1.1 GENERAL

The problem of inventory management is very common and crucial to all the enterprises in any sector of a given economy. The fundamental reason for keeping inventory is that it is either physically impossible or economically unsound to have goods arrive in a given system precisely when their demands occur. Inventories are inevitable not only for the smooth and efficient running of the business but also for keeping the goodwill of the customers. Similarly the market price of some raw materials as used by the manufacturer may exhibit considerable seasonal fluctuations. Thus when the price is low, it may be profitable for the businessman to procure sufficient quantity, so that it may last through the high priced season. It is also very common and essential for any retail establishments as the subsequent sales may exhibit the profit contribution quite reasonably.

It was Ford Harris [22] who invented for the first time what is often called the "Simple lotsize formula", in 1950. This formula is also derived by Wilson R.H. [68]. Since then it has become extremely popular as "Wilson's formula". It was Raymand [44] who provided the first full length book dealing with inventory problems. Since then, many researchers as well
as O.R. practitioners have contributed greatly in the area of inventory control. Some standard texts that are found today are by Arrow, Karlins and Scarf [6], Hanssman [24], Hadley and Whitin [23], Naddor [41], Peterson, Rein and Silver [42], Starr and Miller [58], Wanger [62], Whitin [66], Buffa and Miller [15], Brown [11] etc.

In the standard literature on inventory control, a discussion is given on several deterministic & probabilistic models. As per passage of time, the inventory theory has literally become the full blown academic discipline, nonetheless with little regard to the practical applicability of the models developed. In spite of this wide gap between theory and practice, the invasion of Industrial Management area by Operations Research techniques paved the way of enhanced research efforts to tie up academic and practical fields. Most of the research work carried out in the earlier time assume that the products that are stored as inventory would not perish, deteriorate, decay or become obsolete as per passage of time. However later on during last 25 to 30 years time, the researchers have developed the inventory models for describing the optimal policies for products that would deteriorate, decay, perish or become obsolete. As a result, there are large number of applications in this area. Some studies pertaining to this field is mentioned in the works carried out by Shih [52], Sachan [47], Shah Y.K. [50], Shoukath Ali and Ramaswamy [53], Jani [30], Dave & Patel [21] etc.

It is very often found in practice that the inventory sub-systems are subjected to interaction between the marketing
policies and economic ordering quantities. Kotler [33] has developed managerial decision making approach pertaining to the above aspects. He has also indicated in his exhaustive study several methods to deal with the model building approach pertaining to this area. A well known model of inventory control involves lotsizing with quantity discount possibilities. Factors such as convenience, industry practice and marketing have made the use of price quantity discount schedules common in many business (see Wilcox [67]). Quantity discounts can provide economic advantages for buyer and vendor both. Quantity discounts can provide the buyer lower per unit purchase cost, lower ordering costs, the decreased likelihood of shortages. The advantages that can be realised by the supplier are reductions in setup cost, transportation cost and the supplier's inventory.

Several authors have contributed in this area. Some of these works may be mentioned by Ladany and Sternlieb [35], Subramanyam and Kumaraswamy [60], Wayland [65], Zangwill [69], Agrawal and Ayalwadi [3], Monahan [39], Rosenblatt and Lee [45], Banerjee [7], Kuzdrall and Britney [34], Rubin, Dilts and Barron [46], Budhbhatti [14], Brahmbhatt [10], Buzacott [16, 17], Das [18], Jaiswal and Jani [29], Shah C.M. [49], Manne [36], Michael and Klein Morton [38], etc.

To summarise, we may note that the cost and production analysis plays a significant role in the managerial decision making problems. One of the most important tasks of the management is to choose an optimum course of action from among a given set of identified actions. In recent times, many
concepts, tools & techniques have been developed by the Economists, Statisticians and Mathematicians which help the management in resolving these choice of problems efficiently and effectively.

