j c_j \right) \]  

\[ i = 1, 2, \ldots n \]

where \((q_1, q_2, \ldots q_n)\) is the vector of quantities purchased by the consumer at prices \((p_1, p_2, \ldots p_n)\) and income (in fact, total expenditure), \(m\). The Slutsky-Hicks substitution matrix would be negative semidefinite if

\[ 0 < b_i < 1 \quad \text{for all } i \]

\[ \sum_{i=1}^{n} b_i = 1 \]  

(3.1.2)

and \(m - \sum_{j=1}^{n} p_j c_j > 0\)

The underlying utility function for LES, introduced by Samuelson (1947-48) and Geary (1950-51), can be written as
subject to the conditions (3.1.2). This function (3.1.3), obviously belongs to the additive class of utility functions. Samuelson (1947-48) has given a suggestive interpretation for the parameters in LHS. The vector \((b_1; b_2; \ldots; b_n)\) is called 'marginal budget shares' and \((c_1; c_2; \ldots; c_n)\) as 'committed quantities'. This nomenclature suggests that the expenditure on each commodity consists of two parts - a committed expenditure, \(p_i c_i\), and a portion of supernumerary income, \(b_i (m - \sum_{j=1}^{n} p_j c_j)\).

It is worth mentioning that the above interpretation does not imply that \(c_i = 0\) for all \(i\). The income and price elasticities of demand for (3.1.1) can be shown to be equal to

\[
e_i = \frac{b_i m}{p_i q_i} \quad (3.1.4)
\]

\[
\begin{bmatrix}
- \frac{b_i p_i c_i}{p_i q_i} & \text{for } j \neq 1 \\
-1 + \frac{(1-b_i)c_i}{q_i} & \text{for } j = 1
\end{bmatrix}
\]

\(i, j: 1, 2, \ldots n\)

The compensated price effect between \(i\)th and \(j\)th commodities is
The relations (3.1.4) to (3.1.6) summarise all the properties of the system. From second order conditions, we can see that inferior goods are excluded and that goods which are price elastic will have negative $c'$. We also note that all price elastic goods are gross substitutes with all price inelastic goods. From (3.1.4) it follows that all income elasticities will tend to unity as income rises. These limitations stem from the twin assumptions of linearity and additivity inherent in the model.

We have noted in chapter that under direct additivity, there are only $n$ independent responses to be estimated. Thus, given the income derivatives, only one parameter is needed to derive all price effects. Also, the structure of substitution matrix, apart from a scaling factor, is determined by the income derivatives alone.

3.2 Direct Addilog System: The underlying utility function for DAS has been proposed byouthaker (1960). This also belongs to the additive class of utility functions and the commodity specific utility functions are logarithmic in nature. Accordingly, the derived demand system is called as Direct Addilog Model. The utility function for DAS is of the form

$$s_{ij} = \frac{b_i b_j}{B^i B^j} (m - \sum_{j=1}^{n} p_j c_j) \text{ for } j \neq i$$  (3.1.6)
where \((a_i, b_i, i = 1, 2, \ldots, n)\) are the parameters. In order that \((3.2.1)\) should represent a convex preference ordering, \(b_i\) must be less than unity for all \(i\). One of the \(a_i\) can be arbitrarily fixed as a scaling factor of the utility indicator. The derived demand system for \((3.2.1)\) can be shown to be equal to

\[
p_i q_i = \frac{a_i q_i^{b_i}}{\sum_{j=1}^{n} a_j q_j^{b_j}} \quad i = 1, 2, \ldots, n \quad (3.2.2)
\]

By taking ratio of expenditures for \(i\)th and \(j\)th goods and simplifying, \((3.2.2)\) can conveniently be written as:

\[
(b_i - 1) \log q_i - (b_j - 1) \log q_j = \log p_i - \log p_j - \log \left(\frac{a_i}{a_j}\right)
\]

\((3.2.3)\)

From \((3.2.3)\), it can easily be shown that the income and price elasticities of demand are given by

\[
e_i = -\frac{c_i}{k} \quad (3.2.4)
\]

\[
E_{ij} = \begin{cases} 
\frac{1}{k} c_i w_j (1 - c_j) & \text{for } j \neq i \\
\frac{1}{k} c_i w_j (1 - c_j) - c_i & \text{for } j = i
\end{cases} \quad (3.2.5)
\]
where
\[ c_i = \frac{1}{1 - b_i}, \quad k = -\sum_{j=1}^{n} v_j c_j, \quad v_1 = \frac{p_1 q_1}{m} \quad (3.2.6) \]

\[ i, j = 1, 2, \ldots n. \]

From (3.2.4) to (3.2.6) along with convexity conditions, it follows that income elasticity decreases with increase in income for all necessary goods and the range of own-price elasticity is unrestricted unlike in LES. It is also clear that DAS do not allow inferior or complementary goods in to its fold. This limitation is common to all directly additive models.  

