CHAPTER III

COMMON FIXED POINT OF MAPPINGS

(3.1) 'A generalization of Banach contraction principle'.

A self map $T$ defined on a metric space $(X, d)$ is called contraction map if for some $0 < k < 1$, such that for all $x, y$ in $X$

(3.1.1) $d(T(x), T(y)) \leq k \cdot d(x, y)$

A well known Banach contraction principle states that if $X$ is complete and $T$ is contraction mapping, then $T$ has a unique fixed point.


[5] Reich, S. (2)
Kannan [9] proved that if $T$ is self-mapping of a complete metric space $X$ satisfying

$$d(T(x), T(y)) \leq \alpha [d(x, T(x)) + d(y, T(y))]$$

for all $x, y$ in $X$, where $0 \leq \alpha < \frac{1}{2}$, then $T$ has a unique fixed point in $X$.

3.2 Now we shall define a new-type of mapping and prove the following theorem.

Theorem (1) : Let $T$ be a continuous self mapping of a complete metric space $X$ satisfying

$$d(T(x), T(y)) \leq \alpha \frac{d(y, T^2(x)) d(y, T(y))}{d(T(x), T^2(x))} + \beta d(x, y)$$

for all $x, y$ in $X$, $x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, then $T$ has a unique fixed point in $X$.

Proof : Let $x_0$ be an arbitrary point of $X$, and let sequence $\{x_n\}$ of $X$ be defined as $x_n = T(x_{n-1})$ for $n = 0, 1, 2, \ldots \ldots \ldots \ldots$

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If \( x_n = x_{n+1} \) for some \( n \), then the result follows immediately, so let \( x_n \neq x_{n+1} \) for all \( n \). Now by (3.2.1)

\[
d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n))
\]

\[
\leq \frac{d(x_n, T^2(x_{n-1})) d(x_n, T(x_n))}{d(T(x_{n-1}), T^2(x_{n-1}))}
\]

\[
+ \beta d(x_{n-1}, x_n)
\]

which implies

\[
d(x_n, x_{n+1}) \leq \frac{d(x_n, x_{n+1}) d(x_n, x_{n+1})}{d(x_n, x_{n+1})}
\]

\[
+ \beta d(x_{n-1}, x_n)
\]

\[
(1-\alpha) d(x_n, x_{n+1}) \leq \beta d(x_{n-1}, x_n)
\]

\[
d(x_n, x_{n+1}) \leq \left( \frac{\beta}{1-\alpha} \right) d(x_{n-1}, x_n)
\]

\[
\leq \left( \frac{\beta}{1-\alpha} \right)^n d(x_0, x_1).
\]

Thus

\[
d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots \cdots + d(x_{n+p-1}, x_{n+p})
\]
\[ \begin{align*}
&\leq \left( \frac{\beta}{1-\alpha} \right)^n d(x_0, x_1) + \left( \frac{\beta}{1-\alpha} \right)^{n+1} d(x_0, x_1) + \cdots \\
&\quad \cdots \\
&\quad \cdots \\
&\quad \cdots \\
&\quad \cdots \\
&\quad \Rightarrow 0 \quad \text{as} \quad n \to \infty
\end{align*} \]

Hence \( \{x_n\} \) is Cauchy. Since \( X \) being complete, there exists some \( u \in X \) such that

\[ \lim_{n \to \infty} (x_n) = u. \]

Further the continuity of \( T \) implies

\[ T(u) = T \lim_{n \to \infty} (x_n) = \lim_{n \to \infty} T(x_n) \]

\[ = \lim_{n \to \infty} x_{n+1} = u. \]

So \( u \) is the fixed point of \( T \).
To prove uniqueness, if there exists another fixed point \( v \) in \( X \), such that \( v \neq u \), then

\[
d(u, v) = d(T(u), T(v)) \\
\leq \alpha \frac{d(v, T^2(u)) d(v, T(v))}{d(T(u), T^2(u))} + \beta d(u, v) \\
\leq \beta d(u, v) < d(u, v)
\]

as \( \beta < 1 \).

This contradicts our assumption.
Hence \( u = v \).
This completes the proof.

3.3 Now we assume that each pair \( T_i, T_j \) satisfies the same contraction condition, and concludes \( \{ T_n \} \) has a common fixed point.