In this dissertation, some studies regarding certain inventory models are considered. In actual practice, there are several commodities which are stored in a warehouse, so that the aggregation problem becomes very prominent for the storage of these goods. There is also an interconnecting network of prices which is influenced by the marketing behaviour of these commodities. Accordingly the mark-up of prices, quantity discount and the profit maximisation approach for the inventory system under consideration are studied in Chapters II, III & IV. The aggregation problems concerning the several commodities are considered for the respective inventory models. In Chapters V to VIII, the inventory models for restrictions are considered specifying the different types of constraints due to the limited storage facility, limited budgetary constraint, restrictions on the total number of orders etc. In each case corresponding to the respective models considered, the profit maximisation approach is taken into account.

Section 1.2 of this Chapter contains a brief presentation of some concepts and definitions associated with the study in which a general layout of the inventory system along with its cost structure is presented.

In section 1.3, a brief introduction about the quantity discount cost functions along with their specific properties and economic interpretations are given.
In section 1.4, a brief account for the profit maximisation approach and the mark-up of pricing is presented.

In section 1.5 and 1.6, a brief presentation about the aggregation problem and the inventory models under restrictions is given.

Section 1.7 presents briefly the objectives of this study.

Thus, the entire study is presented in details through the Chapters II to VIII.

An exhaustive list of references is presented at the end.

1.2 SOME CONCEPTS AND BASIC DEFINITIONS:

An inventory may be defined as the physical stock of goods or services which is held for the purpose of future production or sales.

An inventory system is a system in which the following three kinds of costs are significant, and in which any two or all the three of them are subject to control [41].

1. The cost of carrying inventories.
2. The cost of incurring shortages.
3. The cost of replenishing inventories.

A production system is one in which these costs arise, and are subject to control. In this system, the costs may be controlled by making appropriate decisions about ordering raw materials, manufacturing semi-finished and finished goods, and inventorying goods in readiness for shipment to customers.
The first kind of cost in this production system, that of *carrying inventories*, is the cost of investment in inventories, of storage, of handling items in storage of obsolescence, etc.

The second kind of cost, that of *incurring shortages*, is the cost of lost sales, of loss of goodwill, of overtime payments, of special administrative efforts (telephone calls, memos, letters) etc.

The third kind of cost, that of *replenishing inventories*, is the cost of machine setups for production runs, of preparing orders, of handling shipments etc.

These three kinds of costs will be referred to in general as the *carrying cost*, *shortage cost*, and *replenishing cost* respectively. The sum of these costs will be referred to as the *total inventory cost*.

In inventory control problems, the optimality decisions are to be taken with respect to (A) When should the inventory be replenished? (B) How much should be added to inventory?

The first question is usually answered in one of the two ways:

(i) Inventory should be replenished when the amount in inventory is equal to or below some quantity units \( (s) \) which is called the **Re-order Point** (For non-zero lead time it is specified by \( (z) \)).
(ii) Inventory should be replenished after every time units \( t \) which is called **Scheduling Period**. It is the length of time between consecutive decision with respect to replenishments.

The answer to second question can be given in one of the two ways:

(i) The quantity to be ordered is equal to some number of units which is called the **replenishment size**. If the replenishment size is the same for every scheduling period, it is called **lot-size** \( (q) \).

(ii) A quantity should be ordered so that the amount in inventory is brought to some level of quantity units \( (S) \) which is called **order level**. (For non-zero lead time, it is specified by \( (Z) \)).

The length of time between scheduling a replenishment and its actual addition to the stock is called the **lead time**. The kind of mathematical analysis used in solving an inventory system depends upon a large extent - whether the units involved are continuous or discrete. For example, inventory problems involving liquids or gasoline are basically represented by **continuous units** while others, for example automobiles, are represented by **discrete units**.

