CHAPTER I

INTRODUCTION

1.1. The present work has for its main object, the study of certain problems regarding the summability of double Fourier series and some allied topics.

Let

\[
(1.1.1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \sum_{n=1}^{\infty} A_n(x),
\]

where

\[
\begin{align*}
    a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \\
    b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt,
\end{align*}
\]

be the Fourier series associated with the function \( f(x) \) which is integrable in the sense of Lebesgue over the interval \((-\pi, \pi)\) and is defined outside this range by periodicity with a period \(2\pi\).

We denote by

\[
\sum_{n=-\infty}^{\infty} f_n e^{ni\theta},
\]

the complex Fourier series of \( f(\theta) \) where

\[
f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-nit} \, dt.
\]

The series

\[
(1.1.2) \quad \sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right) = \sum_{n=1}^{\infty} B_n(x)
\]

is called the 'allied series' or the 'conjugate series' of the
Fourier series (1.1.1). The Fourier series and the conjugate series form respectively the real and imaginary part of the power series,
\[ \left\{ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n - i b_n) z^n \right\} \]
where \( z = e^{ix} \). It is known that the conjugate series of a Fourier series is not necessarily a Fourier series.

Let
\[ \Phi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) - 2s \right\} , \]
\[ \Psi(t) = \frac{1}{2} \left\{ f(x+t) - f(x-t) \right\} . \]

The conjugate function associated with the conjugate series (1.1.2) is
\[ (1.1.3) \quad \frac{1}{W} \int_{-W}^{W} \Psi(t) c(t) e^{-it} dt , \]
the integral being Cauchy's generalised integral at the origin. It is known\(^2\) that the conjugate function of an integrable function exists almost everywhere.

1.2 Just as a function of one variable is associated with an ordinary Fourier series, similarly a function of two variables may be associated with a double Fourier series.

1: E.C. Titchmarsh (78); W.H. Young (87).
2: G.H. Hardy and W.W. Rogosinsky (16); G. Sansone (56).
The double Fourier series, associated with the function \( f(x, y) \), which is integrable in the Lebesgue sense over the square \( Q(-\pi, -\pi \ ; \ \pi, \pi) \) and is doubly periodic with period \( 2\pi \) in each variable, is given by

\[
\sum_{m,n=0}^{\infty} \lambda_{m,n} \left( a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny + c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny \right)
\]

\[
= \sum_{\alpha, \beta, \gamma, \delta} (\alpha, \beta, \gamma, \delta; x, y)_{m,n}^{3} = \sum_{m,n} A_{m,n}(x, y)
\]

where

\[
\lambda_{m,n} = \begin{cases} 
\frac{1}{4} & \text{for } m = 0, n = 0 \\
\frac{1}{2} & \text{for } m = 0, n > 0; m > 0, n = 0 \\
1 & \text{for } m > 0, n > 0
\end{cases}
\]

The coefficients are given by the formula

\[
a_{m,n} = \frac{1}{\pi^2} \iint_{Q} f(x, y) \cos mx \cos ny \, dx \, dy,
\]

e tc., obtained by term by term integration as in any ordinary Fourier series.

It is clear that the formal expression of the series \((1.2.1)\) may be obtained by expressing \( f(x, y) \) as a single Fourier series of cosines and sines of multiple of \( y \) and then

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3: G.M. Merriman (34); E.W. Hobson (20)
each coefficients in that series as a Fourier series of cosines
and sines of multiples of x.

We also denote by

\[
\sum_{m,n=-\infty}^{\infty} C_{m,n} e^{i(mx+ny)}
\]

the complex form of Fourier series of \( f(x,y) \), where

\[
L C_{m,n} = \frac{1}{\pi^2} \iint \! f(u,v) e^{i(mu+nv)} \, du \, dv.
\]

In the case of two variables there are three allied series or
conjugate series obtained by differentiating the given double
Fourier series (1.2.1) once with respect to x, once with
respect to y and successively with respect to x and y and
in each case omitting the numerical coefficients thus introduced
and retaining the new signs.

Thus we have the following three allied series

\begin{align*}
(1.2.2) & \quad \sum_{m}^{\infty} \sum_{n}^{\infty} (d,-c,-b,a,x,y)_{m,n} ; \\
(1.2.3) & \quad \sum_{m}^{\infty} \sum_{n}^{\infty} (c,d,-a,-b,x,y)_{m,n} ; \\
(1.2.4) & \quad \sum_{m}^{\infty} \sum_{n}^{\infty} (b,-a,d,-c,x,y)_{m,n} .
\end{align*}

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4: G.M. Perriman (33); L. Cesari (5); P.L. Sharma (67).
These will be designated hereafter by the first, second and third allied series respectively.

