3.1. Introduction

A complex random p-vector $x$ is said to have complex elliptical distribution if its characteristic function is represented as

$$\exp(-i (t^* \mu)_R) \psi(t^* \Sigma t) \quad \text{for all } t \in \mathbb{C}^p \quad \ldots(3.1.1)$$

where $(t^* \mu)_R$ denotes the real part of $t_1^* \mu_1 + t_2^* \mu_2$ of $t^* \mu$. Here $\mu_1 + i \mu_2 = \mu \in \mathbb{C}^p$ and $\Sigma$ is a Hermitian non-negative definite. We shall assume throughout that $\Sigma$ is positive definite.

Let \( X = (x_1, x_2, \ldots, x_n) \) be \( n \) independent observations on the complex random vector \( x \) and let us consider the linear growth curve model

\[
X = B \xi A^* + \varepsilon \quad \ldots \quad (3.1.2)
\]

where \( A \) and \( B \) are known complex matrices of full rank (i.e. \( B^*B \) and \( A^*A \) are positive definite), \( \xi : q \times m \) is unknown location parametric complex matrix and the column vector \( \xi_i \)'s of \( \varepsilon \) are independent and identical complex random vectors and

\[
E \xi_i = 0, \quad \text{and} \quad E \xi_i \xi_i^* = \text{Var}(\xi_i) = 2b_1 \Sigma = 2b_1 PP^* \quad \ldots \quad (3.1.3)
\]

where \( \Sigma \) is positive definite and \( P \) is non-singular.

Here \( \xi \) and \( \Sigma \) are unknown and their maximum likelihood estimates (under normality of \( \xi_i \)'s) are \( \hat{\xi} \) and \( \hat{\Sigma} \) and they are given by

\[
n \hat{\Sigma} = (X - B \hat{\xi} A^*) (X - B \hat{\xi} A^*)^* \quad \text{and} \quad \hat{\Sigma} = (B^*S^{-1}B)^{-1}B^*S^{-1}XA(A^*A)^{-1} \quad \ldots \quad (3.1.4)
\]

with \( nS = XX^* - XA(A^*A)^{-1} A^* X^* \).
We shall consider the asymptotic joint distribution of \( Z = XA(A^*A)^{-1} \) and \( S \) under the assumption of complex elliptical distribution of \( \xi_i \)'s (or under certain assumption on the moments of \( \xi_i \)'s), and using these results, asymptotic confidence bounds on the elements of \( \xi \) when \( \Sigma \) is unknown are obtained on the line similar to real variables (see for example, Khatri (1988)).

Further the various likelihood ratio tests of hypothesis concerning \( \Sigma \) similar to the problems developed on the real elliptical distributions (see for example Khatri and Bhavsar (1988b)) are considered and their asymptotic distributions of the likelihood ratio test obtained under complex normality assumption are established for the class of complex elliptical distributions. These asymptotic distributions are either non-central Chi-squares or that of a linear function of non-central Chi-square variates.

3.2. Asymptotic results connected to complex elliptical distributions

Let \( x \) be a complex random \( p \)-vector such that
where is a known positive constant and $\Sigma$ is a Hermitian positive definite matrix. Let $\Sigma = PP^*$ and $y' = (P^{-1}(\bar{x} - y))' = (y_1, y_2, \ldots, y_p)$, $y_1 = y_{11} + \sqrt{1} y_{12}$, where $P$ is a non-singular matrix. Then moments of $y_{1j}$ ($i = 1, 2, \ldots, p; j = 1, 2$) can be assumed to be

$$
E(y) = 0, \quad E(y_{1j}^2) = b_1, \quad E(y_{1j} y_{1'j'}) = 0, \quad E(y_{1j}^2 y_{1'j'}^2) = b_2 \text{ for } i \neq i' \text{ or } j \neq j',
$$

$$
E(y_{1j} y_{1'j'} y_{1''j''}) = 0, \quad E(y_{1j} y_{1'j'} y_{1''j''} y_{1'''j'''}) = 0
$$

for $i, i', i'', i''' = 1, 2, \ldots, p$ and $j, j', j'', j''' = 1$ or $2$.

We shall write $y \sim G(y)$ to denote that the distribution of $y$ possesses the above moment relations. If $U$ is a unitary matrix, then it is easy to verify that $Uy$ possesses the above four moment relations. Further, if $\bar{x}$
has complex elliptical distribution introduced in (3.1.1) then \( \tilde{y} = F^{-1}(x - \mu) \) satisfies the above moment relations, because the characteristic function of \( y \) (or the joint characteristic function of \( y_1 \) and \( y_2 \)) is \( \psi(t^*_1 t_2) \) for all \( t \in \mathbb{C}^p \), and \( b_1 = -2 \psi'(0) \) and \( b_2 = 4 \psi''(0) \). The distribution of \( y \) is known as complex spherical distribution.

Thus for a complex random \( p \)-vector \( x \) having the first two moments given by (3.2.1), if we assume \( \tilde{y} = F^{-1}(x - \mu) \) is distributed as \( G(y) \), which satisfies the above moment relations (3.2.2), then we shall denote \( x \sim G(\mu, \Sigma) \) and \( G(y) = G(y; \Omega, I_p) \). This distribution includes the class of complex elliptical distributions.

Let \( y_i \) \((i = 1, 2, \ldots, n)\) be \( n \) independent observations from \( G(y) \), denoted by \( y_i \sim IG(y) \) for all \( i \), and let

\[
W = \sum_{i=1}^{n} \left( y_i y_i^* - 2b_1 I_p \right)/2/n b_1. \quad \ldots (3.2.3)
\]

\[
W = (w_1, w_2, \ldots, w_n) \text{ and } [\text{vec}(W)]' = (w_1', w_2', \ldots, w_n')
\]

Hence from (3.2.3),
\[ \sqrt{n} \{\text{vec}(W)\} = \sum_{i=1}^{n} \left((2b_1)^{-1}\{\text{vec}(y_i^* y_i^*)\} - \{\text{vec}(I_p)\}\right) = \sum_{i=1}^{n} v_i \]

then, we get

\[ \text{vec}(W) = \sqrt{n} \frac{v_i}{y_i} = 2b_1^{-1}\{\text{vec}(y_i^* y_i^*)\} - \{\text{vec}(I_p)\}, \]

for \( i = 1, 2, \ldots, n \). Hence, \( v_1, v_2, \ldots, v_n \) are distributed as \( y = (2b_1)^{-1}\{\text{vec}(y^* y^*)\} - \{\text{vec}(I_p)\} \) with \( y = y_1^* + \sqrt{-1} y_2^* \) being distributed as \( G(y) \). Then, denoting the \( (k, l) \)th component of \( y \) as \( v_{(k, l)} \), we get

\[ v_{(k, l)} = (2b_1)^{-1}\left[(y_1 y_2 + y_2 y_1) + \sqrt{-1}(y_1 y_1 - y_2 y_1)\right] - \delta_{k l} \ldots (3.2.4) \]

with \( \delta_{k l} = 1 \) if \( k = l \) and \( \delta_{k l} = 0 \) if \( k \neq l \). Then using the conditions on moments given by (3.2.2), we get

\[ E(y_i) = 0 \quad \text{and taking}, K+1 = \frac{b_2}{b_1}, \]

\[ \text{Var}[v_{(k,k)}] = 2K+1, \quad \text{for all } k = 1, 2, \ldots, p, \]

\[ \text{Cov}[v_{(k,k)}, v_{(l,l)}] = K, \quad \text{for } k \neq l; k, l = 1, 2, \ldots, p, \]
Var[Re(v_{k,l})] = Var[Im(v_{k,l})] = \frac{K+1}{2},
for all k \neq l; k, l = 1,2,\ldots, p, \ldots(3.2.5)

Cov[v_{(k,k)}, Re(v_{(i,l)})] = 0, for k \neq l, i \neq l,

Cov[v_{(k,k)}, Im(v_{(i,l)})] = 0, for k \neq l, i \neq l,

and Cov[Re(v_{(k,l)}), Im(v_{(k',l')})] = 0, for k \neq l, k' = l'.

Thus, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n are independently and identically distributed complex random vectors with the existence of the first two moments and hence by the central limit theorem (see for example Cramer, (1951)),

\text{vec}(\mathbf{w}) = \sqrt{n} \bar{\mathbf{v}} has a limiting normal distribution.

If \mathbf{w}_c(\mathbf{1}) = (w_{11}, w_{22}, \ldots, w_{pp}), then asymptotically

\mathbf{w}(\mathbf{1}) and w_{ij} (for i > j; i,j = 1,2,\ldots, p) are independently distributed as,

\mathbf{w}(\mathbf{1}) \xrightarrow{\text{asy}} N[0, (K+1)I_p + K I_p I_p']

and \ w_{ij} \xrightarrow{\text{asy}} \text{ICN}[0, K+1], for i > j; i,j = 1,2,\ldots, p.

(Here, ICN means independent complex normal).
This means that the real part of $w_{ij} = (w_{ij})_R$ and the imaginary part of $w_{ij} = (w_{ij})_I$ are asymptotic independent normal variates $N(0, \frac{K+1}{2})$. Thus, we have established the following

**Lemma 3.1:** Let $y_i$ ($i = 1, 2, \ldots, n$) be independently and identically distributed as $G(y)$, and

$$W = \sum_{i=1}^{n} \left( \frac{y_i y_i^* - 2b_1 I_p}{2/n b_1} \right) = (w_{ij}).$$

Then

$$\mathbb{W}(1) = (w_{11}, w_{22}, \ldots, w_{pp}), \quad (w_{ij})_R \text{ and } (w_{ij})_I$$

for $i > j$ ($i, j = 1, 2, \ldots, p$) are asymptotic independent normal variates, $\mathbb{W}(1) \sim N(g, \Sigma_0)$,

$$(w_{ij})_R \sim N(0, \frac{K+1}{2}), \quad (w_{ij})_I \sim N(0, \frac{K+1}{2}),$$

where

$$\Sigma_0 = (K+1)I_p + K_1 I_p^1.$$

Let $b = Uy = U(y_1 + \sqrt{-1} y_2) = b_1 + \sqrt{-1} b_2$,

where $U$ is a unitary matrix. Then, we have with

$$U = (u_{il}'; l, l' = 1, 2, \ldots, p)$$
Then, using (3.2.2) one can observe that

\[ E(b) = \theta, \quad \text{Var}(b) = 2b_1I_p, \quad \text{Cov}(b_1, b_2) = 0 \]

\[ E(b_{4j}^2) = 3b_2, \quad \text{for } j = 1, 2, \ldots, p; \quad j = 1, 2, \]

\[ E(b_{4j}^2 b_{4j'}^2) = b_2, \quad \text{for } j \neq j' \]

and

\[ E(b_{4j} b_{4j'} b_{4j''} b_{4j'''} b_{4j''''}) = 0, \]

\[ \text{for } j, j', j'', j''' = 1, 2, \ldots, p \text{ and } j, j', j'', j''' = 1 \text{ or } 2. \]

Thus the first four moments of \( b = Uy \) are same as those of \( y \) for any unitary matrix \( U \) and, hence, we can have the following

**Lemma 3.2:** Let \( U \) be any pxp unitary matrix and \( V = UWU^* \),
where \( W \) is defined in Lemma 3.1. Then, asymptotically, \( V \) and \( W \) are identically distributed.