The broad categories of inventory models are classified as -  
(i) Deterministic models and  
(ii) Probabilistic models.
The Deterministic models consist of those systems having known demand pattern whereas the Probabilistic models consist of the stochastic pattern of variation in demand. In each of the above categories, there are models with

(i) zero lead time,
(ii) with fixed lead time and
(iii) with variable lead time.

Using the above concepts, both the above types of inventory models can be very broadly classified into different systems like: \((s,q)\), \((t,S)\), \((s,S)\) and \((t,q)\) policies corresponding to zero lead time and into \((z,q)\), \((t,Z)\), \((z,Z)\) and \((t,q)\) policies corresponding to non-zero lead time.

The demand of a given commodity can be represented in terms of a functional form known as the demand function which expresses the relationship between the demand and the price or the marketing effort. Popular formulations of the inverse demand-price relationship include the linear function and the constant elasticity function. Occasionally, positive demand-price relationships are observed and they can be explained in terms of either the income effect, or the expectation effect, or the quantity effect [33]. Popular formulations of the demand-marketing effort relationships are usually the concave functions.

The quantity required to satisfy the demand for inventory will be called the demand size. i.e. demand rate is the demand size per unit time. For this study the demand functions for the
several commodities are of varying nature but the demand is
deterministic. i.e. the respective demand functions for the
commodities are known. In general, the demand function for the
jth commodity is specified by \( R_j = f(p_j) \), where \( R_j \) is the
quantity demanded per unit time and \( p_j \) is the price of the jth
commodity. We further assume that these demand laws are
ordinary or Marshallian demand functions having their respective
price elasticities of demand given by

\[
\eta_j = - \left[ \frac{\partial \log(R_j)}{\partial \log(p_j)} \right] 
\]

Due to the nature of these demand functions it is assumed that

(i) The market prices of the successive commodities are
mutually independent.

(ii) The consumer's behaviour does not change for purchasing his
goods required during the given period of time.

In particular, for this study we can consider the following
specific forms of the demand functions for the several
commodities that are stored in the warehouse.

(1) \( R_j = Kp_j^{-\alpha_j} \) ... (1.1)

\( (j = 1, 2, \ldots, n) \)

where \( p_j > 0, R_j > 0 \)

so that \( \eta_j = \alpha_j \) and note that with \( \eta_j = 1 \), we have the constant
outlay curves.

(2) \( R_j = A - Bp_j \) ... (1.2)

\( (j = 1, 2, \ldots, n) \)

where \( A > 0, B > 0. \)

A is the intercept and B is the gradient of the demand curve.
For this demand function
\[ \eta_j = \left( \frac{A}{Bp_j} - 1 \right)^{-1} \] ... (1.3)

\((j = 1, 2, \ldots, n)\)

1.3 QUANTITY DISCOUNT COST FUNCTIONS:

While discussing the particular forms of the cost functions, which can be allowed for quantity discount, one should also keep in mind the economic considerations for these cost functions. In the standard inventory models, the inventory holding cost per unit time is considered to be a constant. However in practice, situations arise where this cost may no longer remain as-constant, but it becomes a varying cost. Such a Varying cost function is expressed by the symbol \(C(q_j)\), so that the inventory holding cost for \(j\)th commodity is given by
\[ C_{1j} = I.C(q_j) \] ... (1.4)

\((j = 1, 2, \ldots, n)\)

where \(I = \text{Inventory carrying charge in Rs./unit time.}\)
\((0 < I < 1)\).

\(C(q_j) = \text{Cost per unit.}\)

Thus if \(C(q_j) = C_0 = \text{Constant, then } C_{1j} = I.C_0\) which represents the situation when the unit inventory holding cost per unit time is fixed, which is generally considered in the standard literature on inventory control models.

Two specific forms of the cost functions which are generally allowed for the quantity discount can be considered as under.
1.3.1 Case - (a) Linear Quantity Discount:

Here we use the unit cost function $C(q_j)$ given by

$$C(q_j) = C_0 - \alpha q_j \quad \ldots \quad (1.5)$$

where $C_0 > 0$, $\alpha > 0$ ($C_0 > \alpha$) are constants and $C(q_j) > 0$ for all $q_j > 0$.

