CHAPTER IV

SEMI FIELD VALUED ALMOST PERIODIC FUNCTIONS

4.1.  TOPOLOGICAL SEMI FIELD

DEFINITION 4.1.1. We shall call a commutative associated topological ring \( E \) a topological semi-field if there is isolated in \( E \) some set \( K \) satisfying the conditions:

1. \( K + \bar{K} \subseteq K \), \( K \circ K \subseteq E \) (\( \bar{K} \) denotes the closure of \( K \))
2. \( K - K = E \).
3. Every set \( M \subseteq E \) that is bounded above has a least upper bound \( \gamma_M \).
4. For \( \alpha, \beta \in K \), the equation \( \alpha x = \beta \) has at most one solution in \( K \).
5. The intersection \( \bar{K} \cap (-K) \) contains only the zero element of the ring \( E \).
6. We denote by \( F_{\alpha} (\alpha \in E) \) the totality of all elements \( x \in E \) satisfying the condition \( \alpha x \in \bar{F} \). The totality of all sets of the form \( \beta + F_{\alpha} (\alpha, \beta \in E) \) forms a basis system of closed sets of topological space \( E \).

REMARK

1. The axioms for a topological semi field are so chosen that its properties recall those of the fields of real numbers.

1) Antonovskii, M. Ya; Boltyanski, V. G. and Sarymsakov, T. A. (1)
2. We assume here that multiplication require its partial invertibility. It is from this fact that we name it 'semi field'.

3. An interesting example due to I.M. Dektyarev\(^2\) shows that the requirement \( K + \bar{K} \subseteq K \) (axiom 1) in the definition can not be weakened to \( K + K = K \).

4. It is known\(^3\) that a commutative topological ring admits at most one semi field structure.

5. Axiom 6 demands that an arbitrary set which is closed in \( E \) can be obtained from sets of the form \( \beta + F_\alpha \) with the help of the operations of intersection and finite union.

We shall call elements of the set \( K \) positive elements of the semi field \( E \) and elements of the set \( \bar{K}/K \) will be called boundary elements of the semi field \( E \). We agree to write the relations \( x - y \in K, x - y \notin \bar{K} \) also in the form \( x > y, x \gg y \) (or in the form \( y < x, y \ll x \)). In particular, the inequality \( x > 0 \) means that \( x \in K \) and \( x \gg 0 \) means that \( x \in \bar{K} \).

It is easy to see that the ring \( E \) consisting of the zero element only satisfies all the semi field axioms.

\(^{2}\) Dektyarev, I.M.

\(^{3}\) Antonovskii, M.Ya., Boltyanski, V.G., and Sarymsakov, T.A.
The set $K$ contains elements which are different from zero. This follows at once from axiom 2. It follows from the relations $x \gg y$ and $y \ll x$ that $x = y$. It is known that the relations $>$ and $\gg$ are transitive.

Existence of the identity and of the inverse element 4.1.2.

In every semi field $E$ there exists an identity; it is uniquely defined and is a positive element.

The equation $ax = 1$ ($a > 0$) has in $F$, a unique solution; it is a positive element. We shall denote it by $a^{-1}$.

Least Upper Bounds and Greatest Lower Bounds. 4.1.3.

A subset $M$ of the topological semi field $E$ is said to be bounded below if there exists an element $x \in E$ such that $x \ll m$ for all elements $m \in M$. Such an element is called minorant of the set $M$ for all elements $m$ of $M$. Similarly the set $M$ is said to be bounded above if there exists an element $y \in E$ such that $y \gg m$ for all elements $m \in M$. Such an element is called majorant of the set $M$ for all elements $m$ of $M$. A set which is bounded both above and below will be called bounded.

If the set $M$ is bounded below, then among its minorants there exists a largest one (supremum $\sup M$). Analogously,
if the set \( M \) is bounded above, then among its majorants there exists a smallest majorant (Infimum \( M \)).

**Positive and Negative parts, Modulus 4.1.4.**

Let \( x \) be an arbitrary element of \( E \). We set \( x_+ = \text{Sup}(x, 0) \), \( x_- = \text{Sup}(-x, 0) \). The elements \( x_+ \) and \( x_- \) are called the **positive and negative parts** respectively of the element \( x \). It is obvious that

\[
x_+ \gg 0, \quad x_- \gg 0.
\]

The element

\[
|x| = x_+ + x_-
\]

is called the modulus of the element \( x \).

It is important to note that the element \( x \in E \) possesses an inverse element \( x^{-1} \) if and only if \( |x| \in K \).