We write

$$
\phi(u,v) = \frac{1}{4} \left[ f(x+u, y+v) + f(x+u, y-v) - f(x-u, y+v) + f(x-u, y-v) - 4 S \right],
$$

$$
\psi_1(u,v) = \frac{1}{4} \left[ f(x+u, y+v) - f(x-u, y-v) - f(x-u, y+v) + f(x-u, y-v) \right],
$$

$$
\psi_2(u,v) = \frac{1}{4} \left[ f(x+u, y+v) + f(x+u, y-v) - f(x-u, y+v) - f(x-u, y-v) \right],
$$

$$
\psi_3(u,v) = \frac{1}{4} \left[ f(x+u, y+v) - f(x+u, y-v) + f(x-u, y+v) - f(x-u, y-v) \right].
$$

The conjugate functions associated with the allied series (1.2.2), (1.2.3) and (1.2.4) are
\begin{align}
(1.2.5) \quad \frac{1}{\pi^2} \iint_0^\pi \int_0^\pi \psi_1(x, y) \cot \frac{x}{2} \cot \frac{y}{2} \, dx \, dy,
\end{align}

\begin{align}
(1.2.6) \quad \frac{1}{\pi^2} \iint_0^\pi \int_0^\pi \psi_2(x, y) \cot \frac{x}{2} \, dx \, dy,
\end{align}

\begin{align}
(1.2.7) \quad \iint_0^\pi \int_0^\pi \psi_3(x, y) \cot \frac{y}{2} \, dx \, dy,
\end{align}

respectively, each integral being the generalised double integral in the sense of Cauchy. It is known\(^5\) that the conjugate function of a function of two variables in the \(L^p\) class \((p>1)\) exists almost everywhere.

As we are concerned with the applications of various methods of summation, we shall state their definitions.

1.3 Toeplitz matrices. Toeplitz\(^6\) was the first to consider linear methods of summation systematically. Given an infinite matrix

\[ \Lambda = \left\{(d_{n,k})\right\}, \quad (n = 0, 1, 2, \ldots; \quad k = 0, 1, 2, \ldots) \]

the \(\Lambda\)-transforms \(t_n\) of the sequence \(\{s_k\}\) are defined by

\[ t_n = \sum_{k=0}^{\infty} d_{n,k} s_k, \]

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5: K. Sokot Sokolowski (71)
6: O. Toeplitz (79); Cooke, R.G. (8).
provided that all the series on the right converges. If \( t_n \to s \) as \( n \to \infty \), \( S_n \) is said to have the \( \Lambda \)-limit \( s \) and the sequence \( \{ S_n \} \) is said to be summable by \( \Lambda \) process to the sum \( S \). We then write \( S_n \to S (\Lambda) \). An infinite series \( \sum u_n \) with partial sums \( S_n = u_0 + u_1 + u_2 + \ldots + u_n \) is said to be summable \( (\Lambda) \) to \( S \), if \( S_n \to S (\Lambda) \). It is clear that \( S_n \to S (\Lambda) \) and \( \nu_n \to \nu (\Lambda) \) imply

\[
\Lambda S_n + B \nu_n \to \Lambda s + B \nu (\Lambda).
\]

This is a characteristic of all linear methods. Such methods are obviously an extension of the classical notion of convergence, which may be obtained from \( \Lambda \)-process by putting

\[
d_{n,k} = 0 \text{ unless } k = n \text{ and } d_{n,n} = 1.
\]

If \( S_n \to s \) simply \( S_n \to s (\Lambda) \), then the method is said to possess the property of regularity and \( \Lambda \) is called a Toeplitz matrix, or a T-matrix. The conditions on \( d_{n,k} \) which make \( \Lambda \) a regular or T-matrix are given by Silverman-Toeplitz theorem. The necessary and sufficient conditions that \( t_n = \sum_{k=0}^{\infty} d_{n,k} s_k \) should tend to a finite limit \( s \) as \( n \to \infty \), whenever \( S_n \to s \), are that

\[
R_n = \sum_{k} |d_{n,k}| < R
\]

where \( R \) is independent of \( n \),

\[
d_{n,k} \to 0
\]

when \( n \to \infty \), for every \( k \); and

\[
\gamma_n = \sum_k d_{n,k} \to 1
\]
These conditions are necessary in the sense, that, if they are not satisfied, there are sequences \( \{ s_n \} \) convergent to \( S \), but not having \( \wedge \)-limit \( S \).

The sufficiency of the conditions was first established for triangular matrices by Silverman\(^7\); the necessity and sufficiency of the conditions were then proved for row-infinite matrices by Toeplitz\(^8\); and finally the theorem was published in full generality for general infinite matrices by Schur\(^9\).

R.P. Agnew\(^10\) has extended this method for the summability of double sequences. Given two infinite matrices \( \{(a_{mi})\} \) and \( \{(b_{nj})\} \), the T-transforms \( t_{m,n} \) of the sequence \( \{s_{ij}\} \) are defined by

\[
(1.3.5) \quad t_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi} b_{nj} s_{ij}.
\]

The sequence \( \{s_{ij}\} \) is said to be 'summable T' to \( S \) if its transform \( t_{m,n} \) converges to \( S \). The regularity conditions are:

\[
(1.3.6) \quad \text{for each } i, j, \quad \lim_{m \to \infty} a_{mi} = 0, \quad \lim_{n \to \infty} b_{nj} = 0,
\]

\(7\): L.L. Silverman (62)

\(8\): O. Toeplitz (79)

\(9\): I. Schur (53)

\(10\): R.P. Agnew (1)
(1.3.7) for each \( m, n, \sum_{i=0}^{m} |a_{mi}| < K; \sum_{j=0}^{n} |b_{nj}| < K, \)

\( K \) being constant independently of \( m \) and \( n \), and

\[
\lim_{m,n \to \infty} C_{m,n} = 1 \quad \text{where} \quad C_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mi} b_{nj}.
\]

**Examples of T. Methods**

**The Nörlund matrix:** This is the lower triangular matrix

\[
a_{mi} = \frac{p^{m-i}}{p_m}, \quad b_{nj} = \frac{p^{n-j}}{p_n},
\]

for \( 0 \leq i \leq m, 0 \leq j \leq n; \)

\[
a_{mi} = 0 \quad (i > m), \quad b_{nj} = 0 \quad (j > n),
\]

and

\[
P_v = p_0 + \sum_{k=1}^{v} p_k, \quad \text{for} \quad v = m \quad \text{and} \quad v = n.
\]

The conditions of regularity\(^{11}\) for Nörlund summability are

\[
(1.3.9) \quad \sum_{v=0}^{\tau} |P_v| = O(1) \quad \text{and} \quad \frac{P_{\tau}}{P_T} \to 0 \quad \text{as} \quad \tau \to \infty.
\]

The third condition of regularity is automatically satisfied.