Let \( x_i \sim \mathcal{N}(\mu_i; \Sigma) \), for \( i = 1, 2, \ldots, n \) and let

\[
\mathbf{Z} = (\mu_1, \mu_2, \ldots, \mu_n) = \mathbf{B} \mathbf{F} \mathbf{A}^* \quad \ldots \quad (3.2.6)
\]

and consider the following assumption on the elements of the matrix \( A \) for large \( n \):

Assumption: For large \( n \), \( \mathbf{A}^*/n \) tends to a positive definite matrix \( C \). \( \ldots \)(3.2.7)

If \( \mathbf{A}' = (f_1, f_2, \ldots, f_n) \) and \( (\mathbf{A}^c)' = \mathbf{A}^* \), then by

\( (3.2.7) \),

\[
A^c/n = C_{1n} = \sum_{i=1}^{n} f_i f_i^*/n \to C' \quad \text{as} \quad n \to \infty,
\]

and this implies

\[
\sum_{i=1}^{n} f_i f_i^*/n \to C' \quad \text{as} \quad n \to \infty. \quad \text{Hence, this implies}
\]

\[
|f_i|/n \to 0 \quad \text{as} \quad n \to \infty \Rightarrow \max_i |f_i|/n \to 0 \quad \text{as} \quad n \to \infty. \quad \ldots \quad (3.2.8)
\]

First of all, we shall find the asymptotic joint distribution of \( Z = \mathbf{X}A(A^c)^{-1} \) and \( nS = \mathbf{X}^* - Z(A^*A)Z^* \). Let
\[ \Sigma = PP^*, \text{ where } P \text{ is non-singular and let } C_{1n} = AA/n, \]

\[ C_{1n}^{-1} A^* = (d_1, d_2, \ldots, d_n) \text{ and } \]

\[ Y_1 = \sqrt{n} P^{-1}(Z - B \xi) = \sqrt{n} YA(A^*)^{-1} \ldots \ (3.2.9) \]

Note that \( P^{-1}(X - B \xi A^*) = Y = (y_1, y_2, \ldots, y_n) \), and \( y_i \) (\( i = 1, 2, \ldots, n \)) are independent and identical complex random vectors from \( G(y) \).

Notice that \( \text{vec}(Y_i) = \sqrt{n} \bar{y} \) with \( y_i = (d_i^C \otimes y_i) \) for \( i = 1, 2, \ldots, n \) and \( \bar{y} = \sum_{i=1}^{n} y_i/n \), where \( d_i^C \) denotes the conjugate of \( d_i \) so that the conjugate transpose of \( d_i \) is \( d_i^* \) (= \( (d_i^C)' = (\overline{d_i})^C \)). Thus, \( y_i \)'s are independent observations with

\[ E y_i = 0, \ E y_i y_i^* = d_i^C d_i' \otimes E(y_i y_i^*) = d_i^C d_i' \otimes 2b_i I_P. \]

We observe that

\[ \sum_{i=1}^{n} d_i^C d_i' / n = (A' A^C / n)^{-1} \rightarrow C^{-1} \text{ as } n \rightarrow \infty. \]

\[ \ldots \ (3.2.10) \]

Further inorder to use Liapounoff's central limit theorem, we have to show that
Note that,

\[ S^3 = \sum_{i=1}^{n} \mathbb{E} \left( \left| \frac{1}{n} \sum_{j=1}^{n} y_j^i \right|^3 \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

Further if \( A' = (f_1', f_2', \ldots, f_n') \), then

\[ \frac{d_i'}{d_i} = \frac{f_i' A' A'^c}{A' A'^c n}^{-1/2} d_i, \quad \text{for all } i = 1, 2, \ldots, n \]

and using (3.2.7) we have

\[ \sum_{i=1}^{n} \frac{(d_i' A'^c)}{n} = \text{tr} \left( A' A'^c n \right)^{-1} \rightarrow \text{tr} C'^{-1} \]

Further using (3.2.7) and (3.2.8), we have

\[ \max_i \left( \frac{d_i' A'^c}{n} \right)^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

Therefore,

\[ S^3 \leq \left[ \max_i \left( \frac{d_i' A'^c}{n} \right)^{1/2} \right] \left[ \max_i \mathbb{E} \left( \frac{y_j^i y_i}{y_i} \right)^{3/2} \right] \text{tr} C'^{-1} n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

Hence,
Now, if \( \text{vec}(W) = \frac{1}{n} \sum_{i=1}^{n} (Y_i^c \otimes Y_i - 2b_1 \text{vec} I_p)/2 \sqrt{n} \),

then it is easy to verify that

\[
\text{Cov}(\text{vec}(W), \text{vec}(Y_1)) = \text{Cov}(\frac{1}{n} \sum_{i=1}^{n} (Y_i^c \otimes Y_i - 2b_1 \text{vec} I_p)/2 \sqrt{n} b_1, \frac{1}{n} \sum_{i=1}^{n} (d_i^c \otimes y_i)/n) = 0
\]

because \( E Y_i = 0, E Y_i Y_j' = 0 \) and \( E(d_i^c \otimes y_i) = E y_i = 0 \)

Hence \( \text{vec}(W) \) and \( \sqrt{n} \text{vec}(Y_1) \) are asymptotic independent normal variates.

Now, we have

\[
\sqrt{n} \{ S - P Y A (A^* A)^{-1} A^* Y P \}
\]

\[
\Rightarrow \sqrt{n} F^{-1} S F^{-1} = (Y Y^* - Y A (A^* A)^{-1} A^* Y^*) / \sqrt{n}
\]

\[
\Rightarrow \sqrt{n} \{ F^{-1} S F^{-1} - 2(1-m/n)b_1 I_p / 2b_1 \} = \frac{Y Y^* - Y A (A^* A)^{-1} A^* Y^*}{2b_1 \sqrt{n}} - \frac{(A-B) I_p}{\sqrt{n}}
\]
Hence,

\[ \sqrt{n} \left\{ P^{-1} S P^* - I - 2b_1 \left( \frac{n-m}{n} I_p \right) \right\} / 2b_1 = \]

\[ = \frac{Y Y^* - 2n b_1 I_p}{2/n b_1} - \frac{(Y A^*)^{-1} A^* Y^* - 2m b_1 I_p}{2/n b_1} \]

or

\[ \sqrt{n} \left\{ P^{-1} S P^* - I - 2b_1 \left( \frac{n-m}{n} I_p \right) \right\} / 2b_1 = W - \frac{T_n}{2} \sqrt{n} b_1 \]

\[ \ldots \ldots (3.2.13) \]

where

\[ W = \sum_{i=1}^{n} \left( Y_i Y_i^* - 2b_1 I_p \right) / 2 \sqrt{n} b_1 = \frac{Y Y^* - 2n b_1 I_p}{2 \sqrt{n} b_1} \]

and

\[ T_n = Y A^* A^{-1} A^* Y^* - 2m b_1 I_p = Y_1 C_1 Y_1^* - 2m b_1 I_p \]

\[ \ldots \ldots (3.2.14) \]

and for large \( n \), the distribution of \( T_n \) tends to the limiting distribution of \( (Y_1 C_1 Y_1^* - 2m b_1 I_p) \) and the limiting distribution of \( Y_1 \) is complex normal on the basis of the Liapounoff's central limit theorem (see for example, Cramer, 1951, 213-218). Further, on the basis of (3.2.12) and (3.2.13), we have
\[ \text{Cov}(\text{vec}(Y_1), \text{vec}(W_{-1}/n b_1)) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.2.15) \]

Hence, using the above asymptotic results and Lemma-1, we establish

**Theorem 3.1:** Let \( x_i \) be \( IG(x_i; \mu_i, \Sigma) \) for \( i = 1, 2, \ldots, n \) and \( \bar{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) = B \xi A^* \). If

\[ Y_1 = \sqrt{n} (XA(A^*A)^{-1} - B \xi) \]

\[ W_0 = \sqrt{n} (P^{-1}SP^{-1} - 2b_1(1-m/n)I_p) \]

then under the condition (3.2.7), \( Y_1 \) and \( W_0 \) are asymptotic independent, asymptotic distribution of \( W_0 \) is the same as that of \( \bar{W} \) given in Lemma 3.1, and

\[ \text{vec}(Y_1) \xrightarrow{\text{asy}} \mathcal{N}(0, 2b_1(C^*-1 \otimes I_p)) \quad \ldots. \quad (3.2.16) \]

Now, we consider the asymptotic joint distribution of the estimates \( \hat{i} \) and \( \hat{\xi} \) given in (3.1.4). Using (3.2.13), we have
\[ P^{-1} S_P^{-1} = \frac{2b_1 \, \frac{w}{\sqrt{n}}}{n} + \frac{2b_1(n-m)I_p}{n} - \frac{T_n}{n} \]

\[ = \frac{2b_1(n-m)}{n} \left\{ I_p + \frac{w}{\sqrt{n}} \left( 1 + \frac{m}{n-m} \right) - \frac{T_n}{2b_1(n-m)} \right\} \]

\[ = \frac{2b_1(n-m)}{n} \left( I_p + \frac{w}{\sqrt{n}} + \frac{T_{1n}}{n} \right) \]

\[ \text{or } P^* S^{-1}P = \frac{n}{2b_1(n-m)} \left( I_p + \frac{w}{\sqrt{n}} + \frac{T_{1n}}{n} \right)^{-1} \ldots (3.2.17) \]

where

\[ T_{1n} = \frac{m}{n-m} \frac{\sqrt{n}}{w} - \frac{n}{2b_1(n-m)} T_n. \]