Also

\[
|x + y| < |x| + |y|
\]

**4.2. Metric Spaces Over Semi Fields:**

**Definition 4.2.1.** Let \( E \) be a semi field and suppose that \( K \) is the set of all its positive elements. We shall call the set \( X \) a metric space over the semi field \( E \) if there is given a mapping (called the metric)
\[ d : X \times X \rightarrow \mathbb{R} , \]
satisfying the following conditions (for \( x, y, z \in X \)):

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \);
3. \( d(x, y) + d(y, z) \geq d(x, z) \).

If \( E \) is the field of real numbers, then we arrive at the usual idea of a metric space.

DEFINITION 4.2.2. Let \((X, d)\) be a metric space over \( E \). Let \( U \) be a neighbourhood of zero in \( E \) and \( x \in X \). We denote by \( \bigcap(x, U) \) the set of all those elements \( y \in X \) for which \( d(x, y) \in U \). The collection of all sets of the form \( \bigcap(x, U) \) can be taken as a basis of neighbourhoods in \( X \). The topology obtained in this way is called the natural topology of the metric space \( X \). This natural topology is always completely regular and conversely any completely regular topological space can be metrized over some semi field.

Convergence in Metric space over semi field 4.2.2.

Let \( X \) be a metric space over the semi field \( E \), and let \( D \) be a directed set. A mapping \( x : D \rightarrow X \) is called the sequence of
type D in the space X. The image $x(\bar{z})$ of the element $\bar{z} \in D$ is denoted by $\bar{x}_z$. We write $x = \{ x_{\bar{z}} \}, \bar{z} \in D$.

Let $D$ and $D'$ be two directed sets. A mapping $\phi : D' \to D$ is called cofinal if for any element $\bar{z}_0 \in D$, there is an element $\eta_0 \in D'$ such that $\phi(\eta) > \bar{z}_0$ for $\eta > \eta_0$.

Let $x = \{ x_{\bar{z}} \}$ be an arbitrary sequence of type D in $X$; and $\phi : D' \to D$ is cofinal mapping. We define a sequence $x^\phi$ of type $D'$ on $X$ by putting, for each element $\eta \in D'$, $(x^\phi) = x^\phi(\eta)$

We call $x^\phi$ a subsequence of the sequence $x$.

**DEFINITION 4.2.3.** A sequence $x = \{ x_{\bar{z}} \}$ of type D in $X$ is said to converge to $a \in X$ if for every neighbourhood $U$ of the zero in $E$ there exists an element $\bar{z}_U \in D$ such that

$$d(\bar{z}, a) \in U \text{ for } \bar{z} > \bar{z}_U.$$

We write

$$\lim_{\bar{z} \in D} x_{\bar{z}} = a.$$

**DEFINITION 4.2.4.** Let $x = \{ x_{\bar{z}} \}$ be a sequence of type D in $X$. We call a point $a \in X$ a limit point of the sequence $x$ if for any neighbourhood $U$ of zero in $E$ and $\bar{z} \in D$ there exists an index $\bar{z}' > \bar{z}$ such that

$$d(a, x_{\bar{z}'}) \in U.$$
Completeness in metric space over semi field 4.2.5.

**DEFINITION 4.2.6.** A sequence \( x = \left\{ x_\frac{y}{y} \right\} \) of type \( D \) in \( X \) is called fundamental if for any neighbourhood \( U \) of zero in \( F \), there is an element \( \frac{y}{y} \in D \) such that

\[
d \left( x_\frac{y}{y} , x_\frac{y}{y}'' \right) \in U \quad \text{for} \quad \frac{y}{y} , \frac{y}{y}'' > \frac{y}{y} \cdot
\]

we shall call the metric space \( X \) complete if an arbitrary fundamental sequence of type \( D \) is convergent.

**DEFINITION 4.2.7.** A set \( M \) of the metric space \( X \) is called set bounded if the set consisting of all elements \( d(x', x'') \), where \( x', x'' \in M \) is bounded above in \( F \).

**Theorem 4.2.8.** Every convergent sequence is fundamental.

Compactness in metric space over semi field 4.2.9.

**DEFINITION 4.2.10.** A subset \( S \) of \( X \) is called a U-net in the set \( M \) of \( X \) if, for arbitrary point \( x \in M \), a point \( s \in S \) can be found such that \( d(x, s) \in U \) (here \( U \) denotes a neighbourhood of zero in \( F \)).

**DEFINITION 4.2.11.** The space \( X \) is called totally bounded if, for arbitrary neighbourhood of zero \( U \) in \( F \), there exists a finite U-net in \( X \).
**Theorem 4.2.12.** (Hausdorff's Theorem): A metric space $X$ over the semi field $F$ is compact (in the sense of usual topology) if and only if it is totally bounded and complete.