3.3 Indirect Addilog System:

This model has been first introduced by Houthakker (1960) along with its dual, the DAS. He has analysed the properties of both IAS and DAS on an identical footing. The indirect addilog utility function is of the form:

\[ u^* = \sum_{i=1}^{n} \left( \frac{a_i}{b_i} \right) \left( \frac{m_i}{p_i} \right)^{b_i} \quad (3.3.1) \]

1/ See for example, Houthakker (1960).

2/ The concept of indirect additivity was first introduced by Leser (1941-42) while investigating the property that uncompensated cross price elasticities are identical for all goods affected, and only depend on the good whose price has changed.
where \((a_i, b_i, i = 1, 2, \ldots, n)\) are the parameters. It can be seen that \(u^*\) is homogeneous of degree zero in prices and income. This property is common to all indirect utility functions. As indicated in Chapter 2, using Roy's identity we get the IAS as

\[
p_i q_i = \frac{a_i \left( \frac{m}{p_i} \right)^{b_i} m}{\sum_{j=1}^{n} a_j \left( \frac{m}{p_j} \right)^{b_j}} \tag{3.3.2}
\]

The second order conditions for utility maximisation will be satisfied if \(b_i > -1\) for all \(i\). The parameters \(a_i\) and \(b_i\) are known as 'preference coefficients' and 'reaction intensities' (urgency parameters) respectively. As in the case of IAS, an alternative way of writing IAS is to take the ratio of expenditures of two commodities and simplifying we get,

\[
\log q_i - \log q_j = \left( \log a_i - \log a_j \right) + (b_i - b_j) \log m - (1+b_i) \log p_i + (1+b_j) \log p_j \tag{3.3.3}
\]

The income and price elasticities of demand which can be worked out in the usual way, are
where $w_i$ is the value share of $i$th commodity. The interpretation for $b$ as urgency parameters is natural in the sense that if the $b_i$ for $i$th good is greater than the weighted sum of all the reaction coefficients then the $i$th commodity is income elastic. This follows from (3.3.4).

Similarly, a commodity is price elastic or price inelastic depending on whether its reaction coefficient is positive or negative. IAS has nonlinear Engel curves and permits inferior as well as complementary goods. Nevertheless, with regard to price effects this model is as restrictive as any other additive model. Of its $2n-1$ independent parameters, all but one can be identified by a single cross-section data i.e. all price effects are mostly influenced by income effects. We have only one degree of freedom to determine $(n+1)/2$ price effects. An important limitation of IAS is that the cross price elasticities of all goods are equal and depend only on the good whose price has changed. Also, all income elasticities decline by the same amount with rise in income, irrespective of the nature of commodities. The decline in income elasticity with rise in income is desirable for necessary goods but not for luxuries. As in the case of DAS, one of the $a_i$s can be
fixed apriorily as a scaling factor.

3.4 Transcendental Logarithmic Demand Systems

The demand systems discussed so far underlie a number of general restrictions like aggregation, homogeneity, symmetry etc. derived from consumer theory, as well as particular restrictions like additivity, homotheticity etc. imposed apriorily on the structure of utility. These restrictions are treated as part of maintained hypothesis with the specification of the model itself. The implications of these restrictions on individual's behaviour as well as on the range of estimated income and price responses are well known. The need for testing the empirical validity of these restrictions has prompted the search for more flexible functional forms like Transcendental Logarithmic (or Translog) utility functions. These functional forms are nothing but local second order approximations to any general utility function and subsume the specific forms as special cases.3/ The duality between direct and indirect utility functions can also be well exploited with these forms. The translog forms have been first introduced in

3/ It is to be noted that the translog approximation to an utility function do not, in general, possess the properties of the original utility function. For example, the translog approximation to an additive utility function need not necessarily be additive. On the contrary, if the translog approximation of a utility functions is additive, then the original utility function itself would also be additive.

Corresponding to direct and indirect utility functions there are direct and indirect translog forms.

(a) **Direct Translog System:** Following Christensen, Jorgenson and Lau (1975), the direct translog utility function can be written as

\[ -\log u = a_0 + \sum_{i=1}^{n} a_i \log q_i + \frac{1}{2} \sum_{i,j=1}^{n} B_{ij} \log q_i \log q_j \]

where the negative sign on the left hand side is meant for preserving symmetry between direct and indirect forms. \((q_1, q_2, \ldots, q_n)\) is the vector of quantities consumed, \(a\) and \(B\) are parameters. It is clear that the function in (3.4.1) is quadratic in logarithms of quantities consumed and reduces to additive form when all \(B_{ij} = 0\), for \(i \neq j\) and linear logarithmic form when \(B_{ii} = 0\) for all \(i, j = 1, 2, \ldots, n\). Maximising (3.4.1) subject to the budget constraint

\[ \sum_{i=1}^{n} p_i q_i = m \]

where \((p_1, p_2, \ldots, p_n)\) are the unit prices paid for the commodities \((q_1, q_2, \ldots, q_n)\) and, \(m\), is the total
expenditure, we get the DTS as

\[
\frac{p_i q_i}{m} = \frac{a_1 + \sum_{j=1}^{n} B_{ij} \log q_j}{\sum_{j=1}^{n} a_j + \sum_i \sum_j B_{ij} \log q_j}
\]  (3.4.3)