Although named after M. Nörlund, the method was first

\[11\]: C. N. Moore (43) and (44).
given by G.F.Woronoï\textsuperscript{12}) in 1901. His results having been published in a rare Russian journal, did not attract attention for a long time. Except for a short review Woronoï's work remained practically obscure until Tamarékin\textsuperscript{13}) called attention to it in 1932. In the meanwhile N.E. Nörlund\textsuperscript{14)} rediscovered the same method independently in 1919.

If we put
\[ P_m = \frac{c_1(\alpha+1) \cdots (\alpha+m-1)}{m!} \quad (\alpha > -1) \]
so that
\[ P_m = \frac{(\alpha+1)(\alpha+2)(\alpha+3) \cdots (\alpha+m)}{m!} \]
and
\[ P_n = \frac{\beta(\beta+1) \cdots (\beta+n)}{n!} \]
\[ P_n = \frac{(\beta+1)(\beta+2) \cdots (\beta+n)}{n!} \]

we obtain the well known Césaro method \((C, \alpha, \beta)\) as a special case. If we put \(P_n = \frac{1}{n+1}\), then \(P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1} \sim \log n\), the Nörlund summability reduces to Harmonic summability. This method was introduced by M. Riesz\textsuperscript{15}) in 1924 who proved that an infinite series which is summable by harmonic means to the sum \(S\) is also summable \((C, k)\) to the same sum for every positive \(k\).

\textsuperscript{12)} G.F.Woronoï (34)
\textsuperscript{13)} G.F.Woronoï; J.D.Tamarékin (35)
\textsuperscript{14)} N.E.Nörlund (47)
\textsuperscript{15)} M.Riesz (50)
1.4. We define below some more methods of summability and absolute summability which will be used in subsequent chapters.

1.4.1. Borel summability\(^{16}\).

The series \( \sum_{m,n=0}^{\infty} a_{m,n} \) with the sequence of partial sums \( \{ s_{m,n} \} \) is said to be summable by Borel's exponential method or summable \( B \), if

\[
\mathcal{F}_{p,q} = \lim_{p \to \infty} \lim_{q \to \infty} e^{- (p+q)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{p^m}{m!} \frac{q^n}{n!} s_{m,n}
\]

exists and has a finite value \( S \).

1.4.2. (L) summability:

The series \( \sum_{m,n=0}^{\infty} a_{m,n} \) with the sequence of partial sum \( \{ s_{m,n} \} \) is said to be summable (L) if there exists a constant \( S \) such that the expression

\[
\mathcal{T}_{m,n} = (\log \frac{1}{1-x})^{-1} (\log \frac{1}{1-y})^{-1} \sum_{m,n=1}^{\infty} \frac{s_{m,n}}{m^n n^m} x^m y^n
\]

has a double limit zero as \( x \to +1, y \to +1 \) where \( 0 < x < 1 \) and \( 0 < y < 1 \). This extends the definition of (L) summability for single series given by D. Borwein\(^{17}\) to double series.

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16: A.A. Melencov and E.S. Khurev (31); compare G.H. Hardy (14) for the definition for single series.

17: D. Borwein (3).
1.4.3. **Absolute Cesàro summability**\(^{13}\):

A double series \( \sum_{m,n=1}^{\infty} a_{m,n} \) is said to be absolutely summable \((C, \alpha, \beta)\) or summable \(|C, \alpha, \beta|\), \((\alpha, \beta > 1)\) when

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \sigma_{m,n}^{\alpha,\beta} - \sigma_{m-1,n}^{\alpha,\beta} - \sigma_{m,n-1}^{\alpha,\beta} + \sigma_{m-1,n-1}^{\alpha,\beta} \right| < \infty,
\]

\[
\sum_{m=1}^{\infty} \left| \sigma_{m,0}^{\alpha,\beta} - \sigma_{m-1,0}^{\alpha,\beta} \right| < \infty, \quad \sum_{n=1}^{\infty} \left| \sigma_{0,n}^{\alpha,\beta} - \sigma_{0,n-1}^{\alpha,\beta} \right| < \infty,
\]

where \( \sigma_{m,n}^{\alpha,\beta} \) is \((C, \alpha, \beta)\)-mean of \( a_{m,n} \).

The consistency theorem for absolute Cesàro summability is known.

1.4.4 **Absolute summability by Borel's integral method**:

A double series \( \sum_{m,n=0}^{\infty} a_{m,n} \) is said to be absolutely summable by Borel's integral method or summable \(|B|\) if

\[
\int_0^{\infty} \int_0^{\infty} e^{-(x+y)} \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} \frac{x^m}{m!} \frac{y^n}{n!} \right| \, dx \, dy < \infty,
\]

\[
\int_0^{\infty} e^{-x} \left| \sum_{m=0}^{\infty} a_{m,0} \frac{x^m}{m!} \right| \, dx < \infty, \quad \int_0^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} a_{0,n} \frac{y^n}{n!} \right| < \infty.
\]

1.5. Various aspects of theory of double Fourier series have been considered by a number of writers. Those respects in

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13: N.F. Timan (77).