This is the linear form of the unit cost variation as defined by Naddor [41]. Starting from a fixed maximum value, the cost decreases as larger amounts of lots are purchased. Thus the linear decrease as defined above is due to the quantity discount.

Note that when $\alpha = 0$, $C(q_j) = C_0 = \text{Constant}$.

This will be the case of constant cost with no quantity discount.

For the above linear cost function.

Average cost (AVC) $= C_0 q_j^{-1} - \alpha \quad \text{for} \quad (q_j > 0) \quad \ldots \quad (1.6)$

Marginal cost (MC) $= - \alpha \quad \ldots \quad (1.7)$

Thus, the marginal cost always decreases at a constant rate $\alpha$, though this rate of decrease is very small as compared to the fixed cost ($C_0 > \alpha$).

1.3.2 Case - (b) Hyperbolic Quantity Discount:

Here we take the unit cost function $C(q_j)$ as

$$C(q_j) = \begin{cases} 
C_0 & (0 \leq q_j \leq q_0) \\
 b + \frac{d}{q_j} & (q_0 \leq q_j \leq \infty)
\end{cases} \quad \ldots \quad (1.8)$$

where $C_0 > 0$, $b > 0$ and $d \geq 0$ are the constants and $C(q_j) > 0$ for all $q_j$. 
This cost function represents the hyperbolic form of unit cost function considering the case of quantity discount. It considers the quantity discount starting from a fixed maximum cost. It describes a situation where the producer has a fixed maximum cost and a variable cost, so that as the total amount demanded increases, the unit cost $C(q_j)$ varies (i.e. the variable cost decreases). In above, note that as $q_j \rightarrow \infty$, $C(q_j) \rightarrow b$ (or if $d = 0$, $C(q_j) = b$). Thus $C(q_j) = b$ represents the case of constant cost with no quantity discount, and it also represents an asymptote to the cost curve as $q_j \rightarrow \infty$. For the above hyperbolic cost function the

Average Cost (AVC) = $bq_j^{-1} + dq_j^{-2}$ ; ($q_j > 0$) \hspace{1cm} ... (1.9)

($j = 1, 2, ..., n$)

and

Marginal cost (MC) = $-dq_j^{-2}$ ; ($q_j > 0$) \hspace{1cm} ... (1.10)

($j = 1, 2, ..., n$)

Thus comparatively the marginal cost is smaller than the average cost in the case of hyperbolic cost function as compared to the linear form of cost function with quantity discount. The two cost functions with quantity discount are as shown in the diagram (Refer to diagram : 1.1).
From the cost function given in (1.8) above, we have at $q^= q_0$,

$$C_0 = b + \frac{d}{q_0}$$

which gives

$$q_0 = \frac{d}{(C_0 - b)}.$$  \hfill (1.11)

Note that the two cost curves will intersect at two points for $q = q_1$ and $q = q_2$, where $q_1$ and $q_2$ will be determined from

$$q_1 = \frac{(C_0 - b)}{2 \alpha} - \left[ \left\{ \frac{(C_0 - b)}{2 \alpha} \right\}^2 - \left\{ \frac{d}{\alpha} \right\} \right]^{1/2}$$

and

$$q_2 = \frac{(C_0 - b)}{2 \alpha} + \left[ \left\{ \frac{(C_0 - b)}{2 \alpha} \right\}^2 + \left\{ \frac{d}{\alpha} \right\} \right]^{1/2}.$$  \hfill (1.12)

the constants $C_0$, $b$, $\alpha$ and $d$ are such that $q_2 > q_1 > 0$. The values of $q_1$ and $q_2$ determine the boundary values of $q_j$ of the two cost functions.