\(i = 1,2,\ldots,n\)

Using the substitution,

\[
a_M = \sum_{i=1}^{n} a_i, \quad B_{ij} = \sum_{i=1}^{n} B_{ij}
\]  (3.4.4)

(3.4.3) can conveniently be written as

\[
\frac{p_i q_i}{m} = \frac{a_1 + \sum_{j=1}^{n} B_{ij} \log q_j}{a_M + \sum_{j=1}^{n} B_{ij} \log q_j}
\]  (3.4.5)

\(i = 1,2,\ldots,n\)

Unlike the other systems, (3.4.5) is in value share form and is homogeneous of degree zero in prices and income as well as in parameters. Consequently, a normalisation of the type

\[a_M = -1\]  (3.4.6)

is needed for its estimation. It can be shown that the estimated value shares as well as \(B_{ij}\)s are invariant under rescaling of q variables. However, \(a_1\)s are not invariant.
The income elasticities of demand for BT8 can be given by

\[ e_i = \frac{|J_i|}{|J|} \quad i = 1, 2, \ldots, n \]  

(3.4.6a)

where

\[
J = \begin{bmatrix}
0 & w_1 & w_2 & w_n \\
-\frac{\partial \log u}{\partial \log q_1} & u_{11} & u_{12} & u_{1n} \\
& \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
-\frac{\partial \log u}{\partial \log q_n} & u_{n1} & u_{n2} & u_{nn}
\end{bmatrix}
\]

and \(J_i\) is the matrix obtained by replacing \((i+1)\)th column of \(J\) by \((1,0,\ldots,0)^t\). The Allen partial elasticities of substitution, \(\sigma_{ij}\), are given by

\[
\sigma_{ij} = \frac{n}{k=1} \left( \frac{\partial u}{\partial q_k} \frac{u_{ij}}{|U|} \right)
\]

(3.4.6b)

\[1, j = 1, 2, \ldots, n\]

where \(|U|\) is the determinant of the bordered Hessian matrix and \(|U_{ij}|\) is the cofactor of \((i,j)\)th element of \(U\).

Given the income elasticities and partial elasticities of substitution, we can calculate the price elasticities using
Since the income and price elasticities are independent of 
a, it follows that they are invariant under rescaling.

In view of the budget constraint, one equation in 
(2.4.5) is redundant and can be deleted. The parameters 
of the deleted equation can be estimated by using the 
budget constraint. Since the system in (3.4.5) is highly 
nonlinear, methods like maximum likelihood or direct 
numerical procedures are to be employed for its estimation. 
In the original application by Christensen etc. (1973), 
maximum likelihood method was used. Recently, Christensen 
and Manser (1977) have used iterative Zellner's procedure 
and concluded that the parameter estimates are identical 
with those of MLE.

(b) Indirect Translog System The indirect translog 
utility function can be written as

$$
\log u^* = a_0 + \sum_{i=1}^{n} a_i \log \left( \frac{p_i}{m} \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \log \left( \frac{p_i}{m} \right) \log \left( \frac{p_j}{m} \right) \quad (3.4.7)
$$

For purposes of exposition and convenience the same set 
of parameters is used in (3.4.7) as that of (3.4.1). The
utility function \((3.4.7)\) is a local second order approximation to any general indirect utility function and is quadratic in logarithms of ratio of price to total expenditure. Using Roy's identity, we get the ITS as

\[
\frac{p_{1}q_{1}}{m} = \frac{a_{1} + \sum_{j=1}^{n} R_{1j} \log \left( \frac{p_{1}}{m} \right)}{a_{M} + \sum_{j=1}^{n} R_{Mj} \log \left( \frac{p_{j}}{m} \right)} \tag{3.4.8}
\]

where \(a_{M}\) and \(R_{Mj}\) \((j = 1, 2, \ldots, n)\) are as defined in \((3.4.4)\). The system in \((3.4.8)\) is homogeneous of degree zero in parameters and the normalisation given in \((3.4.6)\) can be used for estimation purposes. The methods of estimation for \((3.4.8)\) are analogous to that of \((3.4.5)\). The income and price elasticities of demand for \((3.4.8)\) can be shown to be equal to,

\[
e_{1} = 1 + \frac{v_{1} \sum_{j} R_{Mj} - \sum_{j} R_{ij}}{v_{1} K} \tag{3.4.9}
\]

\[
e_{ij} = -\delta_{ij} + \frac{R_{ij} - v_{1} R_{Mj}}{v_{1} K} \quad (3.4.10)
\]

where \(K = a_{M} + \sum_{j} R_{Mj} \log \left( \frac{p_{j}}{m} \right)\)

\[
\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}
\]
and \( w_i \) is the value share of the \( i \)th commodity. As in the case of RTS, \( e_i \) and \( R_{ij} \) \((i,j = 1,2,...,n)\) are invariant under rescaling of prices and income.

