7.1. **Conclusions:**

In this thesis asymptotic results are derived for multivariate parameters with the assumption on the existence of finite four moments. An important class of non-normal distributions which have been found increasing in use for robustness studies where the aim is to determine how sensitive existing multivariate techniques are to multivariate normality assumption.

In conclusion it may be mentioned that the asymptotic distributions have surprisingly simple forms in the case of elliptical and/or complex elliptical populations. Finally, it is observed that when the sample is drawn from the elliptical and/or complex elliptical populations, the asymptotic distributions of test statistics used in some multivariate procedures are similarly simple. If one looks at the results derived in Chapter 2 and Chapter 3 then it can be observed that the asymptotic distributions of test procedures depending on
various structure of covariance matrix $\Sigma$, when the normality and/or complex normality assumption is replaced by elliptical and/or complex elliptical distributions respectively, are that of $a_1z_1 + a_2z_2$ where $z_1$ & $z_2$ are independent non-central Chi-square variates and $a_1$ and $a_2$ are some constant depending on kurtosis parameter $K$ of the population. Further it may be concluded that the asymptotic distributions of sample canonical correlations is independent of the order of canonical correlations and the asymptotic simultaneous confidence bounds of the discriminatory values $D_a$ based on the LOO statistics in all the three cases: (i) $\mathcal{H}_1 - \mathcal{H}_2$ is known & $\Sigma$ is unknown, (ii) $\mathcal{H}_1 - \mathcal{H}_2$ is unknown & $\Sigma$ is known and (iii) $\mathcal{H}_1 - \mathcal{H}_2$ & $\Sigma$ are unknown, are surprisingly same for large $n$. At last in all the asymptotic results if we take $K = 0$ then the results for normal and/or complex normal distributions can be obtained as special cases from the results of elliptical and/or complex elliptical distributions respectively.

7.2. Future Work:

The work done in this thesis is mostly connected to elliptical and/or complex elliptical random vector
variables. One can develop the asymptotic results parallel to the real and/or complex case for quaternion normal and/or quaternion elliptical random vector variables as follows:

All the asymptotic inferential problems connected with complex elliptical distributions can be tackled for the class of quaternion elliptical distributions. The various tests of hypothesis concerning covariance matrix $\Sigma$ similar to the problems on real and/or complex case can be considered for the class of quaternion normal distributions and the study of robustness for the likelihood ratio tests obtained under quaternion normality assumptions when the quaternion normality distribution is replaced by the class of quaternion elliptical distributions can be considered. Ahlquist (1988) gave principal component analysis of vector-valued functions of four dimensions. For the algebra of quaternion numbers one can refer Andersson (1975) and for the test procedures concerning covariance matrix has quaternion structure one can refer to Andersson et al (1983). Asymptotic distribution of test statistic under normality assumption is also given by them.
Likelihood ratio tests and their asymptotic distributions for the class of real, complex and quaternion elliptical populations can be derived without using the assumption of real, complex and quaternion normality respectively. For normal population see Andersson et al (1983). The likelihood ratio test and its asymptotic distribution for testing the equality of $k$ covariance matrices $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ of $k$ elliptical populations with $k$ different kurtosis parameters $K_1, K_2, \ldots, K_k$ respectively can be derived but it is much more difficult to perform.

The linear growth curve model $E(X) = B \xi A^*$ where the column vectors of $X$ are independently distributed as $p$-variate quaternion elliptical distribution with covariance matrix $\Sigma$, $A$ and $B$ being known quaternion matrices of full ranks, $\xi$ is an unknown quaternion parametric matrix and $\Sigma$ has the intraclass correlation structure, i.e.

$$\Sigma = \delta^2 \{(1- s)I_p + s \tilde{g} \tilde{g}^*\}$$

with

$$\tilde{g} = \{1, \exp(i \mu_1 + j \mu_2 + k \mu_3), \exp(2i \mu_1 + 2j \mu_2 + 2k \mu_3), \ldots, \exp(i(p-1) \mu_1 + j(p-1) \mu_2 + k(p-1) \mu_3)\}$$

where $i$, $j$, $k$ are unit vectors orthogonal to each other and satisfies the relations
\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = d \]

can be considered. The growth curve model for the complex elliptical variates can be considered for unknown $\Theta$ and the asymptotic distributions of maximum likelihood estimators $\hat{\gamma}$, $\hat{s}$, $\hat{\gamma}^2$ and $\hat{\Theta}$ can be obtained. The asymptotic distribution of test procedure for testing autoregressive structure of first order of $\Sigma$ for GMANOVA of elliptical variates and the asymptotic distribution of test procedure for testing intra-class structure of $\Sigma$ for GMANOVA of complex-elliptical variates can be obtained.

The non-central case for the class of real, complex and quaternion elliptical distribution can be considered, i.e. the moments of elliptical distributions with mean $y$ and covariance matrix $\Sigma$ can be obtained after a lot of mathematical calculations and hence these moments can be used for deriving all the asymptotic results which are derived in central case.

Further probability models on scale invariant functions can be considered for a quaternion random vector similar to real and complex random vector. A new class of quaternion lognormal distribution can be defined. All the distributions
which are derived for the directional and compositional data can be derived for the class of real, complex and quaternion elliptical distributions. Important properties of complex log-normal and/or quaternion log-normal distributions for both univariate & multivariate case can also be developed.

Note that if \( p = p_0 + i p_1 + j p_2 + k p_3 \) and \( q = q_0 + i q_1 + j q_2 + k q_3 \) are two quaternion numbers then \( pq \neq qp \) for any two quaternion numbers and hence a quaternion random variable \( q \) and/or \( p \) does not form a field like real and complex random-variables. Hence all the above problems can be tackled for quaternion random variables if the determinant of a matrix whose elements are quaternion random variables is defined with some restrictions.

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