19: I. E. Zak (38); I. E. Zak and N. F. Timan (91).
which multiple Fourier series differ from ordinary Fourier series are sufficiently represented by the case of double Fourier series and hence we shall confine ourselves to the study of this type of series only.

The convergence of the series of a function of bounded variation was discussed by Hardy. He proved that, if a function \( f(x,y) \) is of bounded variation in Hardy's sense, then the series associated with the function is a convergent double series in Pringsheim sense, i.e., the \((m,n)\)-th. partial sum of the double Fourier series \( S_{mn} \) tends to a limit when \( m,n \) tends to infinity independently but simultaneously; and the connection between the sum of the series and the function is the same as that in the case of ordinary Fourier series of a function of bounded variation.

The general theory of the convergence and summability of a double Fourier series has also been discussed by Young and Küstermann. In 1913, in connection with the study of summation by arithmetic means of the double Fourier series corresponding to functions having discontinuities along a curve, Moore was led to the introduction of the notion of restricted summability of a double series. This differs from summability in the general sense in that the indices of the sequence whose limit is involved, become infinite in such a manner that

20: G.H.Hardy (12)
21: W.H.Young (86)
22: W.W.Küstermann (27)
23: C.N.Moore (41), (42)
their ratios remain bounded by two arbitrary positive constants. Moore's method of summability was also used by Miller and Odams \(^{24}\) in the investigation of positive order summability of double Fourier series.

Corresponding to the classical tests for convergence of ordinary Fourier series, tests for Pringsheim convergence of the double Fourier series have been given by a number of writers. A main point of difference in which double, or multiple, Fourier series differ from ordinary Fourier series is the fact that the behaviour of the former, as regards convergence, divergence, or oscillations, at a point, does not, as in the latter case, depend only on the nature of the function in a neighbourhood of the point, but upon its nature in a cross-neighbourhood of the point.

By using directly or indirectly the concept of the Vitali variation either of the function or of an indefinite integral Hardy\(^{25}\), Young\(^{26}\), Tonelli\(^{27}\), Gergen\(^{28}\) and Herriman\(^{29}\) have given test for the Pringsheim convergence of the double Fourier series of an integrable even even function \(f(u,v)\) and period \(2\pi\) in each variable.

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24: M.L. Miller and A. Odams (35)
25: G.H. Hardy (12)
26: W.H. Young (86)
27: L. Tonelli (30)
28: J.J. Gergen (11)
29: G.H. Herriman (36)
The following example shows the nature of extension of the result of ordinary Fourier series to double Fourier series. Corresponding to the well known theorem of Fejér-Lebesgue for ordinary Fourier series we have for double Fourier series the following result:

**Theorem:** If \( \int \log^+|f| \) is Lebesgue integrable on the square \((-\pi, -\pi; \pi, \pi)\) then the Fejér mean \( S_{m,n}(x,y) \) of \( f(x,y) \) tends to \( f(x,y) \) almost everywhere as \( m \) and \( n \) independently increase indefinitely. Moreover, for every increasing function \( \phi(t) \) satisfying the conditions

\[
\phi(0) = 0, \quad \lim_{t \to \infty} \inf \frac{\phi(t)}{t \log^+ t} = 0
\]

there is a function \( f(x,y) \) such that \( \phi(|f|) \) is integrable and that \( S_{m,n}(x,y) \) does not converge almost everywhere 30).

The latter half of this theorem shows that the analogue, in double Fourier series, of a Fejér Lebesgue theorem is not a trivial extension of that of a function of a single variable.

This dissertation contains twelve chapters, seven chapters are on the summability of double Fourier series and its allied series by Cesáro, Borel, Mörlund, logarithmic type, absolute Cesáro and absolute Borel summabilities. In the last four chapters some problems on matrix, harmonic and logarithmic type of summability of the derived Fourier series and allied series.

30: B. Jessen, J. Marcinkiewicz and Zygmund (26); S. Saks (55).
have been discussed.

1.6. Chapter II is entitled "On the Nörlund summability of Double Fourier series". The application of Nörlund summation to Fourier series has been discussed by Hill and Tamarkin \(^{31}\), Astrachan \(^{32}\), Sahney \(^{33}\) and Singh \(^{34}\).

The definition of Nörlund summability was extended for the first time to double series by Moore \(^{35}\). The application of Nörlund summation to double Fourier series has been discussed by Herriot \(^{36}\), who has considered the restricted double Nörlund summability of the rectangular partial sums of the double Fourier series. We shall prove a general theorem without using any restriction of the type used by Herriot.

