CHAPTER XII

A NOTE ON THE (L) SUM ABILITY OF A SERIES ASSOCIATED
WITH THE DERIVED FOURIER SERIES

12.1. Let \( f(\theta) \) be integrable \( L \) in \((-\pi, \pi)\) and periodic with period \( 2\pi \). Let

\[
(12.1.1) \quad f(\theta) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(\theta)\sin n\theta.
\]

The series conjugate to (12.1.1) is

\[
(12.1.2) \quad \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta).
\]

and the differentiated series of (12.1.1) at some \( \theta \) is

\[
(12.1.3) \quad \sum_{n=1}^{\infty} n B_n(\theta).
\]

Let

\[
(12.1.4) \quad \psi(t) = f(\theta + t) - f(\theta - t), \quad g(t) = \frac{\psi(t)}{4 \sin \frac{1}{2} t}.
\]

We suppose throughout that \( g(t) \) is integrable \( L \) in \((-\pi, \pi)\) and defined outside by periodicity.

Definition: An infinite series \( \sum_{n=1}^{\infty} a_n \) with the sequence of partial sums \( \{S'_n\} \) is said to be summable \( (L) \) if

\[
\left( \log \frac{1}{1-x} \right) \sum_{n=1}^{\infty} \frac{S'_n}{n} x^n
\]
tends to a finite limit \( S \) as \( x \to 1 \), in the open interval
The object of this chapter is to study the 'L' summability of the series

\[ (12.1.5) \quad \sum_{n=1}^{\infty} \frac{S_n(x)}{n}, \]

where

\[ S_n(x) = \sum_{r=1}^{n} rB_r(x). \]

We shall prove the following theorem:

**Theorem:** If

\[ (12.1.6) \quad G(t) = \int_{t}^{\infty} g(u) \csc \frac{1}{2} u \, du = o(\log \frac{1}{t}) \]

and

\[ (12.1.7) \quad \int_{t}^{\infty} \frac{G(u)}{u} \, du = o(\log \frac{1}{t}) \]

then the series \((12.1.5)\) is summable \((L)\).

12.2. We require the following lemma.

**Lemma:** If \((12.1.6)\) holds, then the sequence

\[ \left\{ \int_{0}^{\pi} g(t) \csc \frac{1}{2} t \sin nt \, dt \right\}, \quad (n=1,2,\ldots) \]

is summable \((L)\).

This lemma is a restatement of a theorem due to Nanda.

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1: M. Nanda (45)
on the (L) summability of Fourier series. Here we see that (12.1.6) implies

\[ (12.2.1) \quad \int_{\mathring{t}}^{T} \frac{g(u)}{u} \, du = o(\log \frac{1}{t}). \]

3. Proof of the theorem: Let \( T_n(\theta) \) denote the \( n \)-th partial sum of the series (12.1.5), so that

\[ (12.3.1) \quad T_n(\theta) = \sum_{\mathring{t}=1}^{n} \frac{S_{\mathring{t}}(\theta)}{\mathring{t}}. \]

Now

\[
S_{\mathring{t}}(\theta) = \sum_{m=1}^{\mathring{t}} m B_m(\theta) = \frac{1}{\mathring{t}} \int_{0}^{\mathring{t}} \psi(\mathring{t}) \sum_{m=1}^{\mathring{t}} m \sin \mathring{t} m \mathring{t} dt
\]

\[ = -\frac{1}{\mathring{t}} \int_{0}^{\mathring{t}} \psi(\mathring{t}) \frac{d}{d\mathring{t}} \left\{ \frac{1}{z} + \sum_{m=1}^{\mathring{t}} \cos m \mathring{t} \right\} d\mathring{t} \]

\[ = -\frac{1}{\mathring{t}} \int_{0}^{\mathring{t}} \psi(\mathring{t}) \frac{d}{d\mathring{t}} \left\{ \frac{\sin(\mathring{t}+\frac{1}{2})}{2 \sin \frac{1}{2} \mathring{t}} \right\} d\mathring{t} \]

Hence

\[ T_n(\theta) = -\frac{1}{\mathring{t}} \int_{0}^{\mathring{t}} \psi(\mathring{t}) \frac{d}{d\mathring{t}} \left\{ \frac{1}{z} \cos \frac{1}{2} \mathring{t} n \sum_{\mathring{t}=1}^{n} \frac{\sin(\mathring{t}+\frac{1}{2})}{\mathring{t} \mathring{t}} \right\} d\mathring{t} \]

\[ = -\frac{1}{\mathring{t}} \int_{0}^{\mathring{t}} \psi(\mathring{t}) \frac{\sin(\mathring{t}+\frac{1}{2})}{\sin \frac{1}{2} \mathring{t}} d\mathring{t} + \frac{1}{\mathring{t}} \int_{0}^{\mathring{t}} \frac{g(t)}{\sin \frac{1}{2} \mathring{t}} \sum_{\mathring{t}=1}^{n} \frac{\sin \mathring{t} \mathring{t} d\mathring{t}}{\mathring{t}} \]

\[ + \frac{2}{\mathring{t}} \int_{0}^{\mathring{t}} \psi(t) \cos \frac{1}{2} \mathring{t} d\mathring{t} = T_1 + T_2 + T_3, \text{say.} \]
Plainly we see that \( T_3 \) is a constant.

Therefore

\[
\sum_{n=1}^{\infty} T_3 \cdot \frac{x^n}{n} = o\left( \log \frac{1}{1-x} \right).
\]

Also

\[
T_1 = -\frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \csc \frac{1}{2} t \sin nt \, dt + o(1).
\]

Since (12.1.6) implies (12.2.1), and therefore \( T_1 \) is summable \( L \) by the above Lemma.

Finally differentiating (12.1.6) with respect to \( t \) we have

\[
g'(t) = -g(t) \csc \frac{1}{2} t \]

and therefore

\[
T_2 = -\frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sum_{n=1}^{\infty} \frac{\sin nt}{n} \, dt
\]

\[
= -\frac{1}{\pi} \left[ g(t) \sum_{n=1}^{\infty} \frac{\sin nt}{n} \right]_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \frac{d}{dt} \left\{ \sum_{n=1}^{\infty} \frac{\sin nt}{n} \right\} \, dt.
\]

Since

\[
\sum_{n=1}^{\infty} \frac{\sin nt}{n} = O(nt)
\]

and \( g(t) \) is known to be an even function and integrable \( L(0, \pi) \), the integrated term vanishes.

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2: G.H. Hardy (13)
Thus

\[ T_{2k} = \frac{1}{\pi} \int_0^\pi G(t) \frac{\sin nt}{t} \, dt - \frac{1}{2\pi} \int_0^\pi G(t) \, dt + o(1) \]

\[ = T_{2k,1} - T_{2k,2}, \text{ say.} \]

\( T_{2k,1} \) is summable \((l)\) by the Lemma, putting \( G \) in place of \( g \) with the help of the condition \((2.3)\). \( T_{2k,2} \) is constant.

Hence

\[ \sum_{k=1}^\infty T_{2k,2} \frac{x^n}{n} = o \left( \log \frac{1}{1-x} \right). \]

This completes the proof of the theorem.