CHAPTER IV

SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

4.1 The following fixed point theorems were proved by Pachpatte [1] and Achari [4] for Ćirić type maps ([4], Theorem 1).

**THEOREM A** (Pachpatte [1]). Let \( X \) be an \( T \)-orbitally complete metric space and \( T \) be an orbitally continuous self-mapping of \( X \) satisfying

\[
\min \{d(Tx, Ty), d(x, Tx), d(y, Ty), (d(x, y))^{-1}d(x, Tx) d(y, Ty)\}
- \min \{d(x, y)^{-1}d(x, Ty) d(y, Tx), (d(x, y))^{-1}d(x, Tx) d(x, Ty)\} \leq q d(x, y)
\]

for all distinct \( x, y \in X \) and \( q \in (0,1) \), then, for each \( x \in X \), the sequence \( \{T^n x\}_{n=1}^{\infty} \) converges to a fixed point of \( T \).

**THEOREM B** (Achari [4]). Let \( X \) be an \( T \)-orbitally complete metric space and \( T \) be an orbitally continuous self-mapping of \( X \) satisfying

\[
\min \{d(Tx, Ty) d(x, y), d(x, Tx) d(y, Ty)\} - \min \{d(x, Tx) d(x, Ty),
\]

\[
d(y, Ty) d(y, Tx)\} \leq q d(x, y) \min \{d(x, Tx), d(y, Ty)\}
\]

for all \( x, y \in X, \ 0 < q < 1, \ d(x, Tx) \neq 0 \) and \( d(y, Ty) \neq 0 \). Then for each \( x \in X \), the sequence \( \{T^n x\}_{n=1}^{\infty} \) converges to a fixed point of \( T \).
Using the technique of Taskovic [1], Jain and Bohre [[1], [2]] generalized the above results as follows:

**THEOREM C** (Jain and Bohre [1]). Let $X$ be F-orbitally complete metric space and $T$ be an orbitally continuous self-mapping of $X$ satisfying

$$(4.1.3) \quad \alpha_1 \, d(Tx, Ty) + \alpha_2 \, d(x, Tx) + \alpha_3 \, d(y, Ty) + \alpha_4 \, (d(x, y))^{-1} \, d(x, Tx) \, d(y, Ty)$$

$$- \min \{d(x, y)^{-1} \, d(x, Ty), d(y, Tx), (d(x, y))^{-1} \, d(x, Ty) \, d(x, Tx)\} \leq \beta \, d(x, y)$$

for all distinct $x, y \in X$, where $\alpha_i \ (i = 1, 2, 3, 4)$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \rho$ and $\beta - \alpha_2 \geq 0$. Then for each $x \in X$, the sequence $\{T^n x\}_{n=1}^\infty$ converges to a fixed point of $T$.

**THEOREM D** (Jain and Bohre) [2]). Let $X$ be an F-orbitally complete metric space and $T$ be an orbitally continuous self-mapping of $X$ satisfying

$$(4.1.4) \quad \alpha_1 \, d(Tx, Ty) \, d(x, y) + \alpha_2 \, d(x, Tx) \, d(y, Ty)$$

$$- \min \{d(x, Tx) \, d(x, Ty), d(y, Ty) \, d(y, Tx)\} \leq \beta \, d(x, y) \min \{d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$, $d(x, Tx) \neq 0$ and $d(y, Ty) \neq 0$, where $\alpha_1, \alpha_2$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 > \beta$ and $\beta - \alpha_2 \geq 0$. Then for each $x \in X$, the sequence $\{T^n x\}_{n=1}^\infty$ converges to a fixed point of $T$.