(c) **Restrictions on Translog Demand Systems**

The restrictions to be discussed are common to both DTS and ITS. Accordingly, we shall discuss the possible restrictions with reference to (3.4.5). In view of the budget constraint, one equation in (3.4.5) is redundant and can be deleted. Thus, there are \( n-1 \) linearly independent equations each containing \( 3n-1 \) parameters, one less because of the normalisation (3.4.6).

**Equality and Symmetry Restrictions:** Supposing that we have estimated (3.4.5) equationwise, we have \( n-1 \) sets of values for the \( n \) parameters \( R_{M_i} \) \((i = 1,2,...,n)\). If (3.4.5) is generated by utility maximisation, the parameters \( R_{M_i} \) appearing in each equation must be the same. This results in a set of \( n(n-2) \) restrictions relating the \( n \) parameters. These are known as equality restrictions.\(^1\) The symmetry restrictions arise from the symmetry property of the Hessian matrix. There are \( n(n-1)/2 \) such restrictions given by

\[^1\] However, if (3.4.5) is estimated as a system, these restrictions will be implicit and need not be imposed separately.
where each of the functions $u^i(q^i)$ depends on only one commodity $q^i$ and $F$ is a real-valued function of one variable. The parameters of the translog approximation to a directly additive utility function can be written as

\[
-\frac{\partial \log u}{\partial \log q^i} = - F' \frac{\partial \log u^i}{\partial \log q^i} = a_i \quad (3.4.11)
\]

\[
-\frac{\partial \log u}{\partial \log q^i \partial \log q^j} = - F'' \frac{\partial \log u^i}{\partial \log q^i} \frac{\partial \log u^j}{\partial \log q^j} = B_{ij} \quad (3.4.12)
\]

for $i \neq j \quad (i,j = 1,2,...,n)$

where $F' = \frac{\partial F}{\partial u^i}$, $F'' = \frac{\partial^2 F}{\partial u^i \partial u^j}$.

From (3.4.11) and (3.4.12) it follows that

\[
B_{ij} = \theta a_i a_j \quad \text{for } i \neq j, \quad (i,j = 1,2,...,n) \quad (3.4.13)
\]
where \( e = -F''/(F')^2 \). (3.4.13) gives \((n-1)(n-2)/2\) restrictions known as additivity restrictions. The direct translog approximation would be additive if, and only if,

\[
\theta = 0 \quad (3.4.14)
\]

(3.4.14) along with (3.4.13) give the explicit additivity restrictions. We note that the translog approximation to an explicitly additive utility function is explicitly additive.

Homothereity restrictions: - If the utility function \( u \) is homothetic, we can write

\[
\log u = F(\log H(q_1, q_2, \ldots q_n))
\]

where \( H \) is a homogeneous function of degree one in \( (q_1, q_2, \ldots q_n) \). The parameters of translog approximation to a homothetic direct utility function can be written as

\[
\begin{align*}
-\frac{\partial \log u}{\partial \log q_1} &= \frac{\partial F}{\partial \log H} \quad \frac{\partial \log H}{\partial \log q_1} = a_1 \quad (3.4.15) \\
-\frac{\partial \log u}{\partial \log q_1 \partial \log q_j} &= \frac{\partial F}{\partial \log H} - \frac{\partial \log H}{\partial \log q_1 \partial \log q_j} \\
&\quad + \frac{\partial^2 F}{\partial \log H^2} \frac{\partial \log H}{\partial \log q_1} \frac{\partial \log H}{\partial \log q_j} \quad (3.4.16)
\end{align*}
\]
Since \( H \) is homogeneous of degree one, we have from Euler's theorem,

\[
\sum_{i=1}^{n} \frac{\partial \log H}{\partial \log q_i} = 1
\]  

(3.4.17)

\[
\sum_{i=1}^{n} \frac{\partial \log H}{\partial \log q_i} = 0
\]  

(3.4.18)

Using (3.4.6) and (3.4.15) to (3.4.18) in (3.4.4), we have

\[
R_{ij} = \sigma a_j (j = 1, 2, \ldots, n)
\]  

(3.4.19)

where \( \sigma = \frac{F}{\log H^2} \)

There are \( n-1 \) independent restrictions in (3.4.19), which are called homotheticity restrictions. A necessary and sufficient condition for homotheticity of translog utility function is that it is homogeneous, so that

\[
\sigma = 0
\]  

(3.4.20)

(3.4.20) along with (3.4.19) are known as homogeneity restrictions. The translog approximation of a homogeneous utility function is homogeneous.