Theorem (2) : Let \( \{ T_n \} \) \((n = 1, 2, \ldots)\) be a sequence of mappings defined on a complete metric space \( X \) into itself. Suppose that there are non-negative number \( \alpha, \beta \) such that \( x, y \) in \( X \).

\[
(3.3.1) \quad d(T_i(x), T_j(y)) \leq \alpha \frac{d(y, T^2_i(x)) d(y, T_j(y))}{d(T_i(x), T^2_i(x))} + \beta d(x, y)
\]
where \( \alpha + \beta < 1 \). Then the sequence of mapping \( \{ T_n \} \) has a unique common fixed point.

Proof: Let \( x_0 \in X \), and

let \( x_n = T_n(x_{n-1}) \) (\( n = 1, 2, \ldots \)).

Then we have

\[
d(x_1, x_2) = d(T_1(x_0), T_2(x_1))
\]

\[
\leq \alpha \frac{d(x_1, T_1^2(x_0)) d(x_1, T_2(x_1))}{d(T_1(x_0), T_1^2(x_0))} + \beta d(x_0, x_1)
\]

\[
\leq \alpha \frac{d(x_1, x_2) d(x_1, x_2)}{d(x_1, x_2)} + \beta d(x_0, x_1)
\]

Or

\[
(1-\alpha) d(x_1, x_2) \leq \beta d(x_0, x_1)
\]

\[
d(x_1, x_2) \leq \left( \frac{\beta}{1-\alpha} \right) d(x_0, x_1).
\]

Similarly, we have

\[
d(x_2, x_3) \leq \left( \frac{\beta}{1-\alpha} \right) d(x_1, x_2)
\]
\[ \left( \frac{\beta}{1-\alpha} \right)^2 d(x_0, x_1) \leq \left( \frac{\beta}{1-\alpha} \right)^n d(x_0, x_1) \]

In general

\[ d(x_n, x_{n+1}) \leq \left( \frac{\beta}{1-\alpha} \right)^n d(x_0, x_1) \]

Thus

\[ d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \]

\[ \leq \left( \frac{\beta}{1-\alpha} \right)^n d(x_0, x_1) + \left( \frac{\beta}{1-\alpha} \right)^{n+1} d(x_0, x_1) + \ldots + \left( \frac{\beta}{1-\alpha} \right)^{n+p-1} d(x_0, x_1) \]

\[ \leq \left[ \frac{\beta^n}{1-\alpha} + \frac{\beta^{n+1}}{1-\alpha} + \ldots + \frac{\beta^{n+p-1}}{1-\alpha} \right] d(x_0, x_1) \]

\[ \leq \frac{\beta^n}{1-\alpha} d(x_0, x_1) \]

\[ \to 0 \text{ as } n \to \infty \]

Hence \( \{x_n\} \) is Cauchy.
By the completeness of $X$, \( \{x_n\} \) converges to some point $x$ in $X$. For the point $x$,

\[
d(x, T_n(x)) \leq d(x, x_{m+1}) + d(x_{m+1}, T_n(x))
\]

\[
= d(x, x_{m+1}) + d(T_{m+1}(x_m), T_n(x))
\]

\[
\leq d(x, x_{m+1}) + \frac{d(x, T_{m+1}(x_m))d(x, T_n(x))}{d(T_{m+1}(x_m), T_{m+1}(x_m))}
\]

\[
+ \beta d(x_m, x)
\]

\[
\leq d(x, x_{m+1}) + \frac{d(x, x_{m+2})d(x, x_{m+1})}{d(x_{m+1}, x_{m+2})}
\]

\[
+ \beta d(x_m, x)
\]

\[
\longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty
\]

Then we have

\[
d(x, T_n(x)) = 0, \quad \text{i.e.} \quad x \quad \text{is a common fixed point of all} \quad T_n.
\]

Now if there exists another fixed point $y \neq x$, such that $T_n(y) = y$ for every $n$. 
Then we have

\[ d(x, y) = d(T_n(x), T_n(y)) \]

\[ \leq \alpha \frac{d(y, T_n^2(x)) + d(y, T_n(y))}{d(T_n(x), T_n^2(x))} + \beta d(x, y) \]

\[ \leq \beta d(x, y) \]

Since \( \alpha + \beta < 1 \), then we arrive at a contradiction.

Hence \( d(x, y) = 0 \implies x = y \).

This completes the proof.

3.4 The second assume that each \( T_n \) satisfies the same contractive condition and that \( \{T_n\} \) tends to pointwise to a limit function \( T \). The conclusion is that \( T \) has a fixed point \( x_0 \) which is the limit of each of the fixed point \( x_n \) of \( T_n \).