Further, using (3.1.4) we have

\[ \xi - \hat{\xi} = (B^* S^{-1}B)^{-1} B^* S^{-1}X(A^* A)^{-1} - (B^* S^{-1}B)^{-1} B^* S^{-1}B \xi \]

\[ = (B^* S^{-1}B)^{-1} B^* S^{-1}(X(A^* A)^{-1} - B \xi) \]

\[ = (B_1^* P^* S^{-1}P B_1)^{-1} B_1^* P^* S^{-1}P (X(A^* A)^{-1} - B \xi) \text{ with } \]

\[ B_1 = P^{-1}B \]

\[ \therefore \xi - \hat{\xi} = (B_1^* I_p + \frac{w}{\sqrt{n}} + \frac{T_{1n}}{n})^{-1} B_1^* (I_p + \frac{w}{\sqrt{n}} + \frac{T_{1n}}{n})^{-1} \]

\[ Y A^* A^{-1} \ldots (3.2.18) \]
or \( \sqrt{n} (\bar{X} - \bar{X}) = (B^*_1B_1)^{-1} B^*_1 Y_1 + O(n^{-\frac{1}{2}}) \) \( \ldots (3.2.19) \)

From (3.1.4) we have

\[
\begin{align*}
\hat{\Sigma} & = \{ X - B (B^* S^{-1} B)^{-1} B^* S^{-1} X A (A^* A)^{-1} A^* \} \\
& + \{ X - B (B^* S^{-1} B)^{-1} B^* S^{-1} X A (A^* A)^{-1} A^* \} \ast \\
\end{align*}
\]

\[
= n S + R X A (A^* A)^{-1} A^* X^* R^*
\]

with \( n S = XX^* X A (A^* A)^{-1} A^* X^* \), \( R = I - B (B^* S^{-1} B)^{-1} B^* S^{-1} \).

Hence,

\[
\sqrt{n} \left( F^{-1} \hat{\Sigma} F^{-1} - 2b_1 (1-m/n) I_p \right) / 2b_1
\]

\[
= \sqrt{n} \left\{ F^{-1} S F^{-1} + \frac{F^{-1} R X A (A^* A)^{-1} A^* X^* R F^{-1}}{n} \right\} / 2b_1
\]

Then, using (3.2.9), (3.2.13) and (3.2.14) we have

\[
\sqrt{n} \left( (F^{-1} \hat{\Sigma} F^{-1} - 2b_1 (\frac{H-M}{n}) I_p) / 2b_1 \right) = W - \frac{T_n}{2/n b_1} + \frac{R Y_1 C_1 X^* R^*}{\sqrt{n}} \]

\[ \ldots \ldots (3.2.20) \]

with

\[
R = I - B_1 (B^*_1 I_p + \frac{W}{\sqrt{n}}) (\frac{T_n}{n})^{-1} B_1 (I_p + \frac{W}{\sqrt{n}}) (\frac{T_n}{n})^{-1}
\]

\[
= I - B_1 (B^*_1 B_1)^{-1} B^*_1 + O(n^{-\frac{1}{2}}).
\]
Hence, asymptotically,
\[ \sqrt{n} \left( \hat{\Sigma} - \Sigma \right) = \left( B_1^* B_1 \right)^{-1} B_1^* Y_1 + O(n^{-1/2}) \]
and
\[ \sqrt{n} \left( P^{-1} \hat{\Sigma} P^* - 2b_1 I_p \right)/2b_1 = W + O(n^{-3/2}) \]

because asymptotically \( T_n \) and \( R_Y = Y^*_1 R^*_1 \) converge to random variables which have distributions with finite first moments. Hence, using Theorem 3.1, we get

**Theorem 3.2** Let \( x_i \) be \( IG(x_i; \mu_i, \Sigma) \) for \( i = 1, 2, \ldots, n \) and let \( \hat{\Sigma} = \left( \mu_1, \mu_2, \ldots, \mu_n \right) = B \hat{x} A^* \)

where \( \Sigma \) is Hermitian positive definite, \( \Sigma = P P^* \) and \( A^* A \) and \( B^* B \) are non-singular. If \( \hat{x} = (B^* S^{-1} B)^{-1} A^* X^* R^*/n \) with \( X = (x_1, x_2, \ldots, x_n) \)

and \( \hat{\Sigma} = S + R X A^* A)^{-1} A^* X^* R^*/n \) with \( X = (x_1, x_2, \ldots, x_n) \)

and \( R = I - B (B^* S^{-1} B)^{-1} B^* S^{-1} \), then under the condition \( (3.2.7) \), \( \sqrt{n} \left( \hat{x} - x \right) \) and \( \sqrt{n} \left( \hat{\Sigma} - 2b_1 \Sigma \right)/2b_1 \) are asymptotically independent,

\[ \text{vec} \left( \sqrt{n} \left( \hat{x} - x \right) \right) \sim \text{CN}(0, (C^* - I \otimes (B \Sigma^{-1} B)^{-1})2b_1) \]

and the asymptotic distribution of \( \sqrt{n} \left( P^{-1} \hat{\Sigma} P^* - 2b_1 I_p \right)/2b_1 \) is the same as that of \( W \) given in Lemma 3.1.
3.3. Asymptotic confidence bounds for the location parameters:

Let us consider the statistics

\[ S_1 = (\hat{\xi} - \xi)(A^* A)(\hat{\xi} - \xi)^* \quad \text{and} \quad S_2 = (B^* \Sigma^{-1} B)^{-1}(2b_1) \]

\[ \ldots (3.3.1) \]

and let the non-zero eigen values of \( S_2^{-1} S_1 \) be

\[ a_1 \geq a_2 \geq \ldots \geq a_t > 0 \]

where \( t = \min(q, m) \) and \( \xi : q \times m \) is unknown location parametric complex matrix.

Then, we can write,

\[ \sum_{i=1}^{t} \lambda_i = \tau_0^2 = \text{tr} S_2^{-1} S_1 = [\text{vec}(\hat{\xi} - \xi)]^* [C^t \phi(B^* \Sigma^{-1} B)] \]

\[ [\text{vec}(\hat{\xi} - \xi)] /2b_1 \quad \ldots (3.3.2) \]

Since \( \lim_{n \to \infty} (3^* \Sigma^{-1} B) = (B^* \Sigma^{-1} B)/(2b_1) \), the asymptotic distribution of \( 2 \tau_0^2 \) is Chi-square with \( 2qm \) degrees of freedom (or \( \chi^2_{2qm} \)).

Let \( \Pr(\chi^2_{2qm} \geq c_\alpha) = \alpha \).

Then, the asymptotic confidence bounds on \( \xi \) are obtained from
or for all non-null vector $\mathbf{l} \in \mathbb{C}^m$,

$$
|\text{tr} (\mathbf{\hat{\xi} - \xi}) \mathbf{L}|^2 = \frac{1}{2}(\text{vec} \mathbf{\hat{\xi} - vec} \mathbf{\xi})(\text{vec} \mathbf{\hat{\xi} - vec} \mathbf{\xi})^* \mathbf{L} \\
\leq b_1 c_{\mathbf{u}} \left[ \text{tr}(A^*A)^{-1} L^*(B^* \hat{\Sigma}^{-1} B)L \right] \\
\text{(3.3.3)}
$$

where $L = (\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_m)$ and $\mathbf{L} = (\text{vec} \mathbf{L})' = (\mathbf{l}_1', \mathbf{l}_2', \ldots, \mathbf{l}_m')$. From this, the asymptotic simultaneous confidence bounds on $(g(\text{vec} \mathbf{\hat{\xi}})^* \mathbf{l})_R$ for $g \in \mathbb{C}$ and $\mathbf{l} \in \mathbb{C}^m$ are

$$(g(\text{vec} \mathbf{\hat{\xi}})^* \mathbf{l})_R \pm \left[ b_1 c_{\mathbf{u}} (\bar{g} g) \text{tr} \left\{ (A^*A)^{-1} L^*(B^* \hat{\Sigma}^{-1} B)^{-1} L \right\} \right]^{\frac{1}{2}} \\
\text{(3.3.4)}
$$

If, $(2b_1)^{\frac{1}{2}} X = \sqrt{n} (B^* \hat{\Sigma}^{-1} B)^{\frac{1}{2}}(\mathbf{\hat{\xi} - \xi})(A^*/n)^{\frac{1}{2}}$,

then it can be shown that the asymptotic distribution of $X$ is complex normal, i.e. $X \sim \mathbb{CN}(0, I_q; I_m)$, and that the asymptotic joint distribution of the eigen values of $a_1, a_2, \ldots, a_t$ is the same as those of the txt complex Wishart matrix $S_0$ distributed as $\mathbb{C}W_t(u; I_t)$ where
\( u = \max(q, m) \). The density function of the complex Wishart variate \( S_0 \) is given by

\[
\left[ (\bar{f}_t(u))^{-1} | S_0 \right]^{u-t} \exp(-\text{tr} \; S_0) \quad \cdots \cdots \quad (3.3.5)
\]

where \( S_0 \) is Hermitian positive definite, and

\[
\bar{f}_t(u) = \pi^{t(t-1)/2} (u-j+1), \quad u = \max(q, m)
\]

Using (3.3.5), one can obtain the distribution of \( a_1 \).

Let \( \text{Pr} (a_1 \geq d_u) = \alpha \).

Then, for all non-null vectors \( a \in \mathbb{C}^q \) and \( b \in \mathbb{C}^m \), the asymptotic confidence bounds on \( \delta \) are obtained from

\[
\bar{a}^* (\hat{\delta} - \delta)_{\bar{b}}^* (\hat{\delta} - \delta)^{**} a \leq 2b_1d_u \left( \bar{b}^*(\bar{A}^*A)^{-1} \bar{b} \right) \left( \bar{a}^*(\bar{B}^*\bar{A}^{-1}B)^{-1} \bar{a} \right)
\]

\[
\cdots \cdots \quad (3.3.6)
\]

and as before the asymptotic simultaneous confidence bounds on \( (g \bar{a}^* \hat{\delta} \bar{b})_R \) for all \( g \in \mathbb{C} \), \( \bar{b} \in \mathbb{C}^m \) and \( \bar{a} \in \mathbb{C}^q \) are

\[
(g \bar{a}^* \hat{\delta} \bar{b})_R \pm \left[ 2b_1d_u (g\bar{g}) \left( \bar{b}^*(\bar{A}^*A)^{-1} \bar{b} \right) \left( \bar{a}^*(\bar{B}^*\bar{A}^{-1}B)^{-1} \bar{a} \right) \right]^{\frac{1}{2}}
\]

\[
\cdots \cdots \quad (3.3.7)
\]
From (3.3.4) and (3.3.7), one can obtain some special cases for which one can refer to Khatri (1966a).