Additivity and homotheticity restrictions: The class of additive and homothetic direct utility functions coincides with the class of utility functions with constant
elasticities of substitution among all pairs of commodities. If the direct utility function, \( u \) is additive and homothetic, then we can write it in one of the two forms

\[
\log u = F \left( \sum_i \delta_i q_i^p \right)
\]

\[
\log u = F \left( \sum_i \delta_i \log q_i \right)
\]

The second form is a limiting case of the first, i.e., when \( p \to 0 \), corresponding to unitary elasticities of substitution among all pairs of commodities. The parameters of the translog approximation to an additive and homothetic utility function satisfy the additivity and homotheticity restrictions given in (3.4.13) and (3.4.19), so that

\[
B_{ij} = \theta a_i a_j, \ i \neq j \ (i, j = 1, 2, \ldots, n)
\]

\[
B_{ij} = \sigma a_j
\]

\[
B_{jj} = (\sigma + \theta) a_j + \theta a_j^2
\]

We can identify the parameters of translog approximation with that of an additive and homothetic utility function as follows:

\[
\theta + \sigma = p \tag{3.4.21}
\]

\[
a_j = \xi_j, \ j = 1, 2, \ldots, n
\]

where \( \theta = -F''/(F')^2 \)
The translog approximation to a utility function characterized by unitary elasticities of substitution among all pairs of commodities satisfies the relation:

$$\sigma + \theta = \rho = 0$$  \hspace{1cm} (3.4.22)

Under this restriction, the expenditure shares are constant:

$$\frac{p_i q_i}{m} = \frac{a_i + \theta a_i \sum_j \frac{a_j \log q_j}{1 - \theta \sum_j a_j \log q_j}}{1 - \theta \sum_j a_j \log q_j} = \sigma$$

since

$$\sigma = \theta = \rho = 0$$

Thus, the translog approximation to a utility function with unitary elasticities of substitution has the same empirical implications as a linear logarithmic utility function, which is explicitly additive and homogeneous. It is worthwhile to note that the restrictions in the case of indirect translog form are analogous to the above restrictions. In general, the direct and indirect translog approximations to direct and indirect utility functions represent different preferences, the only exception being linear logarithmic form. In this case, both direct and indirect translog approximations are additive and homothetic, implying thereby unitary elasticities of substitution among all pairs of commodities.
3.5 Constant Elasticity System

Evron (1968, 1970a, 1970b) has experimented with constant elasticity system (also known as linear logarithmic model) in a series of articles. It has the following formulation:

\[ \log q = a + b \log m + c \log p \]  

where \( a, b \) are \( n \times 1 \) vectors and \( c \) is an \( n \times n \) matrix. \( b \) represents the vector of income elasticities and \( c \) the matrix of price elasticities. The above formulation implies that \( b \) and \( c \) are constant, independent of income and prices and hence the name constant elasticity model. The immediate advantage of this model is that, the theoretical postulates become linear restrictions dependent on elasticities (parameters) and budget shares alone. It is also possible to postulate and test various types of separability assumptions. However, in this case, the restrictions become nonlinear and accordingly, iterative procedures like linearisation, Newton-Raphson etc. are to be employed for the estimation of the parameters. An important limitation of the model is that the restrictions can be imposed only at the mean level of the sample. In view of the assumption of constancy of elasticities, lot of criticism is levelled against this model. It is argued that the latter assumption implies unitary elasticities thereby reducing the model to Bergson
family i.e. homothetic utility formulation.$^5$

3.6 Rotterdam Model

The main stimulus to the testing aspect of inference in demand theory came from the works of Theil (1965, 1967, 1975) and Barten (1964, 1967, 1968, 1970) on Rotterdam model. The idea underlying this model is to view the demand theory as an allocation process for the consumer. Accordingly, budget shares and changes in them rather than the actual quantities consumed, are of interest to the demand analyst. Change in value share consists of three components - change in income, prices and quantities consumed. Since prices and income and hence their changes are exogenously given to the consumer, the only component behaviourally determined is changes in quantities consumed. With this view, Theil has developed Rotterdam model extensively.

(a) Formulation of the model:

If $w_i = \frac{p_i q_i}{m}$ (i = 1,2,...,n) is the value share of ith commodity in the consumer's budget, the infinitesimal change in it can be expressed as,

$$d w_i = w_1 d \log p_i + w_1 d \log q_i - w_1 d \log m \quad (3.6.1)$$

$^5$ It has been shown by Yoshihara (1969) that (3.5.1) will satisfy the budget constraint, provided all income elasticities tend to unity as income tends to infinity.
In (3.6.1), only the second term on R.H.S. is behaviourally determined. Accordingly, this term viz. the quantity component of budget share change, is chosen as the dependent variable. This can be expressed in terms of quantity differential, $dq_i$, as

$$w_1 d \log q_i = \frac{p_i}{m} dq_i$$

where $dq_i$ itself consists of changes in income and prices.

$$dq_i = \frac{\partial q_i}{\partial m} dm + \sum_{j=1}^{n} \frac{\partial q_i}{\partial p_j} dp_j$$

and

$$dq_i = \frac{\partial q_i}{\partial m} dm + \sum_{j=1}^{n} \left( u_{ij} - \frac{\lambda}{\partial q_i} \frac{\partial q_i}{\partial m} \frac{\partial q_i}{\partial m} \right) dp_j$$

where the notation is as that of chapter 2. From (3.6.2) and (3.6.3), we have

$$w_1 d \log q_i = \frac{p_i}{m} \frac{\partial q_i}{\partial m} dm + \frac{p_i}{m} \left[ \sum_{j=1}^{n} \left( \lambda u_{ij} - \frac{\partial q_i}{\partial m} \frac{\partial q_i}{\partial m} \right) dp_j \right]$$