Theorem (3) 1 Let \( \{T_n\} \) be a sequence of mapping of a complete metric space \( X \) into itself. Let \( x_n \) be a fixed point of \( T_n \) (\( n = 1, 2, \ldots \)) and suppose that \( T_n \) converges uniformly to \( T_0 \). If
\( T_0 \) satisfies the condition

\[
(3.4.1) \quad d(T_0(x), T_0(y)) \leq \alpha \frac{d(y, T_0^2(x)) d(y, T_0(y))}{d(T_0(x), T_0^2(x))} + \beta d(x, y)
\]

where \( \alpha, \beta \) are non-negative, and \( \alpha + \beta < 1 \).

Then \( \{x_n\} \) converges to the fixed point \( x_0 \) of \( T_0 \).

Proof: Let \( \varepsilon > 0 \), such that

\[
(3.4.2) \quad d(T_n(x), T_0(x)) < \varepsilon
\]

for all \( x \in X \), and \( N \leq n \).

Hence

\[
d(x_n, x_0) = d(T_n(x_n), T_0(x_0)) \\
\leq d(T_n(x_n), T_0(x_n)) + d(T_0(x_n), T_0(x_0)) \\
\leq d(T_n(x_n), T_0(x_n)) \\
\leq d(x_0, T_0^2(x_0)) d(x_0, T_0(x_0)) + \alpha \frac{d(x_0, T_0^2(x_0)) d(x_0, T_0(x_0))}{d(T_0(x_n), T_0^2(x_n))} + \beta d(x_n, x_0) \\
\leq d(T_n(x_n), T_0(x_n)) + \beta d(x_n, x_0)
\]
or

\[(1 - \beta) d(x_n, x_0) \leq d(T_n(x_n), T_o(x_n))\]

\[d(x_n, x_0) \leq \left(\frac{1}{1 - \beta}\right) d(T_n(x_n), T_o(x_n))\]

since \((\alpha + \beta) < 1\), we have

\[d(x_n, x_0) \leq \left(\frac{1}{1 - \beta}\right) e\]

which shows that \(\{x_n\}\) converges to \(x_0\).

Completes the proof.

3.5 The third type assumes that \(\{T_n\}\) converges uniformly to a mapping \(T\) which satisfies a particular contractive condition with \(x_0\) is the fixed point of \(T\), the conclusion is that \(x_n \longrightarrow x_0\).

Theorem (4): Let \(\{T_n\}\) \((n = 1, 2, \ldots, \ldots)\) be a sequence of mappings of \(X\) into itself. Suppose that there are non-negative number \(\alpha, \beta\) and \(\{m_n\}\) be a non-negative integer, such that for all \(x, y\) in \(X\) and every pair \(i, j\) with \(i \neq j\)

\[d(T_{i}^{m_i}(x), T_{j}^{m_j}(y)) \leq \alpha \frac{d(y, T_i^{m_i}(T_i^{m_i}(x))), d(y, T_j^{m_j}(y))}{d(T_i^{m_i}(x), T_i^{m_i}(x))} + \beta d(x, y)\]
where $\alpha + \beta < 1$. Then the sequence of mappings $\{T_n\}$ has a unique common fixed point.

Proof: Let $u_i = T_1^m (1 = 1, 2, \ldots)$. Then for $i \neq j$, we have

\[
(3.5.2) \quad d(u_i(x), u_j(y)) \leq \alpha \frac{d(y, u_i^2(x)) d(y, u_j(y))}{d(u_i(x), u_i^2(x))} + \beta d(x, y)
\]

By the theorem (2) the sequence of mappings $\{u_1\}$ has a unique common fixed point $x_0$ in $X$. Then

\[
x_0 = u_i(x_0) = T_1^m(x_0)
\]

Therefore we have

\[
T_1(x_0) = T_1(T_1^m(x_0)) = T_1^m(T_1(x_0)) = u_i(x_0)
\]

This means that $T_1(x_0)$ is fixed point of $u_i$. By (3.5.2) we have

\[
d(T_1(x_0), T_j(x_0)) = d(u_i(T_1(x_0)), u_j(T_1(x_0)))
\]
\[ d(T_j(x_0), u_2^j(T_1(x_0))) d(T_j(x_0), u_j(T_j(x_0))) \leq a \left( \frac{d(u_1(T_1(x_0)), u_2^j(T_1(x_0)))}{d(u_2^j(T_1(x_0)), u_1^j(T_1(x_0)))} \right) + \beta d(T_1(x_0), T_j(x_0)) \]

which implies that

\[ d(T_1(x_0), T_j(x_0)) \leq \beta d(T_1(x_0), T_j(x_0)) \]

we have

\[ d(T_1(x_0), T_j(x_0)) = 0 \]

i.e.

\[ T_1(x_0) = T_j(x_0) \]

Therefore, \( T_1(x_0) = T_j(x_0) \) is a common fixed point of \( u_1 \) and \( u_j \).

