CHAPTER VI

FIXED POINT THEOREMS FOR MAPPINGS WHICH ARE NOT NECESSARILY CONTINUOUS

6.1 Some sufficient conditions have been obtained for self-mapping of a complete metric space which ensure a unique fixed point.

Let \((X,d)\) be a complete metric space. A self mapping \(T\) of \(X\) is said to be contraction if

\[
(6.1.1) \quad d(Tx, Ty) \leq \alpha d(x, y)
\]

for all \(x, y \in X\) and \(0 \leq \alpha < 1\). Then \(T\) has a unique fixed point in \(X\). Contraction mapping (6.1.1) is necessarily continuous and yet the fixed or common fixed points have been obtained for such type of mappings. For example Kannan [1] obtained a unique fixed point for a self mapping \(T\) of a complete metric space satisfying

\[
(6.1.2) \quad d(Tx, Ty) \leq \alpha [ d(x, Tx) + d(y, Ty) ]
\]

for all \(x, y \in X\), where \(0 \leq \alpha < \frac{1}{2}\). The mapping \(T\) satisfying (6.1.2) is not necessarily continuous.

Further, Ciric [2], Pachpatte [3], Taskovic [4], Gupta and Ranganthan [5] and many others obtained fixed points for different types of mappings of $X$, which are not necessarily continuous. Rhoades [6] gives a comprehensive list of various mappings, and most of these mappings are not necessarily continuous.

6.2 Now we shall prove the following theorem for mapping which is quite new.

Theorem: Let $(X,d)$ be a complete metric space and $T : X \rightarrow X$ satisfy

$$d(T^{n+1}(x), T^{n+2}(y)) \leq d(T^n(x), T^{n+1}(x))$$
$$+ d(T^{n+1}(y), T^{n+2}(x)))$$
$$+ \beta [d(T^n(x), T^{n+2}(x))$$
$$+ d(T^{n+2}(y), T^{n+1}(x))$$
$$+ \gamma d(T^n(x), T^{n+1}(y))$$

for all $x, y$ in $X$, $\alpha$, $\beta$ and $\gamma$ are non-negative and $2\alpha + 2\beta + \gamma < 1$. Then $T$ has a unique fixed point.

[3] Pachpatte, B.G. (1)
Proof : We prove the theorem for \( n = 0 \). The proof in the general case follows on similar lines, condition (6.2.1) is now

\[
(6.2.2) \quad d(Tx, \tau^2 y) \leq \alpha [d(x, Tx) + d(Ty, \tau^2 x)] \\
+ \beta [d(x, T^2 x) + d(T^2 y, \tau^2 x)] \\
+ \gamma d(x, Ty)
\]

We define a sequence \( \{x_n\} \) as follows

\[
x_n = T(x_{n-1}) = T^n(x_0) \quad (n = 1, 2, \ldots)
\]

Then we have

\[
d(x_1, x_2) = d(Tx_0, \tau^2 x_0) \\
\leq \alpha [d(x_0, Tx_0) + d(Tx_0, \tau^2 x_0)] \\
+ \beta [d(x_0, T^2 x_0) + d(T^2 x_0, \tau^2 x_0)] \\
+ \gamma d(x_0, Tx_0) \\
\leq \alpha [d(x_0, x_1) + d(x_1, x_2)] \\
+ \beta [d(x_0, x_2)] + \gamma d(x_0, x_1) \\
\leq (\alpha + \beta + \gamma) d(x_0, x_1) + (\alpha + \beta) d(x_1, x_2)
\]
\[(1-\alpha-\beta) \, d(x_1, x_2) \leq (\alpha+\beta+\gamma) \, d(x_0, x_1)\]

i.e.
\[d(x_1, x_2) \leq \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\beta}\right) \, d(x_0, x_1)\]

similarly
\[d(x_2, x_3) \leq \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\beta}\right)^2 \, d(x_0, x_1)\]

In general
\[d(x_n, x_{n+1}) \leq \left(\frac{\alpha+\beta+\gamma}{1-\alpha-\beta}\right)^n \, d(x_0, x_1)\]

since \[\left(\frac{\alpha+\beta+\gamma}{1-\alpha-\beta}\right) < 1\].