Theorem: If

\[
\Phi(u,v) = \int_0^u \int_0^v |\phi(s,t)| \, dt = O\left[ \frac{uv}{\log^\frac{1}{u} \log^\frac{1}{v}} \right],
\]

\[
\phi_i(u) = \int_0^\pi \int_0^u |\phi(s,t)| \, ds \bigg| = O\left[ \frac{u}{\log^\frac{1}{u}} \right],
\]

\[
\phi_i(v) = \int_0^\pi \int_0^v |\phi(s,t)| \, dt \bigg| = O\left[ \frac{v}{\log^\frac{1}{v}} \right],
\]

\(^{31}\) E.Hill and J.D.Tamarkin (19)

\(^{32}\) M.Astrachan (2)

\(^{33}\) B.N.Sahney (54)

\(^{34}\) T.Singh (63)

\(^{35}\) C.N.Moore (44)

\(^{36}\) J.G.Herriot (18)
then the double Fourier series of the function \( f(u,v) \) is summable by \((N, p_n, p_m)\) to \(S\), where \( \{p_n\} \) and \( \{p_m\} \) are both nonnegative and nonincreasing sequences, such that

\[
\sum_\alpha \frac{P_\alpha}{\nu \log \nu} = O(P_\xi), \text{ for } \xi = m, n,
\]

and \( \alpha \) is a fixed positive integer.

If we put \( p_n = \frac{1}{n^r} \) and \( p_m = \frac{1}{m^r} \), we get the result of Sharma\(^{37}\), on the harmonic summability of double Fourier series.

1.7. In chapter III, "On the Borel summability of double Fourier series and its allied series", we extend the theorems of Sahney\(^{38}\) for double series. Borel summability of ordinary Fourier series was investigated by Hardy and Littlewood\(^{39}\), Stone\(^{40}\), Takahashi and F.T. Wang\(^{41}\) and recently by Sahney\(^{42}\). Also Sinvhal\(^{43}\) has discussed the Borel summability of the conjugate series of the derived Fourier series.

We prove the following theorems:

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37: P.L. Sharma (66)  
38: B.N. Sahney (51)  
39: G.H. Hardy and J.E. Littlewood (15)  
40: M.H. Stone (72)  
41: T. Takahashi and F.T. Wang (76)  
42: B.N. Sahney (51)  
43: S.D. Sinvhal (65)
Theorem 1: If, as $u \to +\infty$, $v \to +\infty$,

$$\Phi(u, v) = \int_0^u \int_0^v |\phi(s, t)| dt = o\left(\frac{uv}{\log u \log v}\right),$$

$$\int_0^{\Pi} dt \int_0^u |\phi(s, t)| ds = o\left(\frac{u}{\log u}\right),$$

$$\int_0^v d\sigma \int_0^v |\phi(s, t)| dt = o\left(\frac{v}{\log v}\right),$$

then the double Fourier series of function $f(u, v)$ is summable by Borel exponential mean to the sum $S$.

Theorem 2: If, as $u \to +\infty$, $v \to +\infty$,

$$\Psi(u, v) = \int_0^u \int_0^v |\psi(s, t)| dt = o\left(\frac{uv}{\log u \log v}\right),$$

$$\int_0^{\Pi} dt \int_0^u |\psi(s, t)| ds = o\left(\frac{u}{\log u}\right);$$

$$\int_0^v d\sigma \int_0^v |\psi(s, t)| dt = o\left(\frac{v}{\log v}\right),$$

then the first allied series is summable by Borel exponential mean to the value of the conjugate integral (1.2.5), provided it exists.

Similar theorems can also be proved for the second and third allied series.
1.3. Chapter IV is entitled "On the summability of double series by Borel's absolute integral method". We prove the theorem:

Theorem: If each of

$$\frac{\partial^2}{\partial u \partial v} g(u,v), \quad \frac{\partial}{\partial u} g(u,v), \quad \frac{\partial}{\partial v} g(u,v),$$

if finite and is of constant sign in the rectangle \((0,\pi; 0,\pi)\), then the double Fourier series (1.2.1) is summable \(1B1\).

This extends a recent result of Mohanty\(^{44}\) on the summability of Fourier series by Borel's absolute integral method, to double Fourier series.

1.9. In chapter V, we discuss the summability (L) of double Fourier series. In a recent paper, Borwein\(^ {45}\) has constructed a new method of summability for an infinite sequence \(\{s_n\}\) which is known as (L) summability. Concerning this kind of summability, Borwein has established a number of fundamental facts. Thus we have the following full inclusive relation:

\[(L) \supset (A) \supset (C, \tau),\]

for \(\tau > -1\) where (A) is the ordinary Abel's summability and is the Cesàro summability of order \(\tau\). K.Ishiguro\(^ {46}\)

\[44: \text{R. Mohanty (37)}\]
\[45: \text{D. Borwein (3)}\]
\[46: \text{K. Ishiguro (24)}\]
proved that if \( \{a_n\} \) is summable by Riesz logarithmic mean of order one, it is also summable (L) to the same sum, but the converse is not true. Further he\(^{47}\) gave a Tauberian theorem concerning this method of summability.

\(^{48}\) Hsiang applied this new method of summability to the Fourier series of \( f(x) \) in order to obtain a corresponding summability criterion for it and proved the following theorem:

**Theorem H:** If

\[
(1.9.1) \quad \int_0^t |\phi(u)| \, du = o\left( t \log \frac{1}{t} \right), \text{ as } t \to 0,
\]

\[
(1.9.2) \quad \int_t^\delta \frac{|\phi(u)|}{u} \, du = o\left( \log \frac{1}{t} \right), \text{ as } t \to +0,
\]

for any arbitrary \( 0 < \delta < \pi \) then the Fourier series of \( f(t) \) is summable (L) to \( S \) at \( \theta \).

The condition (1.9.1) is implied by (1.9.2) and is therefore superfluous.

We shall prove the theorem.