To prove that \( x_0 \) is a unique common fixed point of \( T_1 \) and \( T_j \). Let \( y_0 \) be a common fixed point of \( T_1 \) and \( T_j \).

\[ d(x_0, y_0) = d(T_1^{m_1}(x_0), T_j^{m_j}(y_0)) \leq a \left( \frac{d(y_0, T_1^{m_1}(x_0)) d(y_0, T_j^{m_j}(y_0))}{d(T_1^{m_1}(x_0), T_1^{m_1}(T_1^{m_1}(x_0)))} \right) + \beta d(x_0, y_0) \]
\[ \delta \beta d(x_0, y_0) \]

which implies \( x_0 = y_0 \).

Therefore, \( T_i \) and \( T_j \) have a unique common fixed point in \( X \).

Since \( \{u_i\} \) has a unique common fixed point \( x_0 \), we have \( T_i(x_0) = x_0 \) (\( i = 1, 2, \ldots \)).

Therefore, \( x_0 \) is a common fixed point of the sequence of mappings \( \{T_n\} \). The uniqueness is trivial.

This completes the proof.

3.6 Now we shall prove the following theorem which is a generalization of M.G. Maia [10].

Theorem (5) : Let \( X \) be a metric space with two metric \( d_1 \) and \( d_2 \). If \( X \) satisfies

(i) \( d_1(x, y) \leq d_2(x, y) \) for all \( x, y \) in \( X \).

(ii) \( X \) is complete with respect to \( d_1 \).

(iii) Two mappings \( u, v : X \rightarrow X \) are continuous with respect to \( d_1 \), and

\[
(3.6.1) \quad d_2(u(x), v(y)) \leq \alpha \frac{d_2(y, u^2(x)) \cdot d_2(y, v(y))}{d_2(u(x), u^2(x))} + \beta d_2(x, y)
\]

for all \( x, y \) in \( X \), where \( \alpha, \beta \) are non-negative and \( \alpha + \beta < 1 \), then \( u \) and \( v \) have a unique common fixed point.

**Proof:** Let \( x_0 \in X \), such that in sequence \( \{x_n\} \),

where

\[
x_n = \begin{cases} 
u(x_{n-1}) & \text{n is even} \\ u(x_{n-1}) & \text{n is odd} \end{cases}
\]

\[
d_2(x_{2n}, x_{2n+1}) = d_2(u(x_{2n-1}), v(x_{2n}))
\]

\[
d_2(x_1, x_2) = d_2(u(x_0), v(x_1))
\]

\[
\leq \alpha \frac{d_2(x_1, u^2(x_0)) \cdot d_2(x_1, v(x_1))}{d_2(u(x_0), u^2(x_0))} + \beta d_2(x_0, x_1)
\]
\[ d_2(x_1, x_2) d_2(x_1, x_2) \leq \alpha \frac{d_2(x_1, x_2)}{d_2(x_1, x_2)} + \beta d_2(x_0, x_1) \]

\[ (1-\alpha) d_2(x_1, x_2) \leq \beta d_2(x_0, x_1) \]

\[ d_2(x_1, x_2) \leq \left( \frac{\xi}{1-\alpha} \right) d_2(x_0, x_1) \]

Similarly

\[ d_2(x_2, x_3) \leq \left( \frac{\xi}{1-\alpha} \right) d_2(x_1, x_2) \leq \left( \frac{\xi}{1-\alpha} \right)^2 d_2(x_0, x_1) \]