Therefore, the sequence \(\{x_n\}\) is a Cauchy sequence. Hence by the completeness of \(X\), \(\{x_n\}\) converges to some point \(u\) in \(X\), such that

\[\lim_{n \to \infty} x_n = u.\]

We shall show that \(u\) is the unique fixed point of \(T\).

Let \(t\) be any integer. Then

\[d(u, Tu) \leq d(u, x_t) + d(x_t, Tu)\]

\[= d(u, x_t) + d(Tu, T^2x_{t-2})\]
\[ \leq d(u, x_t) + \alpha [d(u, T_u) + d(x_{t-2}, T^2 u)] \\
+ \beta [d(u, T^2 u) + d(T^2 x_{t-2}, T^2 u)] \\
+ \gamma d(u, T x_{t-2}) \\
\leq d(u, x_t) + \alpha [d(u, T_u) + d(x_{t-1}, T^2 u)] \\
+ \beta [d(u, T^2 u) + d(x_t, T^2 u)] \\
+ \gamma d(u, x_{t-1}) \\
\leq d(u, x_t) + \alpha [d(u, T_u) + d(x_{t-1}, T^2 u)] \\
+ \beta d(u, x_t) + \gamma d(u, x_{t-1}) \\
(1-\alpha) d(u, T_u) \leq d(u, x_t) + \alpha d(x_{t-1}, T^2 u) \\
+ \beta d(u, x_t) + \gamma d(u, x_{t-1}). \]

The expression on the right hand side can be made arbitrary small by choosing \( t \) to be sufficiently large.

Hence \( d(u, T_u) = 0 \)

i.e. \( u = T_u \).
Therefore, \( u \) is a fixed point of \( T \).

Suppose \( v \) be another fixed point of \( T \).

Then

\[
d(u, v) = d(Tu, v^2) \\
\leq \alpha [d(u, Tu) + d(Tv, v^2)] + \beta [d(v, v^2) + d(T^2v, v^2)] + \gamma d(u, Tv)
\]

\[
d(u, v) \leq \alpha d(v, u) + \beta d(v, v) + \gamma d(u, v)
\]

\[
(1 - \alpha - \beta - \gamma) d(u, v) \leq 0
\]

i.e. \( u = v \).

Therefore, \( T \) has a unique common fixed point \( u \).

This completes the proof.

6.3 More recently in 1980, Brain Fisher [7] has proved the following theorem.

Theorem (2) : If \( S \) and \( T \) are mappings of a complete metric space \( X \) into itself satisfying the inequality.

\[(6.3.1) \quad [d(Sx, Ty)]^2 \leq bd(x, Sz) d(x, Ty) + cd(y, Sz) d(y, Ty)\]

for all \(x, y\) in \(X\), where \(b, c > 0\) and

\[(6.3.2) \quad [b + (b^2 + 4b)^2][c + (c^2 + 4c)^2] < 4\]

Then \(S\) and \(T\) have a unique common fixed point.

Further, Fisher obtained the particular case of above theorem by taking \(S = T\) and \(b = c\) given as follows

**Theorem (3)**: If \(T\) is a self-mapping of a complete metric space \((X, d)\) satisfying the inequality

\[(6.3.3) \quad [d(Tx, Ty)]^2 \leq c[d(x, Tx) d(x, Ty)] + d(y, Tx) d(y, Ty)]\]

for all \(x, y\) in \(X\), where \(0 \leq c < \frac{1}{2}\). Then \(T\) has a unique fixed point.