1.4.1 Profit Maximisation Approach:

The mathematical models that are generally used in inventory control theory have found a large number of applications which can be considered to be extremely useful for the managerial decision making. Generally, the inventory models that are used in standard literature correspond to the cost minimisation approach. Kotler [33] has indicated that there is an interaction between the marketing policies and the economic ordering quantities. He has also obtained the optimum price on a given demand curve providing the maximum total revenue independently of the ordering quantity and then derived the economic order quantity assuming the demand and price as the fixed quantities.
The practitioners and the research workers in this field have given reasonable attention to solve the problems involving the maximisation of profit rather than the cost minimisation problem, where the profit is interpreted as the net revenue of the system under consideration.

Ladany and Sternlieb [35] have considered EOQ model for infinite replenishing rate under the influence of the marketing policies. They have considered two types of quantity discounts that usually occur in practice, while determining the value of optimum order quantities. Under quantity discount case, the unit inventory holding cost is no longer constant, but it is discounted by the appropriate cost functions known as "Quantity discount cost structures".

Buzacott [16,17] has considered EOQ models under inflation where the fixed horizon model has been considered along with the different types of pricing policies. Wayland [65] has shown how the Economic Order Quantities are intercorrelated with the decision making based upon the break-even chart analysis.

Some other work related to this area is also considered by Jani [30], Dave [19], Brambhatt [10], Budhbhatti [14], Agrawal and Ayalwadi [3], Shah C.M. [49], Subramanayam and Kumarswamy [60] etc.

In this thesis, a study of certain inventory models is considered for the managerial decision making for the situation in which there is a storage of a large number of items, so that we have an aggregation problem pertaining to all the
commodities. Similarly, the problem is studied for the different restrictions imposed, and the decision variables that are determined from such model can examine the effectiveness upon the profit contribution which is generally the most important concept for the businessman.

1.4.2 **MARK-UP PRICING**

A great number of firms set their prices largely or even wholly on the basis of their costs. Typically, all costs are included usually arbitrary allocation of overheads made on the basis of expected operating levels.

The most elementary examples of this are **mark-up pricing** and **cost-plus pricing**. They are similar, in that the price is determined by adding some fixed percentage to the unit cost. Mark-up price is most commonly found in the retail traders (groceries, furniture, clothing and jewellery etc.) where the retailer adds pre-determined but different mark-ups to various goods that he carries, cost-plus pricing (cost-plus margin) is most often used to describe the pricing of jobs that are non-routine and difficult to "cost" in advance such as construction and military Weapon development.

Many hypothesis have been advanced to explain the variations in mark-ups within selected product groups. Many researches conducted a detailed study to examine how much of the mark-up dispersion is within common product.

Mark-ups should vary inversely with turnover. Mark-ups should be higher and prices lower on reseller's (private) brands than on manufacturer's brand.
In general, there can be three types of mark-up pricing which can be summarised as

(i) Fixed Mark-up
(ii) Variable Mark-up and
(iii) Stochastic Mark-up.

In connection with several commodities, let $p_j$ denote the selling price of the $j$th commodity.

$C(q_j) = \text{unit cost for the } j\text{th commodity and}$

$\theta_j = \text{Mark-up parameter for the } j\text{th commodity} 
(j = 1, 2, \ldots, n)$

then using the fixed mark-up of pricing, we have the relation

$p_j = \theta_j C(q_j)$

which means that the selling price $p_j$ will be proportional to the unit cost function $C(q_j)$ and these prices are regulated by means of the mark-up parameter $\theta_j$.

Similarly, in variable mark-up of pricing, we can assume that

$f(\theta_j) = \text{parametric function for the parameter } \theta_j 
(j = 1, 2, \ldots, n)$

so that

$p_j = f(\theta_j).C(q_j) 
(j = 1, 2, \ldots, n)$

is the corresponding variable mark-up relationship as the selling price varies accordingly.