Theorem: If, as \( u \to 0, \nu \to 0 \),

\[47:\quad \text{K. Ishiguro (25)}\]

\[48:\quad \text{F. C. Hsiang (23)}\]

\[49:\quad \text{M. L. Misra (36)}\]
\[ \int_0^\infty \int_0^\infty \frac{\phi(s,t) ds dt}{st} = o\left( \log \frac{1}{u}, \log \frac{1}{v} \right), \]

\[ \int_0^\infty \left| \int_0^\infty \frac{\phi(s,t)}{s} ds \right| dt = o\left( \log \frac{1}{u} \right); \quad \int_0^\infty \left| \int_0^\infty \frac{\phi(s,t)}{t} dt \right| ds = o\left( \log \frac{1}{v} \right), \]

then the double Fourier series of the function \( f(u,v) \) is summable \( (L) \) to sum \( S \) at \( u = x, \quad v = y \).

This theorem generalizes theorem \( H \) for double Fourier series.

1.10. Chapter VI is entitled "The absolute Cesàro summability of double Fourier series". M.F. Timan \( 50A \) has defined this summability for double series and extended the result of Bosanquet \( 50B \) to double Fourier series. We prove the following theorem:

**Theorem:** If the series

\[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} P_{m,n} \left( \log m, \log n \right)^{1+\varepsilon}, \quad \varepsilon > 0 \]

is convergent, then the series

\[ \sum_{m,n} A_{m,n}(x,y), \]

is summable \( \left| c, \alpha, p \right| \quad (\alpha, p > \frac{1}{2}) \) almost everywhere.

1.11. In chapter VII we prove the following theorem:

**Theorem:** If

\[ \tilde{F}(u,v) = \int_0^u \int_0^v \phi(s,t) ds dt = o\left( \frac{uv}{\log \frac{1}{u} \log \frac{1}{v}} \right) \]

\[ 50A: \quad \text{M.F. Timan (77)} \]

\[ 50B: \quad \text{L.S. Bosanquet (4)} \]
\[ \int_0^\pi \int_0^u \phi(s,t) \, ds \, dt = o\left( \frac{u}{\log u} \right); \quad \int_0^\pi \int_0^v \phi(s,t) \, dt \, ds = o\left( \frac{v}{\log v} \right), \]

then

\[ S_{m,n} = o\left( \log \log m \cdot \log \log n \right) \]

where \( S_{m,n} \) is the \((mn)\)th partial sum of the double Fourier series.

This extends a result of Sőszas 51).

1.12. Chapter VIII is entitled "On the determination of the jump of the function associated with its double Fourier series". If there exists \( \phi(x,y) 52) \) such that

\[ y'(u,v) = f(x+u,y+v) - f(x+u,y-v) \]

\[ - f(x-u,y+v) + f(x-u,y-v) - \phi(x,y), \]

as \( u,v \to 0 \),

where

\[ \phi(x,y) = f(x+0,y+0) - f(x+0,y-0) - f(x-0,y+0) + f(x-0,y-0) \]

then \( \phi(x,y) \) is called the jump of the function \( f(x,y) \) at the point \((x,y)\). Even when (1.12.1) does not hold, \( \phi(x,y) \) may satisfy either the relation

51: O. Szász (75)

52: We have defined \( \phi(x,y) \) for first allied series.

See also Taneresi, Fabio (23).
(1.12.2) \[ \psi(u,v) = \int_0^u \int_0^v |\psi(u,v)| \, du \, dv = o(uv), \]

or still more generally, the relation

(1.12.3) \[ \psi(u,v) = \int_0^u \int_0^v \psi(u,v) \, du \, dv = o(uv). \]

In each of the latter cases, \( f(x,y) \) is said to have a generalised jump \( \phi(x,y) \) at the point \( (x,y) \).

Broadly speaking there are two different methods that have been adopted by different writers in their investigations for the determination of the jump of a function of a single variable in the theory of ordinary Fourier series. The first method\(^{53} \) deals with the determination of the jump by considering the summability of the sequence \( \{ \cap B_{n}(x) \} \). Subsequently Szasz\(^{54} \) devised a new method for the determination of the jump. He considered the difference of the means of the two different orders of the partial sums of the corresponding conjugate series instead of the summability of the sequence \( \{ \cap B_{n}(x) \} \). His line of research has been pursued further by a number of writers\(^{55} \). Chow\(^{56} \) has also considered the relationship between the two types of methods of determination of jump.

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53: O. Szasz (73)
54: O. Szasz (74)
55: M.L. Misra (36); H.C. Chow (6)
56: H.C. Chow (7)
We prove the following theorem:

**Theorem:** If \( 0 < \alpha < 1 \), \( 0 < \beta < 1 \) and \( u, v \to 0 \),

\[
\Theta(u, v) = \int_0^u \int_0^v \Theta_i(s, t) \, ds \, dt = o(u^{\alpha} v^{\beta}),
\]

\[
\int_0^u \left| \int_0^v \Theta_i(s, t) \, ds \right| = O(u^{\alpha}) ; \int_0^v \left| \int_0^u \Theta_i(s, t) \, dt \right| = O(v^{\beta}),
\]

then

\[
\lim_{m,n \to \infty} \left[ \Theta_{2m,2n}^{\alpha,\beta} - \Theta_{m,n}^{\alpha,\beta} - \Theta_{2m,2n}^{\alpha,\beta} + \Theta_{m,n}^{\alpha,\beta} \right] = \frac{(\log 2)^2}{4\pi^2} \phi(x,y)
\]

where

\[
\Theta_i(u, v) = 4 \gamma_i(u, v) - \phi(x,y),
\]

and \( \Theta_{m,n}^{\alpha,\beta} \) is the \((m,n)\)th Cesàro mean of order \((\alpha,\beta)\) of the first allied series.