(3.6.4)
Now, we shall simplify and try to interpret each term on the R.H.S. of (3.6.4). Consider the first term viz. the income term,

\[
\frac{p_1}{m} \frac{\partial q_i}{\partial m} \Delta m = p_1 \frac{\partial q_i}{\partial m} \Delta m = \mu_i \Delta \log m \quad (3.6.5)
\]

where \( \mu_i = \frac{p_1}{m} \frac{\partial q_i}{\partial m} \), known as the marginal budget share, i.e. the additional amount that will be spent on \( i \)th commodity when the income of the consumer increases by one unit. Consider the last term i.e. the income effect of all price changes,

\[
- \frac{p_1}{m} \sum_{j=1}^{n} q_j \frac{\partial q_i}{\partial m} \Delta p_j = - p_1 \frac{\partial q_i}{\partial m} \sum_j \frac{p_i q_i}{m} \frac{\Delta p_j}{p_j}
\]

\[
= - \mu_i \sum_{j=1}^{n} v_j \Delta \log p_j \quad (3.6.6)
\]

Combining (3.6.5) and (3.6.6), we have the change in logarithm of real income given by,

\[
\mu_i \left( \Delta \log m - \sum_k v_k \Delta \log p_k \right) \quad (3.6.7)
\]

From (3.6.7), we can conclude that the income effect of all price changes acts as a deflator which transforms money income into real income. This deflator is a weighted average of changes in log prices with budget shares as weights. Consider the second term in R.H.S. of (3.6.4) i.e. the specific effect of all price changes,
\[
\frac{p_i}{n} \sum_{j=1}^{n} \lambda u_{ij} \frac{d\phi_j}{dp_j} = \sum_{j=1}^{n} \lambda \frac{p_j u_{ij} \phi_j}{m} d\log p_j
\]

\[
= \sum_{j=1}^{n} \psi_{ij} d\log p_j \tag{3.6.8}
\]

where \( \psi_{ij} = \lambda \frac{p_j u_{ij} \phi_j}{m} \), \( i, j = 1, 2, \ldots n \)

such that \( \sum_{j=1}^{n} \psi_{ij} = \phi \mu_i \), \( \phi = \frac{\lambda (\lambda/m)}{\partial \lambda/\partial \mu} \)

being the reciprocal of income elasticity of marginal utility of money. Consider the third term on the R.H.S. of (3.6.4) viz. the general substitution effect,

\[
- \frac{p_i}{n} \frac{\lambda}{m} \left( \frac{\partial q_j}{\partial \lambda} \right) \sum_{j=1}^{n} \frac{\partial q_j}{\partial m} \frac{d\phi_j}{dp_j} = - \phi \mu_i \sum_{j=1}^{n} \mu_j d\log p_j \tag{3.6.9}
\]

Combining (3.6.8) and (3.6.9), we have the total substitution effect of all price changes,

\[
\sum_{j=1}^{n} \psi_{ij} d\log p_j - \phi \mu_i \sum_{j=1}^{n} \mu_j d\log p_j = \sum_{j=1}^{n} \psi_{ij} \left( d\log p_j - \sum_{k}^{n} \mu_k d\log p_k \right) \tag{3.6.10}
\]
In (3.6.10), the general substitution effect acts as a deflator to the specific substitution effect. Combining (3.6.7) and (3.6.10), we have the quantity component of budget share change as,

\[ v_i \delta \log q_i = \mu_i (\delta \log m - \sum_{k=1}^{n} w_k \delta \log p_k) + \sum_{j=1}^{n} \psi_{ij} \left( \delta \log p_j - \sum_{k=1}^{n} \mu_k \delta \log p_k \right) \]

\[ i = 1, 2, \ldots, n \]

The system in (3.6.11) is known as the relative price version of the Rotterdam model, since the explanatory variables are changes in real income and relative prices.

Introducing time suffix, \(t\), the finite change approximation of (3.6.1) can be written as

\[ \Delta \tilde{v}_{it} = \tilde{v}_{it} \Delta \log p_{it} + \tilde{v}_{it} \Delta \log q_{it} - \tilde{v}_{it} \Delta \log m_t \]

(3.6.12)

where \(\Delta\) stands for the first difference operator and

\[ \tilde{v}_{it} = \frac{v_{it} + v_{it-1}}{2} \]

summing over all \(i\) \((i = 1, 2, \ldots, n)\), and simplifying, (3.6.12) becomes

\[ \sum_{i=1}^{n} \tilde{v}_{it} \Delta \log q_{it} = \Delta \log m_t - \sum_{i=1}^{n} \tilde{v}_{it} \Delta \log p_{it} \]

(3.6.13)
Using (3.6.13), the finite change analogue of (3.6.11) can be written in its most celebrated version as,

\[ \bar{v}_{it} \Delta \log q_{it} = \mu_i \sum_{k=1}^{n} \bar{v}_{kt} \Delta \log q_{kt} + \]

\[ \sum_{k} \Pi_{ik} \Delta \log p_{kt} + v_{it} \quad (3.6.14) \]

where \( \Pi_{ik} = v_{ik} - \phi \mu_i \mu_k \) and \( v_{it} \) is the residual for \( i \)th equation and \( t \)th period. The finite approximation version given in (3.6.14) is also known as the absolute price version of Rotterdam model.