**Theorem 4.2.13.** A metric space $X$ over the semi field $F$ is compact if and only if any sequence in $X$ has at least one limit point.

**Theorem 4.2.14.** A metric space $X$ over the semi field $F$ is compact if and only if every sequence of points of $X$ contains a convergent subsequence.

**Metrization Theorem of Uniform Structures 4.2.15.**

Let $F$ be a uniform structure on the set $X$. Then there exists a metric $d : X \times X \to F$ over some semi field $F$ such that the natural uniform structure of the metric space $(X, d, F)$ obtained coincides with the initial uniform structure $F$. Stated briefly, "every uniform structure can be metrized over some semi field $F".

4.3. **SEMI-FIELD VALUED ALMOST PERIODIC FUNCTIONS ON A LOCALLY COMPACT GROUP**

In this section we give a necessary and sufficient condition for a continuous function defined on a locally compact group to be almost periodic (in the sense of Von Neumann$^4$) in the context.

$^4$ Neumann, J. Von. (1)
of a topological semi field. The results obtained here are quite analogous with those formulated by Struble\(^5\).

Let \( f \) be a complex valued function defined on a locally compact group \( G \) with values in a complete topological semi field \( E \). Let \( d \) be a metric defined on \( E \).

**DEFINITION 4.3.1.** An element \( w \) of \( G \) is said to be \( \varepsilon \)-translation of \( f \), if corresponding to every neighbourhood \( U \) of \( 0 \) in \( E \) and for each \( \varepsilon > 0 \) the condition

\[
d(f(xy), f(xwy)) \in U_0^q, \varepsilon
\]

holds for all \( x, y \in G \).

**DEFINITION 4.3.2.** A complex valued function \( f \) is said to be Bohr almost periodic, if for every \( \varepsilon > 0 \), there exists a compact subset \( C \) of \( G \) such that \( zC \) contains an \( \varepsilon \)-translation of \( f \) for every \( z \in G \).

**DEFINITION 4.3.3.** A function \( f \) defined on \( G \) with values in a topological semi field \( E \), is said to be almost periodic, if

\[
\text{Struble, R. a.} \quad (1)
\]

6) we put \( U_0^q, \varepsilon = \{ x \in E \mid 0 < x(q) < \varepsilon \} \)

7) Neumann, J. Von. \quad (1)
corresponding to each \( \varepsilon > 0 \) and for every neighbourhood \( U \) of \( 0 \) in \( E \), there exists a finite family \( \{ A_1, A_2, \ldots, A_n \} \) of subsets of \( G \) such that
\[
G = \bigcup_{i=1}^{n} A_i
\]
and
\[
d \left( f(xuy), f(xvy) \right) \in \bigcup_{i=1}^{q} U_{i, \varepsilon}
\]
for all \( x, y \in G \), whenever \( u, v \) belong to the same \( A_i \).

**Theorem 4.3.4.** A complex valued continuous function defined on a locally compact group \( G \) with values in a topological semi field \( F \), is Bohr almost periodic if and only if it is almost periodic (in the sense of Von Neumann).

To prove the theorem we need the following lemmas.

**Lemma 4.3.5.** If \( f \) is an almost periodic function on a locally compact group \( G \) with values in a topological semi field \( F \), then \( f \) is Bohr almost periodic.

**Proof:** Since \( f \) is almost periodic, for each \( \varepsilon > 0 \) and corresponding to every neighbourhood \( U \) of \( 0 \) in \( F \), there exists a finite family \( \{ A_1, A_2, \ldots, A_n \} \) of subsets of \( G \) such that
\[
d \left( f(xuy), f(xvy) \right) \in \bigcup_{i=1}^{q} U_{i, \varepsilon}
\]
for all \( x, y \) of \( G \), whenever \( u, v \) belong to the same \( A_i \).
For each $i$, we select $w_i \in A_i$. Then the set
\[
\{ w_1^{-1}, w_2^{-1}, \ldots, w_n^{-1} \}
\]
is a compact set. For any $z \in G$, there exists $j, 1 \leq j \leq n$, such that
\[
\text{d} ( f(xw_j y), f(xzy) ) \in U_0^q, \varepsilon
\]
for all $x,y$ of $G$ and $z \in A_i$ ($i=1,2,\ldots,n$) which shows that $zw_j^{-1} \in zC$ is an $\varepsilon$-translation of $f$. Hence $f$ is Bohr almost periodic.

**Lemma 4.3.6.** If $f$ is a continuous Bohr almost periodic function on a locally compact group $G$ with values in a topological semi field $E$, then $f$ is uniformly continuous on $G$.