3.4. **Asymptotic distributions for the scale parameters for one population:**

In this section, we obtain the approximate distribution of the likelihood ratio test statistics depending on some structure of covariance matrices for the class of complex elliptical distributions. For the complex elliptical populations, the asymptotic distributions of canonical correlations, the asymptotic confidence bounds of parameters $\xi$ and the simultaneous asymptotic confidence bounds for the discriminatory values $D_a$ were given by Khatri and Bhavsar (1988a).

In this section we consider one sample problems while in Section 3.5, we consider two or more sample problems. For this Section, let $x_1, x_2, \ldots, x_n$ be $n$ independent observations from $G(x; \mu, \Sigma)$ and

$$S = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^*$$

then $S/2b_1$ can be taken as an estimate of $\Sigma$. The likelihood ratio test statistics available under the assumption of complex normal
distributions for testing some structures on $\Sigma$ are considered for the asymptotic distributions under alternatives close to the null hypothesis for the complex elliptical distributions.

3.4.1. Known $\Sigma$ test:

Consider the null hypothesis $H(\Sigma = \Sigma_1$ (given)) against $K (\Sigma \neq \Sigma_1)$. Then the likelihood ratio test statistic

$$\lambda_1 = \left| \Sigma_1^{-1} S \right|^n \exp \left[ -n \text{tr}(\Sigma_1^{-1} S - I_p) \right] \quad \ldots (3.4.1)$$

The alternatives close to $H$ are

$$\Sigma_1^{-1} \Sigma = Q^{-1} D V Q^*, \quad \nu_i = 1 - \frac{s_i}{n} \text{ for } i = 1, 2, \ldots, p$$

$$\ldots (3.4.2)$$

with $\Sigma_1 = QQ^*, \Sigma = PP^*, P = Q D V$

where $\nu_1, \nu_2, \ldots, \nu_p$ are the eigen values of $\Sigma_1^{-1} \Sigma$ and $s_1, s_2, \ldots, s_p$ tends to finite quantities as $n \to \infty$.

For the normal distribution, one can refer to Khatri and Srivastava (1975, 1976). Using Theorem 3.1, we have
\[ \frac{S}{2b_1} = \Sigma + \frac{P^tP^*}{\sqrt{n}}, \quad \frac{\Sigma^{-1} S}{2b_1} = Q^{-1} D\hat{v} + \frac{Q^{-1} D\sqrt{v} WD\sqrt{v} Q^*}{\sqrt{n}} \]

\[ \text{..... (3.4.3)} \]

and taking \( 2b_1 D\hat{v} = I + Ds/\sqrt{n} \), (3.4.1) can be rewritten as

\[ \lambda_1 = \frac{|I + Ds/\sqrt{n}|^n |I + W/\sqrt{n}|^n}{\exp{n \text{ tr} ((I + Ds)(I + W/\sqrt{n}) - I)}} \]

\[ \therefore - \log \lambda_1 = -n \log |I + Ds/\sqrt{n}| - n \log |I + W/\sqrt{n}| \]

\[ + \frac{1}{\sqrt{n}} \text{ tr}(Ds + W + DaW/\sqrt{n}) \]

Then, using result (2.2.5) we have

\[ - \log \lambda_1 = \text{tr}(W + Da)^2 + O(n^{-\frac{3}{2}}) \]

\[ = \frac{p}{\Sigma_{i=1}^p (w_{1i}a_{1i})^2 + 2 \Sigma_{j=1}^p ((w_{1j})^2 + (w_{2j})^2) + O(n^{-\frac{3}{2}})} \]

\[ \text{..... (3.4.4)} \]

where \( W = (w_{1j}) \).

Thus the asymptotic distribution of \(-2 \log \lambda_1\) is that of \( a_1z_1 + a_2z_2 \) where \( a_1 = \Sigma_{i=1}^p a_{1i} + a_{2i} \), \( a_2 = K + 1 \) and \( z_1 \) and \( z_2 \) are independent non-central Chi-square variates with degrees of freedom 1 and \( p^2 - 1 \) and with non-central.
parameters $\mathcal{C}_1 = p \overline{s}^2/a_1$ and $\mathcal{C}_2 = \sum_{i=1}^p (s_i - \overline{s})^2/a_2$

respectively. Here $\overline{s} = \sum_{i=1}^p s_i/p$. Let

$$t_1 = \lambda_{1/n} |2b_1D_v| |I + W/n| \exp[-2b_1 \text{tr}(D_v W/n + D_v - p/2b_1)]$$

and let

$$t_{o1} = |2b_1D_v| \text{str}(I-2b_1D_v) = \sum_{i=1}^p (2b_1v_i \exp(1-2b_1v_i))$$

where $\overline{v} = \sum_{i=1}^p v_i/p$.

Hence, when $v_1, v_2, \ldots, v_p$ are finite and may not tend to zero as $n \to \infty$, then

$$t(1) = \sqrt{n} \log(t_1/t_{o1}) = \text{tr}(I-2b_1D_v)W + O(n^{-\frac{1}{2}})$$

$$= \sum_{i=1}^p (1-2b_1v_i)w_{ii} + O(n^{-\frac{1}{2}})$$

$$\ldots \ldots \quad (3.4.5)$$

and

$$a_2[\sum_{i=1}^p (1-2b_1v_i)^2 + (a_2-1)b^2(1-2b_1\overline{v})^2]^{-\frac{1}{2}} t(1) \overset{\text{asy}}{\sim} N(0, 1)$$

$$\ldots \ldots \quad (3.4.6)$$

Note that here $(1-2b_1v_i)$ for $i = 1, 2, \ldots, p$ are considered as fixed.
3.4.2. Test for sphericity:

Consider the hypothesis $H \left( \Sigma = \sigma^2 \Sigma_1, \sigma^2 \text{ is unknown and } \Sigma_1 \text{ is known} \right)$ against $K: (\Sigma \neq \sigma^2 \Sigma_1)$. Then, the likelihood ratio test statistic is

$$\lambda_2 = \frac{|\Sigma_1^{-1} S|^n}{(\text{tr} \Sigma_1^{-1} S/p)^{np}} \quad \cdots (3.4.7)$$

The alternatives close to the null hypothesis $H$ are

$$2b_1 v_i = 1 + s_i / \sqrt{n} \quad \text{for } i = 1, 2, \ldots, p. \quad \cdots (3.4.8)$$

where $v_1, v_2, \ldots, v_p$ are the eigen values of $\Sigma_1^{-1} \Sigma$ and $s_1, s_2, \ldots, s_p$ tends to a finite quantities as $n \to \infty$.

For the normal distribution one can refer to Khatri and Srivastava (1974). Hence using (3.4.3) $\lambda_2$ can be re-written as

$$\lambda_2 = \frac{|I + Ds/\sqrt{n}|^n |I + W/\sqrt{n}|^n}{[\text{tr}(I + Ds/\sqrt{n} + W/\sqrt{n} + \frac{Daw}{n}/p)]^{np}}$$

$$\therefore -\log \lambda_2 = -n \log |I + Ds/\sqrt{n}| - n \log |I + W/\sqrt{n}|$$

$$+ np \log \left[ 1 + \text{tr}(Ds + W + \frac{Daw}{n}/p)/n \right]$$
Hence using the results

\[ \log |I + cA| = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} c^j}{j} \text{tr} A^j \]

and

\[ \log(1 + x) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} x^j}{j} \]

If we expand \(-\log \lambda_2\) in powers of \(1/n\) we get

\[ -\log \lambda_2 = \frac{1}{2} \left[ \text{tr}(W + Ds)^2 - \frac{1}{p} \left( \text{tr}(W + Ds) \right)^2 \right] + O(n^{-\frac{1}{2}}) \]

or

\[ -2\log \lambda_2 = \sum_{i=1}^{p} (w_{ii} + s_i)^2 + 2 \sum_{i>j} w_{ij} w_{ij}^* - \frac{1}{p} \left( \sum_{i=1}^{p} (w_{ii} + s_i) \right)^2 \]

\[ + O(n^{-\frac{1}{2}}) \]

\[ = a_z z + O(n^{-\frac{1}{2}}) \]

Thus the asymptotic distribution of \((-2a_z^{-1} \log \lambda_2\) is non-central Chi-square with \((p^2 - 1)\) degrees of freedom and with non-central parameter \((2a_z^{-1} \sum_{i=1}^{p} s_i^2)\) where \(\bar{s} = 0\).

Further for fixed \(v_1, v_2, \ldots, v_p\) let
\[ t_2 = \frac{\lambda_2}{\sqrt{n}} = \frac{|\mathbf{D}_v|}{\sqrt{n}} \frac{|I+W|}{\sqrt{n}} \left( \frac{\text{tr}(\mathbf{D}_v + \mathbf{W}/n)}{p} \right)^p \]

and
\[ t_{o2} = \frac{|\mathbf{D}_v|}{\sqrt{n}} \frac{1}{(\text{tr}(\mathbf{D}_v/p))^p} = \prod_{i=1}^{p} (v_i / \sqrt{v}) . \]

Then, we have
\[ \sqrt{n} \log(t_2/t_{o2}) = t(2) = \text{tr} \left( \frac{\mathbf{W} \cdot p(\text{tr} \mathbf{D}_v)(\text{tr} \mathbf{D}_v)^{-1} + 0(n^{-\frac{1}{2}})}{\Sigma_{i=1}^{p} w_{ii} - \sum_{i=1}^{p} v_i w_{ii} / \sqrt{v} + 0(n^{-\frac{3}{2}})} \right) \]

\[ \ldots \ldots (3.4.9) \]

and
\[ t(2) \left[ \sum_{i=1}^{p} (v_i - \bar{v})^2 / \bar{v}^{\frac{3}{2}} \right] \sim N(0, 1) . \]

\[ \ldots \ldots (3.4.10) \]