In general

\[ d_2(x_{2n}, x_{2n+1}) = d_2(u(x_{2n-1}), v(x_{2n})) \]

\[ d_2(x_{2n}, u^2(x_{2n-1})) d_2(x_{2n}, v(x_{2n})) \leq \alpha \frac{d_2(x_{2n}, u^2(x_{2n-1}))}{d_2(u(x_{2n-1}), u^2(x_{2n-1}))} + \beta d_2(x_{2n-1}, x_{2n}) \]

\[ d_2(x_{2n}, x_{2n+1}) d_2(x_{2n}, x_{2n+1}) \leq \alpha \frac{d_2(x_{2n}, x_{2n+1})}{d_2(x_{2n}, x_{2n+1})} + \beta d_2(x_{2n-1}, x_{2n}) \]
\[(1-\alpha) \, d_2(x_{2n}, x_{2n+1}) \leq \beta \, d_2(x_{2n-1}, x_{2n})\]
\[d_2(x_{2n}, x_{2n+1}) \leq \left(\frac{\beta}{1-\alpha}\right) d_2(x_{2n-1}, x_{2n})\]
\[\leq \left(\frac{\beta}{1-\alpha}\right) d_2(x_0, x_1)\]
\[
\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty
\]

Therefore, \(\{x_n\}\) is a Cauchy sequence. By the completeness of \(X\), \(\{x_n\}\) converges to some point \(x\) in \(X\). Then the sequence \(\{x_{n_k}\} \longrightarrow x\), where \(n_k = 2k\). Now \(u\) and \(v\) continuous

\[u \cdot v(x) = u \cdot v\left(\lim_{k \to \infty} x_{n_k}\right) = \lim_{k \to \infty} x_{n_k} = x.\]

Now we show that \(v(x) = x\).

If \(v(x) \neq x\), then

\[d_2(v(x), x) = d_2(v(x), uv(x))\]
\[\leq \alpha \cdot \frac{d_2(v(x), v^2(x)) \cdot d_2(v(x), uv(x))}{d_2(v(x), v^2(x))}\]
\[+ \beta \, d_2(x, v(x))\]
\[\leq \beta \, d_2(x, v(x)) < d_2(x, v(x))\]

a contradiction.
Hence $x$ is a common fixed point of $u$ and $v$.

This completes the proof.

3.7 Now we shall prove the following theorem which is a generalization of above result.

Theorem (6): Let $X$ be a metric space with two metrics $d_1$ and $d_2$, and $T_i (i = 1, 2, \ldots, k)$ is a finite family of continuous mappings of $X$ into itself. Suppose that

(i) $d_1(x,y) \leq d_2(x,y)$ for all $x,y \in X$

(ii) $X$ is complete with respect to $d_1$

(iii) $T_i T_j = T_j T_i (i, j = 1, 2, \ldots, k)$

(iv) There are two systems of positive integer $(m_1, m_2, \ldots, m_k), (n_1, n_2, \ldots, n_k)$ non-negative $\alpha, \beta$ with $\alpha + \beta < 1$, such that $x,y$ in $X$.

(3.7.1) $d_2(T_1^{m_1}T_2^{m_2} \cdots T_k^{m_k}(x), T_1^{n_1}T_2^{n_2} \cdots T_k^{n_k}(y))$

$\leq \alpha d_2(y, T_1^{m_1}T_2^{n_1} \cdots T_k^{m_k}(T_1^{m_1}T_2^{m_2} \cdots T_k^{m_k}(x))).$
\( d_2(y, T_1^{n_1} T_2^{n_2} \ldots \ldots T_k^{n_k}(y)) \)

\[
= d_2(T_1^{m_1} T_2^{m_2} \ldots \ldots T_k^{m_k}(x), T_1^{m_1} T_2^{m_2} \ldots \ldots T_k^{m_k}(T_1^{m_1} T_2^{m_2} \ldots \ldots T_k^{m_k}(x))) + \beta d_2(x, y)
\]

Then \( T_i (i = 1, 2, \ldots \ldots k) \) have a unique common fixed point.

Proof: As in the proof of the above theorem, let
\[ u = T_1^{m_1} T_2^{m_2} \ldots \ldots T_k^{m_k} \]
\[ v = T_1^{n_1} T_2^{n_2} \ldots \ldots T_k^{n_k} \]

then \( u, v \) are continuous and satisfy the condition (3.7.1). Therefore, by theorem (5) \( uv \) have a unique common fixed point \( x_0 \) in \( X \). Hence \( u(x_0) = v(x_0) = x_0 \). Then we can arrive the technique in the proof of theorem (2). Therefore, \( T_i(x_0) \) \( (i = 1, 2, \ldots \ldots k) \) are common fixed point of \( u \) and \( v \). From the uniqueness of \( u \) and \( v \), we know
\[ T_i(x_0) = x_0 \ (i = 1, 2, \ldots \ldots k) \]

This completes the proof.