**Proof:** We must prove that for any $\varepsilon > 0$ and corresponding to every neighbourhood $U$ of $0$ in $E$, there exists a neighbourhood $V$ of the identity $e$ of $G$, such that
\[
\text{d} ( f(x_1), f(x_2) ) \in U_0^q, \varepsilon
\]
for all $x_1, x_2 \in G$, whenever $x_1 x_2^{-1} \in V$.

Let $C$ be a compact set in $G$ such that $zC$ contains an $\varepsilon/3$-translation of $f$, for every $z \in G$. We select a neighbourhood $W$ of $e$ in $G$ so that its closure is compact. Then $f$ is uniformly continuous on the compact set $\bar{WC}$. Let $V$ be any
neighbourhood of $e$ in $G$ such that

$$d(f(y_1), f(y_2)) \in U_0, \in \epsilon / 3$$

for any $y_1, y_2 \in W$, whenever $y_1 y_2^{-1} \in V$.

We assume that $V \subseteq W$.

Let $w$ be an $\epsilon / 3$-translation of $f$ such that $w \in x_2^{-1} C$.

Then

$$d(f(x_1), f(x_2)) \ll d(f(x_1), f(x_1 w)) + d(f(x_1 w), f(x_2 w)) +$$

$$+ d(f(x_2 w), f(x_2))$$.

But $x_1 w \in (V x_2) (x_2^{-1} C) = VC \subseteq WC$,

$$x_2 w \in WC \text{ and } (x_1 w) (x_2 w)^{-1} = x_1 x_2^{-1} \in V$$

therefore we also have

$$d(f(x_1 w), f(x_2 w)) \in U_0^{q}, \in \epsilon / 3$$.

Hence

$$d(f(x_1), f(x_2)) \in U_0^{q}, \in \epsilon / 3 + U_0^{q}, \in \epsilon / 3 + U_0^{q}, \in \epsilon / 3 = U_0^{q}, \epsilon$$.

Thus $x_1 x_2^{-1} \in V$ implies

$$d(f(x_1), f(x_2)) \in U_0^{q}, \epsilon$$

which proves the lemma.
PROOF OF THEOREM: We need only to prove that if \( f \) is continuous Bohr almost periodic function on \( G \) with values in \( E \), then \( f \) is almost periodic.

Let \( C \) be a compact set in \( G \) such that \( xC \) contains an \( \frac{\epsilon}{3} \) - translation of \( f \), corresponding to \( \epsilon > 0 \), for all \( x \in G \). By Lemma 4.3.6, there corresponds a neighbourhood \( V \) of \( e \) in \( G \) for every neighbourhood \( U \) of \( 0 \) in \( E \) such that

\[
\lambda(x, y_1, y_2) \in U_0, \frac{\epsilon}{3} \]

for all \( y_1, y_2 \in G \), whenever \( y_1 y_2^{-1} \in V \).

We choose a neighbourhood \( V \) of \( e \) in \( G \) such that \( z z^{-1} \subset V \) and \( z = z^{-1} \). Let \( \{x_1, x_2, \ldots, x_n\} \) be a finite family of subsets of \( G \) such that

\[
C \subseteq \bigcup_{i=1}^{n} x_i.
\]

For each \( i, 1 \leq i \leq n \), let \( A_i \) be the union of all sets of the form \( w z z^{-1} \), where \( w \) is any \( \frac{\epsilon}{3} \) - translation of \( f \). We claim that \( G = \bigcup_{i=1}^{n} A_i \). For if \( x \in G \), then \( xC \)

contains an \( \frac{\epsilon}{3} \) - translation \( w \) of \( f \). Since \( x^{-1} w \in C \), for each \( i \) we have \( x^{-1} w \in z^{-1} \).

Thus \( x \in w_{\infty} z^{-1} \subset A_i \).
Let \( u \) and \( v \) belong to same \( A_1 \). Then \( u \in w_1 z x_1^{-1} \) and \( v \in w_1 z x_1^{-1} \), where \( w_1, w_2 \) are any two \( \frac{\epsilon}{3} \) - translations of \( f \). From which we have

\[
(w_1^{-1} u x) (w_2^{-1} v x)^{-1} \in z z^{-1} \subseteq V.
\]

Hence for any \( x \in G \) we have

\[
d(f(ux), f(vx)) \leq d(f(ux), f(w_1^{-1} ux)) + d(f(w_1^{-1} ux), f(w_2^{-1} vx)) + d(f(w_2^{-1} vx), f(vx)) \leq \frac{\epsilon}{3} \cdot U_0, \frac{\epsilon}{3} + U_0, \frac{\epsilon}{3} - U_0, \frac{\epsilon}{3} = U_0, \epsilon.
\]

Hence the proof is completed.