### 3.4.3. Test for independence of two sets of vectors:

Let us partition \( \Sigma \), \( S \) and \( W \) as

\[ \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \]

where \( \Sigma_{ij} \), \( S_{ij} \) and \( W_{ij} \) are \( p_1 \times p_2 \) matrices for \( i, j = 1, 2 \) and \( p = p_1 + p_2 \), \( p_1 < p_2 \). Then, the statistic
\begin{align*}
\lambda_3 &= \frac{|S|_n}{|S_{11}|_n |S_{22}|_n} \quad \text{.... (3.4.11)}
\end{align*}

is used for testing the hypothesis \( H (\Sigma_{12} = 0) \) against \( K (\Sigma_{12} \neq 0) \). The alternatives close to \( H \) are

\[ \Sigma = PP^*, \quad P = \begin{bmatrix} P_1R_1 \\ P_2R_2 \end{bmatrix}, \quad R_1 = (I_{p_1}, 0), \]

\[ R_2 = (T/n, (I_{p_2} - TT^*/n)^\frac{1}{2}) \]

\[ \text{.... (3.4.12)} \]

where \( P_1 \) is a \( p_1 \times p_1 \) non-singular matrix and \( T \) tends to a finite matrix as \( n \to \infty \). One can refer to Khatri and Bhavsar (1988a) for the asymptotic distribution of the eigen values of \( S^{-1}_{11} S_{12} S^{-1}_{22} S_{21} \), and Srivastava and Khatri (1979) or Anderson (1985) for the asymptotic distribution of \( \lambda_3 \) under normality. Now using theorem 3.1, we have

\[ \frac{S}{2b_1} = PP^* + \frac{PWP^*}{\sqrt{n}}, \quad \frac{S_{11}}{2b_1} = P_1P_1^* + \frac{P_1W_{11}P_1^*}{\sqrt{n}} \]

and

\[ \frac{S_{22}}{2b_1} = P_2P_2^* + \frac{P_2R_{22}R_{22}^*P_2}{\sqrt{n}}. \]

Then, similar to the earlier cases, we have
\[-\log \lambda_3 = -n \log |I - T^*/n| - n \log |I + W/\sqrt{n}| + n \log |I + \frac{W_1}{\sqrt{n}}| + n \log |I + \frac{R_2WR^*_2}{\sqrt{n}}|
\]

Expanding \((-\log \lambda_3)\) in powers of \(1/\sqrt{n}\) and observing

\[
\begin{align*}
\text{tr} R_2WR^*_2 &= \text{tr}(W_{22} + TW_{12}/\sqrt{n} + W_{21}^*/\sqrt{n}) + O(n^{-1}) \\
\text{tr} W^2 &= \text{tr}(W_{11}^2 + W_{22}^2 + 2W_{12}W_{21}) \\
\text{tr}(R_2WR^*_2)^2 &= \text{tr} W_{22}^2 + O(n^{-3/2})
\end{align*}
\]

we have,

\[
-\log \lambda_3 = \text{tr} T^* - n \left\{ \frac{\text{tr} W}{\sqrt{n}} - \frac{\text{tr} W^2}{2n} \right\} + n \left\{ \frac{\text{tr} W_{11}}{\sqrt{n}} - \frac{\text{tr} W_{11}^2}{2n} \right\} + n \left\{ \frac{\text{tr} R_2WR^*_2}{\sqrt{n}} - \frac{\text{tr}(R_2WR^*_2)^2}{2n} \right\} + O(n^{-3/2})
\]

\[
= \text{tr} T^* + \text{tr} W_{12} W_{21} - \text{tr} TW_{12} - \text{tr} W_{21}^* + O(n^{-3/2})
\]

or,

\[
-\log \lambda_3 = \text{tr}(W_{12} - T^*)(W_{12} - T^*)^* + O(n^{-3/2}) \quad \text{...(3.4.13)}
\]

\[
= \left( \sum_{i=1}^{p_1} (w_{12,ii-t_{11}}i)_{R}^2 + (w_{12,ii-t_{11}}i)_{I}^2 \right) + \left( \sum_{i \neq j} ((w_{12,ij-t_{13}}j^2_R + (w_{12,ij-t_{13}}j^2_I) + O(n^{-3/2}).
\]

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\]
Thus \((-2 \sigma^{-2} \log \lambda_3)\) has a non-central Chi-square distribution with degrees of freedom \(2p_1p_2\) and non-central parameter \((2\sigma^{-2} \text{tr } TT^*)\). Let

\[
\frac{1}{n} = \frac{|S|}{|S_{11}| |S_{22}|} & \text{ and } t_{03} = |T(1) T(1)^*|
\]

and let us assume

\[
\Sigma = \begin{bmatrix}
    P_1 & 0 \\
    P_2 T_0 & P_2 T(1)
\end{bmatrix} \begin{bmatrix}
    P_1^* & T_0^* P_2^* \\
    0 & T(1)^* P_2^*
\end{bmatrix}, \quad T_0^* = (D_V, 0), \quad T(1) = (I - T_0 T_0^*)^{1/2}
\]

\(D_V = \text{diag}(v_1, v_2, \ldots, v_{p_1})\) where \(v_1, v_2, \ldots, v_{p_1}\) are the canonical correlations between the two sets of variates which are fixed and do not tend to zero as \(n \to \infty\). Then

\[
t(3) = \sqrt{n} \log(t_3/t_{03}) = \text{tr}(W - W_{11} - R_2 W R_2^*) + O(n^{-\frac{1}{2}}) \ldots (3.4.14)
\]

where \(\Sigma = PP^*, \quad P = \begin{bmatrix}
    P_1 & R_1 \\
    P_2 & R_2
\end{bmatrix}, \quad R_1 = (I_{p_1}, 0), \quad R_2 = (T_0, T(1))\) and

\[
T(1) = \begin{bmatrix}
    I_{p_1} - D_V^2 & 0 \\
    0 & I_{p_2-p_1}
\end{bmatrix}^{1/2}
\]

Note that
\[ \text{tr } R_2^W = \text{tr } [D_2 W_1] W_{12} \left[ \begin{array}{cc} I_{p_1} - D_v^2 & 0 \\ 0 & I_{p_2 - p_1} \end{array} \right] \frac{1}{2}
+ \left[ \begin{array}{cc} I_{p_1} - D_v^2 & 0 \\ 0 & I_{p_2 - p_1} \end{array} \right] \cdot W_{21}(D_v, 0) + W_{22} - D_v^2 W_{22} \].

Hence (3.4.14) can be rewritten as
\[ t(3) = \text{tr } D_v^2 (w_{22} - w_{11}) - 2 \text{tr}(D_v, o) \left[ \begin{array}{cc} I_{p_1} - D_v^2 & 0 \\ 0 & I_{p_2 - p_1} \end{array} \right] \left( w_{12} \right)_R^* (n^{-\frac{1}{2}}) \]

\[ = \sum_{i=1}^{p_1} v_i^2 (w_{22, ii} - w_{11, ii}) - 2 \sum_{i=1}^{p_1} v_i (1 - v_i^2) (w_{12, ii}) R + o(n^{-\frac{1}{2}}) \]

Then,
\[ t(3) [2a_2 (\sum_{i=1}^{p_1} v_i^2)]^{\frac{1}{2}} \xrightarrow{\text{asy}} N(0, 1). \] .... (3.4.15)

3.4.4. Test for quaternion structure:

Let \( p_1 = p_2 = q \) (say). Then, the likelihood ratio test statistic
\[ \lambda_4 = \frac{|S|^n}{|S + J SJ'|^n}, \quad J = \begin{bmatrix} 0 & -I_q \\ I_q & 0 \end{bmatrix} \] ...

(3.4.16)

is used for testing the hypothesis \( H (\Sigma_{11} = \Sigma_{22} \& \Sigma_{12} + \Sigma_{21} = 0) \) against \( K : (\Sigma_{11} \neq \Sigma_{22} \text{ or } \Sigma_{12} + \Sigma_{21} \neq 0) \).

For the case of normal population it is given by Andersson et al (1983) and its null distribution is given by them.

The alternatives close to \( H \) are

\[ \Sigma = PP^* = Q \left[ \text{diag} \{ (I+D_v), (I-D_v) \} \right] Q^*, \quad D_v = D_s/\sqrt{n} \]

where \( 0 \leq v_i \leq 1 \) for every \( i \) and \( Q = \begin{bmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{bmatrix} \) is a non-singular matrix. Using the results of Andersson et al (1983), we can write

\[ \frac{S}{2b_1} = \Sigma + \frac{PW^*}{n}, \quad \Sigma = PP^*, \quad P = QR, \quad R = \text{diag}(R_1, R_2), \]

\[ R_1 = (I_p + D_s/\sqrt{n})^{1/2}, \quad R_2 = (I - D_s/\sqrt{n})^{1/2} \]
and if, $J = \begin{bmatrix} 0 & -I_q \\ I_q & 0 \end{bmatrix}$ then, $JJ' = J'J = I_{2q}$, $JQJ' = Q$

\[ ... (3.4.18) \]

and

\[ \frac{S + JSJ'}{2b_1} = \frac{2\hat{P}_F^* + \frac{Q(RWR^* + J RWR^* J')Q^*}{\sqrt{n}}}{Q(RWR + J RWR J')} \]

Hence from (3.4.16) we have

\[-\log \lambda_4 = -n \log |I - D_s^2/n| - n \log |I + W/n| \]
\[ + n \log |I + (RWR^* + JRWR^* J')/2n| \]
\[ = \text{tr} \left[ \frac{1}{2}(W_{11} - W_{22}) + Ds \right]^2 + \text{tr} \left[ \frac{1}{2}(W_{12} + W_{21}) \right]^2 + O(n^{-1/2}) \]

\[ ... (3.4.19) \]

where $\text{tr} (RWR^* + JRWR^* J')$ and $\text{tr}(RWR^* + JRWR^* J')^2$ can be obtained similar to real variable case given in Sub-section (2.2.5). Note that $\text{tr} \left[ \frac{1}{2}(W_{11} - W_{22}) + Ds \right]^2/a_2$ has a non-central Chi-square distribution with $q^2$ degrees of freedom and with non-central parameter $(2 \sum_{i=1}^q s_i^2)/a_2$ and $\text{tr} \left[ \frac{1}{2}(W_{12} + W_{21}) \right]^2/a_2$ has a central Chi-square distribution with $q^2$ degrees of freedom.

Thus $(-2 a_2^{-1} \log \lambda_4)$ has a non-central Chi-square
distribution with degrees of freedom $2q^2$ and with non-central parameter $(2 \sum_{i=1}^{q} a_i^2)$. 