In this section, we obtain some fixed point theorems for multivalued mappings on an orbitally complete metric space which include the above results.
THEOREM 1. Let \( X \) be an \( F \)-orbitally complete metric space and \( F: X \to B(X) \) be continuous mapping satisfying

\[
(4.15) \quad \alpha_1 \delta(\bar{F}x, \bar{F}y) + \alpha_2 \delta(x, \bar{F}x) \delta(y, \bar{F}y) - \alpha_3 \delta(y, \bar{F}y) + \alpha_4 \frac{\delta(x, \bar{F}x) \delta(y, \bar{F}y)}{d(x, y)} \leq \beta d(x, y) d(y, \bar{F}y)^{r-1}
\]

for all distinct \( x, y \in X \), where \( r \geq 1 \) is an integer, \( \alpha_i (i = 1, 2, 3, 4) \) and \( \beta \) are real numbers with \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \beta \) and \( \beta - \alpha_2 \geq 0 \), then there exists \( x \in X \) such that \( x \in \bar{F}x \) where \( \bar{F} \) denotes the closure of \( F \). If \( F \) is point closed mapping, then \( F \) has fixed point.

PROOF. Let \( x_0 \in X \) be an arbitrary point in \( X \) and define sequence \( \{x_n\} \) in \( X \) by

\[
x_1 \in \bar{F}x_0, \quad x_2 \in \bar{F}x_1, \quad \ldots \ldots, \quad x_n \in \bar{F}x_{n-1}.
\]

Let us suppose that \( d(x_n, \bar{F}x_n) > 0 \) for all \( n = 0, 1, 2, \ldots \) (otherwise for some positive integer \( n, x_n \in \bar{F}x_n \)).

Applying the condition (4.1.5) for \( x = x_{n-1} \) and \( y = y_n \), we have

\[
\alpha_1 \delta(\bar{F}x_{n-1}, \bar{F}x_n) + \alpha_2 \delta(x_{n-1}, \bar{F}x_n) \delta(x_n, \bar{F}x_n) - \alpha_3 \delta(x_n, \bar{F}x_n) + \alpha_4 \frac{\delta(x_{n-1}, \bar{F}x_n) \delta(x_n, \bar{F}x_n)}{d(x_{n-1}, x_n)} \leq \beta d(x_{n-1}, x_n) d(x_n, \bar{F}x_n)^{r-1}
\]

\[
- \min \left\{ \frac{d(x_{n-1}, \bar{F}x_n) d(x_n, \bar{F}x_n)}{d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, \bar{F}x_n) d(x_n, \bar{F}x_n)}{d(x_{n-1}, x_n)} \right\} \leq \beta d(x_{n-1}, x_n) d(x_n, \bar{F}x_n)^{r-1}
\]

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or,
\[
\alpha_1 d(x_n, x_{n+1})^r + \alpha_2 d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1} + \alpha_3 d(x_n, x_{n+1})^r
\]
\[
+ \alpha_4 \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} - \min \left\{ \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \right\}
\]
\[
\leq \beta d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1}
\]

or,
\[
(\alpha_1 + \alpha_3 + \alpha_4) d(x_n, x_{n+1})^r + \alpha_2 d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1} - \min \{0, d(x_{n-1}, x_n)\}
\]
\[
\leq \beta d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1}
\]

or,
\[
(\alpha_1 + \alpha_2 + \alpha_3) d(x_n, x_{n+1})^r \leq (\beta - \alpha_2) d(x_{n-1}, x_n) d(x_n, x_{n+1})^{r-1}
\]

or,
\[
d(x_n, x_{n+1}) \leq \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3 + \alpha_4} d(x_{n-1}, x_n).
\]

Proceeding in this manner, we obtain
\[
d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \ldots \leq k^n d(x_0, x_1),
\]

where
\[
k = \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3 + \alpha_4} < 1.
\]

Since \(0 < k < 1\), it follows that \(\{x_n\}\) is a Cauchy sequence in \(X\) and since \(X\) is orbitally complete, there is a point \(x \in X\) such that \(x_n \to x\). Now the continuity of \(F\) implies that \(F x_n \to F x\) in \(B(X)\).