Adding an additive constant and renaming the parameters with an obvious correspondence, a typical equation of the Rotterdam demand system can be written as,

\[ \bar{v}_{it} \Delta \log q_{it} = b_i \sum_{k=1}^{n} \bar{v}_{kt} \Delta \log q_{kt} + \]

\[ \sum_{k} s_{ik} \Delta \log p_{kt} + a_i + v_{it} \quad (3.6.15) \]

\[ i = 1, 2, \ldots, n \]

\[ t = 1, 2, \ldots, T \]

where \( a_i, b_i, s_{ik} \) (\( i, k = 1, 2, \ldots, n \)) are the parameters. The variables and parameters in (3.6.15) have the following meaning:

(1) The Left Hand Side (L.H.S.) variable is the change in logarithm of quantity demanded weighted by the average value share of expenditure on \( i \)th commodity.
(ii) The first term on Right Hand Side (R.H.S.) represents both the effect of changes in total expenditure and income effect of changes in prices on L.H.S. variable. It adds up to L.H.S. variable also.

(iii) The second term on R.H.S. represents the combined substitution effect of price changes.

(iv) Since the equations are in first difference form, the constant term 'a\_t' represents the combined trend like shifts in preferences of all commodities.

(v) The \( b\_i = p\_i \frac{\partial q\_i}{\partial m} \), is the income coefficient, also known as marginal budget share (propensity to consume) on \( i\)th good.

\[
S\_k = \frac{p\_kp\_k}{m} \left( \frac{\partial q\_i}{\partial p\_k} q\_k + q\_k \frac{\partial q\_i}{\partial m} \right)
\]

are price coefficients known as substitution effects, decomposable into specific and general components.

(vi) The last variable on R.H.S. viz. \( v\_t \) is the residual which includes all factors affecting market demand other than total expenditure, prices and trend like shifts.

We can write (3.6.15) in matrix notation:

\[
Y\_t = b \cdot (Y\_t + S Z\_t + a + V\_t) \quad (3.6.16)
\]

\[
t = 1,2,\ldots,T
\]
where \( Y_t \) is the \( n \) element column vector of observations on L.H.S. variable in period \( t \). \( Z_t, V_t \) are \( n \) element column vectors of log changes in prices and of disturbances in period \( t \) respectively. \( a, b, S \) are matrices of parameters of appropriate order and \( \ell' \) is a \( n \)-component row vector of unities. The postulates of complete demand system have the following formulation in the case of (3.6.16):

\[
\begin{align*}
\ell' b &= 1 \quad \text{(Adding-up property)} \quad (3.6.17) \\
S \ell &= 0 \quad \text{(Homogeneity property)} \quad (3.6.18) \\
S' &= S' \quad \text{(Symmetry property)} \quad (3.6.19) \\
X' S X &\leq 0 \quad \text{(Negativity property)} \quad (3.6.20)
\end{align*}
\]

for all \( X \neq \lambda p \), \( \lambda \) being a scalar. The decomposition of total substitution effect into specific and general components would give us, \( S = C - \phi \lambda b' \) where \( C \) is the matrix of specific substitution effects. In view of (3.6.18) and (3.6.19), we have \( C \ell = \phi \lambda b \), \( \ell' C \ell = \phi \) and \( C = C' \). In addition, if we assume groupwise independence, we have \( c_{ik} = 0 \) for \( i \neq k \) and hence

\[
\sum_{k=1}^{n} c_{ii} = \phi \quad \text{and} \quad \ell' C = \phi \quad b' \quad (3.6.21)
\]

In the following chapter, we shall discuss methods of imposing the constraints (3.6.18), (3.6.19) and (3.6.21), in maximum likelihood framework. Conditions (3.6.20) are omitted because they are inequality constraints and used only for checking results.\(^6\)

\(^6\) The negativity condition has been imposed for the above model by Parten and Geyskens (1975) using Cholesky decomposition of the substitution matrix.
The major criticism against this model is that it does not satisfy integrability conditions. Since the model is in first differences, an additional set of integrability conditions, apart from the usual symmetry conditions, are needed for this model to be consistent with utility maximisation. These additional conditions are of the form:

\[ \frac{\partial q_p}{\partial m} = \frac{\partial q_m}{\partial p} \]

where \( q_p = \frac{\partial q}{\partial p} \) , \( q_m = \frac{\partial q}{\partial m} \)

McFadden (1967) has shown that the imposition of the above constraints reduces the Rotterdam system to unitary elasticity model. Countering this criticism, Barten (1969) maintains that the system may be considered as a reasonable approximation to reality in narrow ranges of income. In addition, since the data available is of very aggregative in nature, at best on a household and the theoretical postulates refer to an individual consumer, it is of little interest to look for the underlying utility function.