Let $t_{\lambda} = \lambda_\lambda$ and $t_{\lambda_4} = |I - D_v^2|$ and let us assume that

$$
\Sigma = \Sigma P P^*, \quad P = \begin{bmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{bmatrix}, \quad R_1 = (I + D_v)^{\frac{1}{2}}, \quad R_2 = (I - D_v)^{\frac{1}{2}},
$$

where the diagonal elements of $D_v$ are fixed and do not tend to zero as $n \to \infty$. Hence

$$
t(4) = \frac{1}{n} \log(t_{\lambda}/t_{\lambda_4}) = \frac{1}{n}(t_{\lambda} - t_{\lambda_4})/t_{\lambda_4}
$$

then

$$
t(4) = \text{tr} W - \text{tr}(RWR^* + JRWR^* J') + O(n^{-\frac{1}{2}})
$$

$$
= \text{tr}(W_{11} - W_{22})D_v + O(n^{-\frac{1}{2}}) \ldots (3.4.20)
$$

and

$$
t(4) \left[ 2a_2 (\sum_{i=1}^{q} a_i^2) \right]^{-\frac{1}{2}} \xrightarrow{\text{asy}} N(0, 1).
$$

3.4.5. Test of independence under quaternion structure:

Assume the quaternion structure of (3.4.4). Under this assumption, we test the hypothesis $H_0 (\Sigma_{12} = 0)$. Then the likelihood ratio test statistic under normality is
(See for example, Andersson et al (1983)). For the asymptotic distribution, let

$$ \lambda_5 = \frac{\left| \frac{S + JSJ'}{2} \right|^n}{\left| \frac{S_{11} + S_{22}}{2} \right|^{2n}} \quad \ldots (3.4.21) $$

where $P_1$ is non-singular, $T = \text{diag}(s_1J_1, s_2J_1, \ldots, s_kJ_1)$,

$$ J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad s_1, s_2, \ldots, s_k \text{ are finite as } n \to \infty. $$

Using these we have

$$ \frac{S_{11}}{2b_1} = P_1P_1^* + \frac{P_1W_{11}P_1^*}{\sqrt{n}}, \quad \frac{S_{22}}{2b_1} = \frac{P_1P_2W_{22}P_2^*}{\sqrt{n}} + P_1P_1^* $$

and

$$ \frac{S + JSJ'}{2b_1} = 2QRR^*Q^* + \frac{Q(RWR^* + JWR^*J')Q^*}{\sqrt{n}} $$

where

$$ \Sigma = FF^*, \quad P = [\text{diag}(P_1, P_2)] \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad R_1 = (I_q, 0), $$

$$ R_2 = (T/n, (I_q - TT^*/n)^{1/2}) \quad \text{and} \quad RR^* = I_{2q} + \begin{bmatrix} 0 & -T \\ T & 0 \end{bmatrix}/n. $$
Hence, we have from (3.4.21),
\[
-\log \lambda_5 = -n \log \left| \mathbf{R}_2^* + (\mathbf{R}_2^* \mathbf{W}_2^* \mathbf{J})/2 \right|/n \\
+ 2n \log \left| \mathbf{I}_q + (\mathbf{W}_1^2 + \mathbf{R}_2^* \mathbf{W}_2^*)/2 \right|/n \\
= -n \log \left| \mathbf{I}_q -\mathbf{T}^*/n \right| + 2n \log \left| \mathbf{I}_q + (\mathbf{W}_1^2 + \mathbf{R}_2^* \mathbf{W}_2^*)/2 \right|/n ,
\]

One can observe that
\[
\text{tr } \mathbf{R}_2^* \mathbf{W}_2^* = \text{tr} \left[ \mathbf{W}_2^2 + \mathbf{T}(\mathbf{W}_{12} - \mathbf{W}_{21})^2/n - \mathbf{T}^*(\mathbf{W}_{22} - \mathbf{W}_{11})/n \right] + \mathcal{O}(n^{-3/2})
\]
\[
\text{tr } (\mathbf{R}_2^*)^{-1}(\mathbf{R}_2^* \mathbf{W}_2^* \mathbf{J}) = 2(\mathbf{W}_{11} + \mathbf{W}_{22}) + \mathcal{O}(n^{-1})
\]
and
\[
\text{tr } (\mathbf{R}_2^*)^{-2}(\mathbf{R}_2^* \mathbf{W}_2^* \mathbf{J})^2 = 2(\mathbf{W}_{11} + \mathbf{W}_{22})^2 + \mathcal{O}(n^{-1})
\]
\[
+2\text{tr}((\mathbf{W}_{12} - \mathbf{W}_{21})(\mathbf{W}_{21} - \mathbf{W}_{12}) + \mathcal{O}(n^{-1})
\]
and hence
\[
-\log \lambda_5 = -\text{tr} \left[ \frac{1}{2}(\mathbf{W}_{12} - \mathbf{W}_{21} - \mathbf{T})^2 + \mathcal{O}(n^{-1}) \right] \ldots (3.4.23)
\]

Thus the asymptotic distribution of \(-2a_2^{-1} \log \lambda_5\) is
non-central Chi-square with \(q(q-1)\) degrees of freedom and
with non-central parameter \((2a_2^{-1} \text{tr } \mathbf{T})^2 = (4a_2^{-1} \sum_{i=1}^k s_i^2)\).

Similar to the earlier cases let
\(1/n\)

\[
\lambda_5 = t_5, \quad t_{05} = |I - T_0 T_0^*|
\]

and let us assume that

\[
\Sigma = \Sigma^*, \quad F = \text{diag}(P_1, P_1) \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad R_1 = (I_q, 0),
\]

\[
R_2 = (T_0, T_1),
\]

\[
T_1 = (I - T_0 T_0^*)^{2}, \quad T_0 = \text{diag}(v_1 J_1, v_2 J_2, \ldots, v_k J_1, 0)
\]

and \(v_1, v_2, \ldots, v_k\) are fixed and do not tend to zero as \(n \to \infty\). Hence,

\[
t(5) = n \log(t_5/t_{05})
\]

\[
= \frac{1}{2} \text{tr}(RR^*)^{-1}(RWR^* J^* J^*) \text{tr}(W_{11} + R_2 W_{22}^*) + O(n^{-\frac{3}{2}})
\]

\[
= \text{tr} T_0^2(W_{11} - W_{22}) + \text{tr} T_0 T_1(W_{21} - W_{12}) + O(n^{-\frac{3}{2}})
\]

and

\[
t(5) \left[ 2a_2 \text{tr} T_0 T_0^* \right]^{-\frac{1}{2}} \xrightarrow{\text{asy}} N(0, 1) \quad \ldots \quad (3.4.24)
\]

3.4.6. Test for intra-class correlation for \(k\)-sets of variables:

Consider the null hypothesis \(H(\Sigma = \Sigma_0)\) against \(K(\Sigma \neq \Sigma_0)\) with
where $E$ and $F$ are unknown $q \times q$ Hermitian matrices such that $E$, $E-F$ and $E+(k-1)F$ are Hermitian positive definite matrices and $p = kq$ for $k \geq 2$. Then, the likelihood ratio test statistic under the assumption of normality is

$$\lambda_0 = \frac{|S|^n}{|\hat{E} - F|^{(k-1)n} n |\hat{E} + (k-1)\hat{F}|^n} \quad \ldots (3.4.25)$$

where

$$\hat{E} = \sum_{a=1}^{k} S_{aq} / k \quad \text{and} \quad \hat{F} = \sum_{a \neq \beta = 1}^{k} S_{q\beta} / (k(k-1)),$$

$S = (S_{q\beta}; a, \beta = 1, 2, \ldots, k)$.

The alternatives close to the null hypothesis are

$$\Sigma_{-\frac{1}{2}}^{\Sigma} \Sigma_{-\frac{1}{2}}^{\Sigma} = I_{kq} + R / n, \quad R = (R_{ij}; i, j = 1, 2, \ldots, k),$$

$$R_{11} = 0, \quad \sum_{i=2}^{k} R_{ii} = 0 \quad \ldots (3.4.26)$$

Let $D$ be a $k \times k$ unitary matrix such that its first row is $k^{-\frac{1}{2}} \cdot 1_k$. Then the hypothesis $H (\Sigma = \Sigma_0)$ is equivalent to
hypothesis \( H: \{(D \otimes I_q) \Sigma (D^* \otimes I_q)\} \)
\[= \text{diag} \{E + (k-1)F, I_{k-1} \oplus (E-F)\}.\] Let
\[\Sigma_1 = (D \otimes I_q) \Sigma (D^* \otimes I_q) = (\Sigma_{i,ij}; i, j = 1, 2, \ldots, k)\]
\[S_1 = (D \otimes I_q) S(D^* \otimes I_q) = (S_{i,ij}; i, j = 1, 2, \ldots, k).\]
Hence, the statistic \( \lambda_6 \) can be rewritten as
\[\lambda_6 = \frac{|S_1|^n}{\left| S_{1,11} \right|^n \sqrt[k]{\sum_{i=2}^{k} S_{i,ii}/(k-1)^{n(k-1)}}} \ldots (3.4.27)\]