It remains to show that \(d(x, Fx) = 0\) that is \(x \in F x\). Suppose \(y \in F x\), then for any \(n\),
\[
d(x, y) \leq d(x, x_n) + d(x_n, y)
\]
and therefore,

\[ d(x, Fx) \leq d(x, x_n) + d(x_n, Fx). \]

Since \( x_n \to x \), for given \( \varepsilon > 0 \) we can choose an \( N_1 \) such that \( d(x_n, x) < \frac{\varepsilon}{3} \) for all \( n \geq N_1 \). On the other hand, since \( Fx_n \to Fx \), for the same \( \varepsilon \) we can choose an \( N_2 \) such that

\[ Fx_{n-1} \subset A_{\varepsilon/3} = \bigcup_{x \in Fx} S \left( a, \frac{\varepsilon}{3} \right) \]

for all \( n-1 \geq N_2 \). Further, since \( x_n \in \overline{F} x_{n-1} \), there exists a \( y \in Fx_{n-1} \) such that

\[ d(x_n, y) < \frac{\varepsilon}{3} \quad \text{and} \quad y \in Fx_{n-1} \subset \bigcup_{a \in Fx} S \left( a, \frac{\varepsilon}{3} \right) \]

implies that there exists an \( a \in Fx \) such that \( d(a, y) < \frac{\varepsilon}{3} \). Thus

\[ d(x_n, Fx) \leq d(x_n, a) \leq d(x_n, y) + d(y, a) \]

\[ < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \]

\[ = \frac{2}{3} \varepsilon, \]

for all \( n-1 \geq N_2 \). Let \( N = \max\{N_1, N_2\} \). Then

\[ d(x, Fx) \leq d(x, x_n) + d(x_n, Fx) \]

\[ < \frac{\varepsilon}{3} + \frac{2}{3} \varepsilon \]

\[ = \varepsilon, \]

for all \( n \geq N \) and so

\[ x \in \overline{F} x, \text{ since } \varepsilon \text{ is arbitrary.} \]
If $F$ is a point closed mapping, i.e. $Fx$ is closed for each $x \in X$, then $x \in Fx$ and therefore $F$ has a fixed point. This completes the proof of Theorem 1.

**REMARK 1.** If $F$ is a single valued mapping $T$, $r = 1$ in Theorem 1, we obtain Theorem C.

**THEOREM 2.** Let $X$ be $F$-orbitally complete metric space and $F : X \to B(X)$ be continuous mapping satisfying

\[
\alpha_1 \delta(Fx, Fy) + \alpha_2 \delta(x, Fx) \delta(y, Fy) - \min\{d(x, Fx) \leq \beta d(x, y) d(y, Fy)^{r-1}
\]

\[
\min\{d(x, Fx), d(y, Fy)\}
\]

for all $x, y \in X$, where $r \geq 1$ is an integer, $d(x, Fx) \neq 0$ and $d(y, Fy) \neq 0$, $\alpha_1, \alpha_2$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 > \beta$ and $\beta - \alpha_2 \geq 0$, then there exists $x \in X$ such that $x \in Fx$ where $F$ denotes the closure of $F$. If $F$ is a point closed mapping, then $F$ has fixed point.

**PROOF.** Let $x_0 \in X$ be an arbitrary point is $X$. Define sequence $\{x_n\}$ in $X$ by

\[
x_1 \in Fx_0, \quad x_2 \in Fx_1, \quad \ldots, \quad x_n \in Fx_{n-1}.
\]

Let us suppose that $d(x_n, Fx_n) > 0$ for all $n = 0, 1, 2, \ldots$ (otherwise for some positive integer $n$, $x_n \in Fx_n$).

Applying the condition (4.1.6) for $x = x_{n-1}$ and $y = x_n$, we have

\[
\alpha_1 \delta(Fx_{n-1}, Fx_n) + \alpha_2 \delta(x_{n-1}, Fx_{n-1}) \delta(x_n, Fx_n) - \min\{d(x_{n-1}, Fx_{n-1}) d(x_n, Fx_n) d(x_n, Fx_{n-1})\}
\]

\[
\leq \beta d(x_{n-1}, x_n) d(x_n, Fx_n)^{r-1} \min\{d(x_{n-1}, Fx_{n-1}), d(x_n, Fx_n)\}
\]