In stochastic mark-up of pricing, a probability density function of the mark-up parameter $\theta_j$ can be assumed so that one can think of the expected selling price corresponding to the $j$th commodity.
As per Stiggler [59], "there is the feeling that the mark-up pricing is socially fairer to both the buyer and the seller. The seller does not take advantage of the buyer when his demand becomes acute, yet the seller earns a fair return on his investment. The popularity of a cost oriented approach to pricing rests on considerations of administrative simplicity, competitive harmony and social fairness".

1.5 AGGREGATION PROBLEMS:

Generally in the routine stock control models, the decision variables are to be determined, so that the total or net effectiveness of the system's impact can be improved. We may have cost minimization or a profit maximization problem in which the decision variables can be determined by the appropriate optimization techniques involved for obtaining the solution of the model. In most of the theoretical models, the concentration is generally placed on a single item only. In real life situation, hardly such things can happen because there are very few organizations which have only one item which they may maintain in inventory. As the organizations get themselves expanded, the trend is very decisively in the direction of an enormous number of items which should be maintained in stock. Some organizations number their stock keeping units in hundreds or in thousand or even in lakhs and there can be different items for storage rather than only one item of the produced lots. Thus looking to the practical point of view, there exist a class of single period inventory problems. Which frequently arise in practice and accordingly we can have an aggregation problem for several commodities. Manytimes these items are completely
independent of each other. In such a situation, it is very interesting to examine, whether the existence of a multitude of items changes the problems, adds new problems or increases the range of possible solutions to the problems. It may be argued that for the simple aggregation problem, the same basic equation corresponding to the inventory model that is considered, can be applied to each item, using the data for that item and the optimal amount to order for that item can be determined by the corresponding equation. This gives us the optimal policy corresponding to each item and the final picture reflects the best overall solution to the total inventory problem.

1.6 CONSTRAINED OPTIMISATION PROBLEMS:

Most real world inventory systems deal with storage of several number of items rather than only a single item. It might be possible to study each item individually as long as there are no interactions among the items. However in real life problems, it is generally found that the successive items are not completely independent but they are related by some constraints. These constraints might be due to several factors as considered below:

(1) Constraint about the limited floor area that can be utilised.

(2) Constraint about the limited budget capacity.

(3) Constraint about the number of orders that can be placed taken commoditywise.

(4) Constraint about the total number of orders taken for the whole group of commodities etc.
The only interaction between this aggregate of items is assumed to be through such constraints. It is interesting to decide about the optimal inventory policy under such constraints and these problems are called the constrained optimisation problems. Generally, the technique used to solve such problems is to use of Langarange's multipliers and then solve the optimisation problems as per routine. A more general and comprehensive approach in this direction may not perhaps be quite relevant, so that it can be applied for the solution for such a model straightway as we apply usually Wilson Harris [68] and other formulae. Hence a need arises to obtain solutions for such problems in different situations. If solutions to such problems can be obtained then it can definitely help for its actual applications in practice.

1.7 OBJECTIVES OF THIS STUDY:

In the present research work, the broad objectives of this study are mentioned briefly as under.

(i) To make a decision about obtaining the optimum values of the variables that are included in the respective models considered, so that the enterpreneur's net revenue (profit) is maximised.

(ii) To examine the influence of the mark-up parameter upon the pricing of the several commodities that are stored in a warehouse with an objective of maximising the supplier's net return on investment (ROI).
(iii) To study the interaction between the decision variables for the inventory system which is subjected to the influence of the marketing policies.

(iv) To examine the aggregation approach to the respective models considered, so that the proposed models can have a wider range of applications in the actual life.

(v) To analyse the inventory systems under the given restrictions about the limited floor space, budgetary constraint or order constraints etc. with an objective of maximising the supplier's net ROI.

(vi) To carry out the total and partial sensitivity analysis for the inventory models, so that the role of more effective and influential parameters can be highlighted.