Further let \( \Sigma_1 = PP^*, P = QT \) with \( Q = \text{diag}(P_1, I_{k-1} \otimes P_2) \)
and \( T \) is a lower triangular matrix with
\[
T = \begin{bmatrix}
I_q & o & o & \cdots & o \\
T_{21}/n & T_{22} & o & \cdots & o \\
& & \cdots & \cdots & \cdots \\
T_{k1}/n & T_{k2}/n & \cdots & T_{kk}
\end{bmatrix}
\]
Hence using Theorem 3.1,
\[
\frac{S_1}{2b_1} = \Sigma_1 + \frac{QTW^*Q^*}{\sqrt{n}}; \quad \frac{S_{1,11}}{2b_1} = P_1P_1^* + \frac{P_1W_{11}P_1^*}{\sqrt{n}},
\]
\[
\frac{S_{1,ii}}{2b_1} = P_2P_2^* + \frac{P_2R_{ii}W_{ii}R_{ii}^*P_2^*}{\sqrt{n}}.
\]
where
\[ R(1) = \left( \frac{T_{11}}{\sqrt{n}}, \frac{T_{12}}{\sqrt{n}}, \ldots, \frac{T_{1i}}{\sqrt{n}}, 0, \ldots, 0 \right) \]
and
\[ Q^{-1} \Sigma_1 Q^{-1} = T \Sigma T^* = I + R/n, \quad P_1^{-1} \Sigma_1 P_1^{-1} = I_q, \]
\[ P_2^{-1} \Sigma_{1,i} P_2^{-1} = T_{1i} T_{1i}^* + \sum_{j=1}^{i-1} T_{1j} T_{1j}^*/n = I + R_{1i}/n \]
so that \( \sum_{i=2}^{k} R_{1i} = 0. \) Hence,
\[ -\log \lambda_6 = -n \log |I+R/n| - n \log |I+W/n| + n \log |I+W_{1i}/n| + n(k-1) \log |I_q + \sum_{i=2}^{k} R(i) W R(i)^*/n (k-1)|, \]
and if we expand \(-\log \lambda_6\) in powers on \(1/n\) then similar to the real case developed in Subsection (2.2.4) of Chapter 2 we have
\[ -\log \lambda_6 = \frac{1}{2} \left[ \text{tr} \sum_{i=2}^{k} \left( R_{1i} W_{1i} \right)^2 - \frac{1}{(k-1)} \text{tr} \left( \sum_{i=2}^{k} W_{1i} \right)^2 \right] \\
+ \sum_{i=2}^{k} \sum_{j=1}^{i-1} \text{tr}(W_{1j}+R_{1j})(W_{1i}+R_{1i}) + O(n^{-1}) \]
\[ \cdots \cdots \text{(3.4.28)} \]
Let us consider the transformation \( Y_2 = \sum_{i=2}^{k} \frac{W_{1i}}{(k-1)} \) and \( Y_i = \sum_{\alpha=2}^{k} \delta_{i\alpha} (W_{\alpha \alpha} + R_{\alpha \alpha}) \) for \( i = 3, 4, \ldots, k \) where
\[ G = (\delta_{i\alpha}; i, \alpha = 2, 3, \ldots, k) \text{ is a } (k-1)x(k-1) \text{ unitary} \]
matrix with its first row as \( g_{2a} = (k-1)^{-\frac{1}{2}} I_{k-1} \). Then (3.4.28) can be rewritten as
\[
-\log \lambda_6 = \frac{1}{2} \sum_{i=3}^{k} \text{tr} Y_i^2 + \sum_{i=2}^{k} \sum_{j=1}^{i-1} \text{tr}(W_{ij} + R_{ij})(W_{ij} + R_{ij}) + o(n^{-\frac{1}{2}})
\]

Now, it is easy to verify that the elements of \( Y_i \)'s (\( i = 3, 4, ..., k \)) are independent normals and hence
\[
2a_2^{-1} \text{tr} Y_i^2 \]
has a non-central Chi-square distribution with degrees of freedom \( q^2 \) and with non-central parameter
\[
\frac{\text{tr} R_{ii}}{a_2^2} \text{ and } \left[ 2a_2^{-1} \sum_{i=2}^{k} \sum_{j=1}^{i-1} \text{tr}(W_{ij} + R_{ij})(W_{ij} + R_{ij}) \right]
\]
has non-central Chi-square distribution with degrees of freedom \( (k(k-1)q^2) \) and with non-central parameter
\[
\left\{ 2a_2^{-1} \sum_{i \neq j} \text{tr} R_{ij}^2 \right\}
\]

Thus the asymptotic distribution of \((-2a_2^{-1} \log \lambda_6)\) is non-central Chi-square with degrees of freedom \( (p^2 - 2q^2) \) and with non-central parameter \( (2a_2^{-1} \text{tr} R^2) \).

Similar to the earlier cases, let
\[
\frac{1}{n} \lambda_6 = t_6, \quad t_0 = |T_0^*|
\]
where \( T_0 \) is assumed to be fixed and does not tend to \( I p \) as \( n \to \infty \). Then,
$$t_6 = \sqrt{n(t_0/t_\infty)} = \text{tr} W - \text{tr} W_{11} - \sum_{i=2}^{k} \text{tr} T(i)WT(i) + O(n^{-\frac{1}{2}})$$

$$= \text{tr}(I - R_0)W + O(n^{-\frac{1}{2}}) \quad \ldots \quad (3.4.29)$$

where $R_0 = T_0T_0^*$, $\text{tr} R_0 = p$, $T(i)T(i) = I_q$.

$T(i) = (T_{i1}, T_{i2}, \ldots, T_{ii}, 0, \ldots, 0)$.

Hence

$$t_6 \left[ a_2 (\text{tr} R_0^2 - p) \right]^{-\frac{1}{2}} \overset{\text{asy}}{\sim} N(0, 1) \quad \ldots \quad (3.4.30)$$

### 3.4.7. Test for independence of k-sets of vector variables:

The likelihood ratio test for testing

$H \{ \Sigma = \text{diag}(\Sigma_{11}, \Sigma_{22}, \ldots, \Sigma_{kk}) \}$ against

$K \{ \Sigma \neq \text{diag}(\Sigma_{11}, \Sigma_{22}, \ldots, \Sigma_{kk}) \}$ under normality is

$$\lambda_7 = \frac{|S|^{n}}{\prod_{i=1}^{k} |S_{ii}|^{n}} \quad \ldots \quad (3.4.31)$$

where $S = (S_{ij}; \ i,j = 1,2,\ldots,k)$. The alternatives close to $H$ are

$$[\text{diag}(P_1, P_2, \ldots, P_k)]^{-\infty}[\text{diag}(P_1, P_2, \ldots, P_k)]^{-1} = I_p + R/n \quad \ldots \quad (3.4.32)$$
where \( \Sigma = PP^* \), \( \Sigma_{ii} = P_i P_i^* \) for \( i = 1,2,\ldots,k \) and

\[
R = (R_{ij}, i,j = 1,2,\ldots,k) \text{ with } R_{ii} = 0 \text{ for all } i = 1,2,\ldots,k; \quad F = QT, \quad Q = \text{diag}(P_1, P_2,\ldots, P_k) \text{ and } T \text{ is a lower triangular matrix with } T_{ii}'s \quad (i = 1,2,\ldots,k)
\]

being square matrices and

\[
T = \begin{bmatrix}
T_{11} & 0 & 0 & \cdots & 0 \\
T_{21}/n & T_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
T_{k1}/n & T_{k2}/n & \cdots & T_{kk}
\end{bmatrix}
\]

Hence using Theorem 3.1 we have

\[
\frac{S}{2b_1} = PP^* + \frac{PW^*}{\sqrt{n}}, \quad \frac{S_{ii}}{2b_1} = P_i P_i^* + \frac{P_i T_{ii} W_{ii} T_{ii}^* P_i^*}{\sqrt{n}} \quad \text{for } i = 1,2,\ldots,k. \quad \ldots \tag{3.4.33}
\]

Then using (3.4.33) in (3.4.31) we have

\[
-\log \lambda_7 = -n \log |I + R/n| - n \log |I + W/n| \\
+ n \log \sum_{i=1}^k |I_{q_i} + T_{ii} W_{ii} T_{ii}^*/n| \\
= \frac{1}{2} \text{ tr } \sum_{i,j} (R_{ij} R_{ji} + W_{ij} W_{ji}) + o(n^{-\frac{3}{2}})
\]

where \( Q^{-1}Q^* - 1 = TT^* = I + R/n, \quad T_{ii} T_{ii}^* = I_{q_i} \) as \( n \to \infty \),
Thus the asymptotic distribution of $\sum_{i=1}^{k} q_i = p$. Thus the asymptotic distribution of $\sum_{i=1}^{k} (-2a_2^{-1} \log \lambda_i)$ is non-central Chi-square with degrees of freedom $p(p+1) - \sum_{i=1}^{k} (q_i^2 + q_i)$ and with non-central parameter $(2a_2^{-1} \text{tr } R^2)$. Further, let

$$\frac{1}{n} t_7 = \lambda_7 \text{ and } t_{07} = |I + R_0|,$$

and let us assume that $Q^{-1} \sum Q^{-1} = TT^* = I + R_0$ is a fixed matrix as $n \to \infty$ then,

$$t(7) = \frac{1}{n} \log(t_7/t_{07}) = \text{tr}(I - \sum_{i=1}^{k} T(i)^* T(i))W + O(n^{-\frac{3}{2}}) \quad (3.4.34)$$

with $T(i) = (T_{i1}, T_{i2}, \ldots, T_{ii}, 0, \ldots, 0)$ and

$$t(7) [a_2 \text{tr } R_0^2]^{-\frac{1}{2}} \text{ asy } \sim N(0, 1) \quad (3.4.35)$$

3.5. Asymptotic distributions for the scale parameters for the two or more populations:

Let $\chi_{ij}$ ($i = 1, 2, \ldots, k; j = 1, 2, \ldots, n_i$) be $n$ independent observations from $G(\chi_{ij}) = G(\chi_{ij}; \mu_i, \Sigma_i)$ and let
The likelihood ratio test for testing \( H (\Sigma_1 = \Sigma_2 = \ldots = \Sigma_k) \) against \( K(\Sigma_i \neq \Sigma_i', \text{for at least one pair } (i, i')) \) is

\[
\lambda_8^* = \frac{\prod_{i=1}^{k} |S_{ii}|^{\frac{n_i}{2}}}{|S|^{n}} \quad \ldots \quad (3.5.1)
\]

3.5.1. Two populations:

The alternatives close to \( H \) when \( k = 2 \) are

\[
\Sigma_1 = P\Sigma P^*, \quad \Sigma_2 = \Sigma P^* D \Sigma P^*, \quad \nu_i = 1 + \xi_i/n \quad \text{for } i = 1, 2, \ldots, p
\]

\ldots \quad (3.5.2)

where the diagonal elements of \( D \) are the eigenvalues of \( \Sigma_1^{-1} \Sigma_2 \) and \( n_1 \) and \( n_2 \) are so large that \( n_i/n = r_i \) (\( i = 1, 2 \)) are fixed with \( r_1 + r_2 = 1 \). Now using Theorem 3.1, we have

\[
\frac{S_1}{2b_1} = PP^* + \frac{W_1^* P^*}{\sqrt{n_1}}, \quad \frac{S_2}{2b_1} = \Sigma P^* D + \frac{D_{\Sigma} P^* W_{\Sigma} D_{\Sigma} P^*}{\sqrt{n_2}}
\]

and

\[
\frac{S}{2b_1} = P(r_1 I + r_2 D_{\Sigma}) P^* + P(\sqrt{r_1} W_1 + \sqrt{r_2} D_{\Sigma} W_{\Sigma} D_{\Sigma}) P^* / n
\]

\ldots \quad (3.5.3)

Hence for \( k = 2 \) if we denote \( \lambda_8^* \) by \( \lambda_8 \) then
\[ -\log \lambda_8 = -n_1 \log \left| I + W_1 / n_1 \right| - n_2 \log \left| I + D_2 / n_2 \right| \]
\[ + n \log \left| I + \left( r_1 W_1 + r_2 D_1 + r_2 D_2 \right) / n \right| \]
\[ = \frac{1}{2} \text{tr} \{ r_2 W_1 - r_1 W_2 + (r_1 r_2) D_3 \} + o(n^{-\frac{1}{2}}) \]
\[ \text{.... (3.5.4)} \]

Hence, the asymptotic distribution of \(-2 \log \lambda_8\) is that of \(a^2 z_1 + a_2 z_2\) type where \(a_1 = pK + a_2\) and \(a_2 = K + 1\) and \(z_1\) and \(z_2\) are independent non-central Chi-square variates with respective degrees of freedom \(1\) and \((p^2 - 1)\) and with respective non-central parameters \((pr_2 \bar{s}^2 / a_1)\) and 
\[
\left( r_1 r_2 \sum_{i=1}^p \left( s_i - \bar{s} \right)^2 / a_2 \right).
\]

As usual let \(\lambda_8 = 1/n\) and \(\lambda_8 = \left| D_1 \right| r_2^2 / r_1 I_1 + r_2 D_2 |^{-1}\).