6.4 Now we shall prove the following theorem:

**Theorem (4)**: Let \((X, d)\) be a complete metric space and let \(T : X \rightarrow X\) satisfying the inequality
(6.4.1) \[ [d(Tx, T^2y)]^2 \leq c [d(x, Tx) d(x, T^2x) \]
\[ + d(T^2y, T^2x) d(Ty, T^2x)] \]
for all \( x, y \) in \( X \), and \( c \) a non-negative constant such that \( 0 \leq c < \frac{1}{2} \). Then \( T \) has a unique fixed point in \( X \).

Proof: We define a sequence of elements \( \{x_n\} \) in \( X \) as follows:

Let \( x_0 \) be any arbitrary element in \( X \), and let \( x_n = T(x_{n-1}) = T^n(x_0) \) for \( n = 1, 2, \ldots \).

Then by (6.4.1), we have

\[
[d(x_1, x_2)]^2 = [d(Tx_0, T^2x_0)]^2
\]
\[ \leq c [d(x_0, Tx_0) d(x_0, T^2x_0) \]
\[ + d(T^2x_0, T^2x_0) d(Tx_0, T^2x_0)] \]
\[ \leq c d(x_0, x_1) d(x_0, x_2) \]
\[ \leq c d(x_0, x_1) [d(x_0, x_1) + d(x_1, x_2)] \]
\[ \leq c[d(x_0, x_1)]^2 + c d(x_0, x_1) d(x_1, x_2) \]

By noting that the quadratic equation \( x^2 = \alpha x + \alpha \) has the roots...
\[ x = \left[ a \pm \left( a^2 + 4a \right) \right] / 2. \]

Hence it follows that

\[ d(x_1, x_2) \leq q \cdot d(x_0, x_1), \]

where \( q = \left[ c + (c^2 + 4c)^{1/2} \right] / 2 < 1 \)

for \( 0 \leq c < 1/2 \).

Similarly

\[ \left[ d(x_2, x_3) \right]^2 = \left[ d(Tx_1, T^2x_1) \right]^2 \]

\[ \left[ d(x_2, x_3) \right]^2 \leq c \left[ d(x_1, x_2) \cdot d(x_1, x_3) \right. \]

\[ \left. + d(x_2, x_2) \cdot d(x_2, x_3) \right] \]

which gives on further simplification,

\[ d(x_2, x_3) \leq q \cdot d(x_1, x_2) \]

\[ \leq q^2 \cdot d(x_0, x_1). \]

So, in general, we have

\[ d(x_n, x_{n+1}) \leq q^n \cdot d(x_0, x_1) \]

since \( q < 1 \), it follows that the sequence \( \{x_n\} \) is a Cauchy sequence. Now from the completeness of \( X \), there exists some \( u \) in \( X \), such that

\[ \lim_{n \to \infty} (x_n) = u. \]
Now, we shall show that \( u \) is the fixed point of \( T \). Consider

\[
[d(u_n, x_n)]^2 = [d(u_n, T x_{n-1})]^2 = [d(u_n, T^2 x_{n-2})]^2
\]

\[
\leq c [d(u_n, u) d(u, T^2 u)
+ d(T^2 x_{n-2}, T^2 u) d(T x_{n-2}, T^2 u)]
\]

\[
\leq c [d(u_n, u) d(u, T^2 u)
+ d(x_n, T^2 u) d(x_{n-1}, T^2 u)]
\]

On letting \( n \) tends to infinity, we get \( [d(u_n, u)]^2 = 0 \) which implies that \( u \) is the fixed point of \( T \).

Now, suppose that \( v \) is another fixed point of \( T \) in \( X \) such that \( v \neq u \), then \( d(u, v) > 0 \).

Further, consider

\[
[d(u, v)]^2 = [d(u_n, T^2 v)]^2
\]

\[
\leq c [d(u_n, u) d(u, T^2 u)
+ d(T^2 v, T^2 u) d(T v, T^2 u)]
\]

\[ \leq 0, \]

a contradiction to our supposition. Hence it follows that \( u = v \) and so, \( u \) is the unique fixed point of \( T \).

This completes the proof.