1.13. Chapter IX is entitled "On the matrix summability of the derived Fourier series and its conjugate series". The series

\[(1.13.1) \quad \sum_{n=1}^{\infty} n (b_n \cos(nt) - a_n \sin(nt)) = \sum_{n=1}^{\infty} n B_n(t)\]

which is obtained by differentiating \((1.1.1)\) formally term by term, is called the first derived series of \( f(t) \). The series \((1.13.1)\) will, however, be called the derived Fourier series. The series conjugate to \((1.13.1)\) is

\[(1.13.2) \quad \sum_{n=1}^{\infty} n (a_n \cos nt + b_n \sin nt).\]
We write
\[ \phi(t) = f(x+t) + f(x-t) - 2f(x) \]
\[ \psi(t) = f(x+t) - f(x-t) \]
\[ \Delta \lambda_{n,v} = \lambda_{n,v} - \lambda_{n,v+1} \]
\[ \Delta^2 \lambda_{n,v} = \Delta \lambda_{n,v} - \Delta \lambda_{n,v+1} \]
\[ h(t) = \phi(t) \frac{1}{4 \sin \frac{1}{2} t} \]
\[ g(t) = \psi(t) \frac{-c}{4 \sin \frac{1}{2} t} \]
and
\[ \mu(t) = 2h(t) \sec \frac{1}{2} t \]
where \( c \) is a function of \( x \).

Fejér\(^{57}\) has proved that if \( f(x) \) is of bounded variation in \((0, 2\pi)\), then the sequence \( \{nB_n(x)\}\) is summable \((C, r)\) to the jump
\[ \ell(x) = \left\{ f(x+) - f(x-) \right\} / \pi \]
for every \( r > 0 \) at each point of \( x \). Further Siddiqui\(^{58}\) proved

\(^{57}\) L. Fejér (10)

\(^{58}\) J.A. Siddiqui (59)
that \( \{ n B_n(x) \} \) sequence is summable \((N, p_n)\) to the jump \( \ell(x) \) provided
\[
\sum_{k=1}^{n} k |\Delta k| = O(P_n), \quad |P_n| \to \infty \quad \text{as} \quad n \to \infty,
\]
and \( f(x) \) is of bounded variation.

These results were extended for the triangular Toeplitz matrix by Siddiqui \(^{59}\) in the following theorem:

**Theorem S** Let \( \Lambda: (\lambda_{n,v}) \) be a triangular Toeplitz matrix, i.e. \( \lambda_{n,v} = 0 \) for \( v > n \). If the function is of bounded variation and in addition it satisfies the condition
\[
(1.13.3) \quad \sum_{v=0}^{n} |\Delta \lambda_{n,v}| = o(1),
\]
then the sequence \( \{ n B_n(x) \} \) is summable \((A)\) to the sum \( \ell(x) \).

Further Siddiqui \(^{60}\) has given a necessary and sufficient condition on \( \Lambda \) for the validity of the theorem \( S \) and derived certain consequences for the Fourier coefficients of continuous functions of bounded variation. Recently F. Hsiang \(^{61}\) has replaced condition \((1.13.3)\) by a more general condition,
\[
(1.3.4) \quad \sum_{v=0}^{n} |\Delta^2 \lambda_{n,v}| = o(1).
\]

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59: J.A. Siddiqui \((60)\)
60: J.A. Siddiqui \((61)\)
61: F.C. Hsiang \((21)\)
Hsiang \(^{62}\) has further developed theorem \(\Sigma\) in a more general form for the general infinite Toeplitz matrix method of summability or simply \((T)\) method.

Varshney \(^{63}\) very recently considered the \(\Lambda\) summability of the derived Fourier series. He proved the following theorem:

**Theorem \(V\)**: If \(g(t)\) is of bounded variation in \((0, \pi)\) and \(g(t) \rightarrow 0\) as \(t \rightarrow 0\), then the series \(\sum \theta B_n(x)\) is summable\((\Lambda)\) to the value \(C\), provided that \(\Lambda\) is regular and

\[
(1.13.5) \quad \sum_{v=0}^{n} |\triangle \lambda_{n,v}| = o(1)
\]

We shall extend the theorem \(V\) in two directions. First we shall develop it for the general summability \((T)\) of the derived Fourier series, and secondly, we shall replace \((1.13.5)\) by a less stringent condition,

\[
(1.13.6) \quad \sum_{v=0}^{n} |\triangle^{2} \lambda_{n,v}| = o(1)
\]

The condition \((1.13.6)\) is more general than \((1.13.5)\), since

\[
\sum_{v=0}^{n} |\triangle^{2} \lambda_{n,v}| = \sum_{v=0}^{n} |\triangle \lambda_{n,v} - \triangle \lambda_{n,v+1}|
\]

\[
\leq 2 \sum_{v=0}^{n} |\triangle \lambda_{n,v}|.
\]

Analogous theorems for conjugate derived series are also proved.

We shall prove the following theorems:

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\(^{62}\): F.C. Hsiang (32)

\(^{63}\): O.P. Varshney (32)
Theorem 1: If \( q(t) \) is of bounded variation in the neighbourhood of \( t = 0 \) and absolutely continuous in \((\eta, \pi)\) for any \( 0 < \eta < \pi \), then the derived Fourier series (1.13.1) is summable \((\text{T})\) to the value \( C \).