Then
\[
t(8) = n \log(t_8 / \lambda_8) = r_1 \text{tr}[I - (r_1 I_1 + r_2 D_2)^{-1}] W_1
\]
\[ + r_2 \text{tr}[I - D_1 (r_1 I_1 + r_2 D_2)^{-1}] W_2 + o(n^{-\frac{1}{2}}) \]
\[ \text{.... (3.5.5)} \]

and 
\[
t(8) \overset{\text{asy}}{\sim} N(0, \sigma_8^2) \]
\[ \text{.... (3.5.6)} \]

where
Further, let $t_8^* = \text{tr} S^{-1} S_2$ & $t_{08}^* = \frac{1}{\sqrt{r_2}} \sum_{i=1}^{p} \frac{v_i}{r_1 + r_2 v_i}$. Then,

$$\sqrt{n}(t_8^* - t_{08}^*) = \frac{1}{\sqrt{r_2}} \text{tr} \{ (r_1 I_p + r_2 D_2)^{-1} D_2 - r_2 D_2 (r_1 I_p + r_2 D_2)^{-2} \} W_2$$

$$- \sqrt{r_1} \text{tr} D_2 (r_1 I_p + r_2 D_2)^{-2} W_1 + O(n^{-\frac{1}{2}})$$

..., (3.5.7)

and

$$\sqrt{n}(t_8^* - t_{08}^*) / \delta_8^* \xrightarrow{asy} N(o, 1)$$

with $\delta_8^2 = \frac{r_1}{r_2} \left[ a_2 \left( \sum_{i=1}^{p} \frac{v_i^2}{r_1 + r_2 v_i} \right)^4 \right. + (a_2 - 1) \left( \sum_{i=1}^{p} \frac{v_i}{r_1 + r_2 v_i} \right)^2 \left. \right]$. (3.5.8)

Further, if we take

$$t_{8**} = \text{tr} S^{-1} S_2 \text{ and } t_{08**} = \frac{\sqrt{r_2}}{\sqrt{r_1}}$$

Then

$$\sqrt{n} (t_{8**} - t_{08**}) = \text{tr} D_2 \left( \frac{W_2}{\sqrt{r_2}} - \frac{W_1}{\sqrt{r_1}} \right) + O(n^{-\frac{1}{2}})$$

..., (3.5.9)

and

$$\sqrt{n}(t_{8**} - t_{08**}) / \delta_{8**} \xrightarrow{asy} N(o, 1)$$
with \( G^{**} = \frac{1}{r_1 r_2} \{ a_2 \sum_{i=1}^{p} v_i^2 + (a_2-1)p_2 \frac{v^2}{v} \} \) \((3.5.10)\)

### 3.5.2. More than two populations:

For \( k > 2 \), we have \( \lambda^*_B = \frac{k}{1} |S_1|^{n_1} / |S|^n \). Hence for the asymptotic distribution under normality and under \( H \), one can refer to Anderson (1985)* while the asymptotic distribution under normality and under alternatives close to \( H \) when \( k = 2 \) is given by Khatri and Srivastava (1974).

For \( k > 2 \) the alternatives close to \( H \) are

\[
\Sigma = \sum_{i=1}^{k} r_i \Sigma_1 = PP^*, \quad F^{-1} \Sigma_1 F^*-1 = T_1 D_1 v T_1^* \quad \ldots (3.5.11)
\]

and \( v_{i\alpha} = 1 + s_{i\alpha} / n \) for \( \alpha = 1, 2, \ldots, p \) and \( i = 1, 2, \ldots, k \), where the diagonal elements of \( D_1 v \) are the eigen values of \( \Sigma^{-1} \Sigma_1 \) for \( i = 1, 2, \ldots, k \) and \( n_1, n_2, \ldots, n_k \) are so large that \( n_i / n = r_i \) are fixed, \( \Sigma \) \( r_i = 1 \) and \( T_i \)'s are unitary matrices for all \( i \). Now using Theorem 3.1, for \( k > 2 \) we have,

\[
\frac{S}{2b_1} = PP^* + \frac{F(\Sigma \sqrt{T_1 T_i D_i/v W_i D_i/v T_i^*})P^*}{\sqrt{n}}
\]

and \( \frac{S_i}{2b_1} = P(T_i D_1 v T_i^* + T_i D_1/v W_i D_1/v T_i^*/(nr_i))^P. \)

\ldots (3.5.12)
Hence,
\[-\log \lambda^*_B = -n \sum_{i=1}^{k} r_i \log |I + (R_i + W(i)/\sqrt{n})/\sqrt{n}| \]
\[+ n \log |I + \sum_{i=1}^{k} \sqrt{r_i} W(i)/\sqrt{n}| \]
\[= \frac{1}{2} \left\{ \sum_{i=1}^{k} \text{tr} \left( /r_i R_i + W(i) \right)^2 \right\} - \text{tr} \left( \sum_{i=1}^{k} /r_i W(i) \right)^2 \} + o(n^{-\frac{1}{2}}) \]
\[= \text{(3.5.13)} \]

where \( R_i = T_i D_{iv} T_i^* \), \( D_{iv} = I + D_{iv}/\sqrt{n} \), \( W(i) = T_i W_i T_i^* \).

Note that \( W(i) \) has the same distribution as \( W_i \) since \( T_i \)'s ( \( i = 1, 2, \ldots, k \) ) are unitary matrices. Let us consider the transformation \( Y_i = \sum_{a=1}^{k} j_{ia} \left( W(a) + \sqrt{r_a} R_a \right) \)
for \( i = 1, 2, \ldots, k \) where \( (j_{ia}; i, a = 1, 2, \ldots, k) \) is a \( k \times k \) unitary matrix and its first row is \( j_{i1} = \frac{1}{\sqrt{r_1}}, \frac{1}{\sqrt{r_2}}, \ldots, \frac{1}{\sqrt{r_k}} \). Then
\[ Y_i = \sum_{a=1}^{k} \sqrt{r_a} \left( W(a) + \sqrt{r_a} R_a \right) = \sum_{a=1}^{k} \sqrt{r_a} W(a) \]

and hence
\[ -2 \log \lambda^*_B = \sum_{i=2}^{k} \text{tr} Y_i^2 + o(n^{-\frac{1}{2}}) \quad \text{(3.5.14)} \]

The asymptotic distribution of \( \text{tr} Y_i^2 \) is that of \( a_1 z_{1i} + a_2 z_{2i} \) where \( z_{1i} \) and \( z_{2i} \) are independent non-central Chi-squares with respective degrees of freedom 1 and \( (\beta^2 - 1) \).
and with respective non-central parameters \( \tau_1 = (\text{tr } L_1)^2/pa_1 \)
and \( \tau_2 = [\text{tr } L_1^2 - (\text{tr } L_1)^2/p]/a_2 \) where
\[ L_1 = \sum_{i=1}^{k} Y_i = \sum_{a=1}^{k} J_{ia} J_{ia} R_a. \]

Thus the asymptotic distribution of \(-2 \log \lambda^*_g\) is that of \(a_1z_1 + a_2z_2\) where \(z_1\) and \(z_2\) are independent non-central Chi-squares with degrees of freedom \((k-1)\) and \((k-1)(p^2-1)\)
and with non-central parameter \( \tau_1 = p \sum_{a=1}^{k} r_a \bar{s}_a^2/a_1 \) and
\[ \tau_2 = \sum_{i=1}^{k} r_i (s_{ia} - \bar{s}_i)^2/a_2 \] respectively where
\[ \sum_{a=1}^{k} s_{ia}/p = \bar{s}_i \] and \[ \sum_{i=1}^{k} r_i \bar{s}_i = 0 \] on account of \[ \sum_{i=1}^{k} r_i R_i = 0. \]

Let \( \Sigma_i P_i^{*-1} = Q_i \) \((i = 1,2,\ldots, k)\) be fixed matrices and let \( t_{g}^* = \lambda^*_g/n \) and let \( t_{o8}^* = \prod_{i=1}^{k} Q_i r_i. \)

Then
\[ t(8) = n \log(t_{g}^*/t_{o8}^*) = \text{tr} \sum_{i=1}^{k} r_i (I - Q_i) W_i + O(n^{-1/2}) \]
and
\[ t^*_g / \varepsilon_g^* \quad \text{asy} \quad N(0, 1) \]
with \( \varepsilon_g^* = a_2 \sum_{i=1}^{k} r_i (\text{tr } Q_i - \frac{(\text{tr } Q_i)^2}{p}) + \frac{a_1}{p} \sum_{i=1}^{k} r_i ((\text{tr } Q_i)^2 - p^2) \)
\[ \quad \text{(3.5.15)} \]

\[ \text{and t}^*_g / \varepsilon_g^* \quad \text{asy} \quad N(0, 1) \]
with \( \varepsilon_g^* = a_2 \sum_{i=1}^{k} r_i (\text{tr } Q_i - \frac{(\text{tr } Q_i)^2}{p}) + \frac{a_1}{p} \sum_{i=1}^{k} r_i ((\text{tr } Q_i)^2 - p^2) \)
\[ \quad \text{(3.5.16)} \]