3.7 Other Models:

The other models of demand include, Frisch's complete scheme, Pearce System, Extended Linear Expenditure System (ELES), Quadratic utility function and a number of variants
of LBS which can collectively be called as Australian models.

In Frisch's scheme, there is no formal utility function as such. Given the budget shares and income elasticities it is possible to compute all price elasticities, provided we know exogenously one parameter called income flexibility (Frisch parameter), say $\hat{W}$. Based on the results of a number of cross-country studies, Frisch identified the average value of $\hat{W}$ as -2 for countries in middle income range. The absolute value of $\hat{W}$ would be higher for underdeveloped countries and lower for developed nations. Under the assumption of 'want independence', i.e. the utility function is additive and measurable up to a linear transformation, the following formula can be used to calculate the matrix of price elasticities ($E$), given the budget shares ($W$) and income elasticities ($e$).

$$E = \phi (\hat{e} - ee' W) - e W$$

where $\phi$ is the reciprocal of $\hat{W}$ and $\hat{e}$ is the diagonal form of $e$. If we ignore higher order terms starting with second, we have from the above relation,

$$E = \phi \hat{e}$$

implying thereby that the ratio of price to income elasticities is constant and equals to $-\frac{1}{2}$ approximately for countries where $\hat{W}$ is -2. This is the famous Pigou's law. Needless to say, that the above simplification is valid in the absence of reliable price information.
In LES, prices and total expenditure are the explanatory variables. Instead, we can take personal disposable income in the place of total expenditure. The resultant system is known as ELES, which takes care of individual's saving as well. ELES has two possible approaches - first through an instantaneous utility maximisation (as in LES) combined with a Keynesian consumption function; secondly through an intertemporal utility maximisation with personal disposable income and prices as explanatory variables. In the former approach, LES is to be stochastically specified whereas in the latter ELES is to be stochastically specified. The ELES has been found to give more stable estimates for 'committed quantity' parameters rather than its counterpart viz. LES. Also an estimate of marginal propensity to consume (mpc) is obtainable from ELES.

The quadratic utility function can be written as

\[ u(q_1, q_2, \ldots, q_n) = \frac{1}{2} q' a q + b' q + c \quad (3.7.1) \]

where \( a \) is a \( n \times n \) symmetric matrix, \( b \) and \( c \) are \( n \) component vectors of parameters. The demand system that can be obtained from (3.7.1) is of the form,

\[
\begin{bmatrix}
q \\
-\lambda
\end{bmatrix} =
\begin{bmatrix}
a & p' \\
p' & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
-b \\
m
\end{bmatrix} \quad (3.7.2)
\]
where \( p \) is the vector of prices and \( m \), total expenditure of the consumer. \( \lambda \) is the Lagrangian multiplier or marginal utility of money. It can be seen that the Engel functions associated with (3.7.2) are linear and do not dependent on the additive constant, \( C \) of the utility indicator. However, the parameters in (3.7.2) are not amenable for easy interpretation. One redeeming feature of (3.7.2) is that the Hessian matrix is independent of prices and income viz. the \( a \) matrix itself. Wald (1940) has shown how a preference function of the form (3.7.1) can be estimated, apart from an arbitrary proportionality factor and an additive constant, using family budget data of at least \((n+1)/2\) periods observed in different price structures. This exacting data requirement can be relaxed if we make simplifying assumptions like block independence, complete additivity etc. for the preference structure.²/ Mathur (1964) has developed an alternative method of estimating (3.7.1) using residuals around Engel curves. Unfortunately, neither the utility indicator nor the derived demand system has attracted enough attention in empirical applications.³/

²/ See Radhakrishna and Murty (1975).

³/ For India, Radhakrishna (1969) and Mahajan (1972) have estimated quadratic utility function using National Sample Survey data. Their results indicate, mainly the violation of convexity conditions.
Demand Models for Durable Goods

The models discussed so far are essentially static in nature and attempts to make the parameters to depend on time, past consumption or even explicit inclusion of time as an explanatory variable, can not change this character. In reality, the current purchases of durable goods depend not only on current income and prices, but also on income - profile, level of stocks held, habit formation, income expectations and so on. These factors introduce elements of dynamism into the consumer's decision making process. Since in practice, it is not possible to incorporate all these factors, some of which are not directly observable, attempts are made only at simple models.

There are two approaches to the analysis of dynamic demand models. In the first one, variables such as stocks, current purchases of durable commodities etc. are included in the utility function to be maximised subject to budget constraint and depreciation rules. The studies of Cramer (1957), Houthakker and Taylor (1970), Philips (1973) etc. belong to this category. The other approach is to postulate the demand model itself in terms of stocks, state variables etc. The models of Stone and Rowe (1957), Houthakker and Taylor (1970) etc. are some among these.