Theorem 2: If \( h(t) \) is of bounded variation in the neighbourhood of \( t = 0 \) and absolutely continuous in \((\eta, \pi)\) for any \( 0 < \eta < \pi \), then the series (1.13.2) is summable \((\text{T})\) to the value of the integral

\[
-\frac{1}{\pi} \int_{0}^{\pi} h(t) \cos \frac{t}{2} \, dt
\]

at a point \( x \), where the integral exists in the Cauchy sense.

Theorem 3: If \( q(t) \) is of bounded variation in \((0, \pi)\) and \( q(t) \to 0 \) as \( t \to 0 \), then the series \( \sum \eta B_n(x) \) is summable \((\text{A})\) to the value \( C \), provided that (1.13.7)

(1.13.7) \( \sum_{n=0}^{\infty} |\Delta^2 \lambda_{n,\nu}| = o(1) \).

Theorem 4: If \( h(t) \) is of bounded variation in \((0, \pi)\) and \( h(t) \to 0 \) as \( t \to 0 \), then the series \( \sum \eta A_n(x) \) is summable\((\text{A})\) to the value of the integral

\[
-\frac{1}{\pi} \int_{0}^{\pi} h(t) \cos \frac{t}{2} \, dt
\]

at a point \( x \), where the integral exists in the Cauchy sense, provided that (1.13.7) holds.

1.14. The title of chapter 6 is "On the sequence of Fourier coefficients". Mohanty and Nanda \(^{64}\) have proved that if,

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\(^{64}\) R. Mohanty and M. Nanda (38)
(1.14.1) \( f(x+t) - f(x-t) - t = \psi(t) = o\left(\left(\log t\right)^\gamma\right), a_5 t \to 0 \),
and \( a_n \) and \( b_n \) are \( O(n^\gamma), 0 < \gamma < 1 \), then the sequence \( \{nB_n(\infty)\} \)
is summable \((C,1)\) to the value \( \frac{t_n}{n} \).

Further Singh \( (65) \) has proved the same result under Lebesgue's criterion. It is well known \( (66) \) that Lebesgue's
test includes the Jordan's test, thus Singh's theorem includes the result of Fejér. Varshney \( (67) \) has defined the \((H,1),(C,1)\) method of summability by superimposing the harmonic summability method \((H,1)\) on the Cesàro mean of order one and has studied the summability \((H,1),(C,1)\) of the sequence \( \{nB_n(\infty)\} \).
From this he deduces the above result of Mohanty and Nanda using the well known Tauberian theorem associated with harmonic summability.

Very recently Sharma \( (68) \) generalised the result of Varshney and proved the following theorem:

**Theorem 5**: If

(1.14.2) \( \psi(t) = o\left(\left(\log t\right)^\gamma\right), a_5 t \to 0 \)

and for some \( \gamma \) with \( 0 < \gamma < 1 \),

(1.14.3) \( \lim_{n \to \infty} \sum n^\gamma |a_{n,m} - a_{n+1,m}| = 0 \)

then the sequence \( \{nB_n(\infty)\} \) is summable \((A,C,1)\) to the

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65: Basudeo Singh (64)
66: G. Sansone (57)
67: O.P. Varshney (81)
63: P.L. Sharma (70)
sum \( \ell/\Pi \), where \( A = (a_{n,m}) \) is row infinite matrix method of summability.

Our object here is to generalize theorem 5 by replacing both the conditions (1.14.2) and (1.14.3) by weaker ones. In fact we prove the theorem:

**Theorem:** If
\[
\psi(t) = \int_0^t \psi(u)\,du = o\left(\frac{t}{\log t}\right), \text{ as } t \to 0
\]
and for some \( r \) with \( 0 < r < 1 \),
\[
\sum n^r |a_{n,m} - a_{n+1,m}| = O(1)
\]
then the sequence \( \{ nB_n(x) \} \) is summable \( A: (c,1) \) to the sum \( \ell/\Pi \).

1.15. In chapter IV we study the harmonic summability of a series associated with Fourier series. We shall be concerned with the series

\[
(1.15.1) \quad \sum_{\gamma} \frac{S_{n-S}}{n}
\]

where

\[
(1.15.2) \quad S_n = \sum_{i} A_{\mu},
\]

\( S \) is constant and \( \sum A_{\mu} \) is the Fourier series.

We shall observe that the relation of (1.15.1) to the integral
\[
\int_0^\pi \frac{f(t)}{2} \csc \frac{t}{2} \, dt
\]
is very similar to that between the conjugate series and the conjugate integral.
1.16. In chapter XII we shall study the 'L' summability of a series associated with the derived Fourier series. We are concerned with the series

\[(1.16.1) \quad \sum_{n=1}^{\infty} \frac{S_n(x)}{\eta}\]

where

\[S_n(x) = \sum_{r=1}^{n} r B_r(x), \quad g_r(t) = \frac{f(x+t) - f(x-t)}{4 \sin \frac{x}{2}}.\]

we shall prove the following theorem:

**Theorem:** If

\[G(t) = \int_{t}^{\pi} g_\eta(u) \csc \frac{u}{2} \, du = o \left( \log \frac{1}{t} \right)\]

and

\[\int_{t}^{\pi} \frac{G(u)}{u} \, du = o \left( \log \frac{1}{t} \right)\]

then the series \((1.16.1)\) is summable \((L)\).