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or, \[ \alpha_1 \, d(x_n, x_{n+1})^T \, d(x_{n-1}, x_n) + \alpha_2 \, d(x_{n-1}, x_n) \, d(x_n, x_{n+1})^T \]
\[ - \min \{ d(x_{n-1}, x_n) \, d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1})^T \, d(x_n, x_n) \} \]
\[ \leq \beta \, d(x_{n-1}, x_n) \, d(x_n, x_{n+1})^{r-1} \, \min \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \]
or, \[ (\alpha_1 + \alpha_2) \, d(x_n, x_{n+1})^T \, d(x_{n-1}, x_n) - \min \{ d(x_{n-1}, x_n) \, d(x_{n-1}, x_{n+1}), 0 \} \]
\[ \leq \beta \, d(x_{n-1}, x_n) \, d(x_n, x_{n+1})^{r-1} \, \min \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \]
or, \[ (\alpha_1 + \alpha_2) \, d(x_n, x_{n+1})^T \, d(x_{n-1}, x_n) \leq \beta \, d(x_{n-1}, x_n) \, d(x_n, x_{n+1})^{r-1} \]
\[ \min \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \]
or, \[ (\alpha_1 + \alpha_2) \, d(x_n, x_{n+1}) \leq \beta \, \min \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \]
or, \[ d(x_n, x_{n+1}) \leq \frac{\beta}{(\alpha_1 + \alpha_2)} \min \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \]
\[ = k \min \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \]
where,
\[ k = \frac{\beta}{(\alpha_1 + \alpha_2)} < 1. \]

Now, if \( d(x_{n-1}, x_n) \) is minimum, then we get
\[ d(x_n, x_{n+1}) \leq k \, d(x_{n-1}, x_n) \]
and if \( d(x_n, x_{n+1}) \) is minimum, then we have
\[ d(x_n, x_{n+1}) \leq k \, d(x_n, x_{n+1}) \]

which is contradiction, since \( k < 1 \)
so we obtain
\[ d(x_n, x_{n+1}) \leq k \ d(x_{n-1}, x_n). \]

Proceeding in this manner we obtain
\[ d(x_n, x_{n+1}) \leq k \ d(x_{n-1}, x_n) \leq k^2 \ d(x_{n-2}, x_{n-1}) \leq \ldots \leq k^n \ d(x_0, x_1). \]

Since 0 < k < 1, it follows that \( \{x_n\} \) is a Cauchy sequence in \( X \) and since \( X \) is orbitally complete, there is a point \( x \in X \) such that \( x_n \to x \). Now the continuity of \( F \) implies that \( Fx_n \to Fx \) in \( B(X) \).

It remains to show that \( d(x, Fx) = 0 \) that is \( x \in \overline{F}x \). Suppose \( y \in \overline{F}x \), then for any \( n \),
\[ d(x, y) \leq d(x, x_n) + d(x_n, y) \]
and therefore,
\[ d(x, Fx) \leq d(x, x_n) + d(x_n, Fx). \]

Since \( x_n \to x \), for given \( \varepsilon > 0 \) we can choose an \( N_1 \) such that \( d(x_n, x) < \varepsilon /3 \) for all \( n \geq N_1 \). On the other hand, since \( Fx_n \to Fx \), for the same \( \varepsilon \) we can choose an \( N_2 \) such that
\[ Fx_{n-1} \subset A_{\varepsilon/3} = \bigcup_{x \in Fx} S\left(a, \frac{\varepsilon}{3}\right) \]
for all \( n-1 \geq N_2 \). Further, since \( x_n \in \overline{F}x_{n-1} \), there exists a \( y \in Fx_{n-1} \) such that
\[ d(x_n, y) < \frac{\varepsilon}{3} \quad \text{and} \quad y \in Fx_{n-1} \subset \bigcup_{a \in Fx} S\left(a, \frac{\varepsilon}{3}\right) \]
implies that there exists an \( a \in Fx \) such that \( d(a, y) < \frac{\varepsilon}{3} \). Thus
\[ d(x_n, Fx) \leq d(x_n, a) \leq d(x_n, y) + d(y, a) \]
\[ \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \]
\[ = \frac{2}{3} \varepsilon, \]
for all \( n-1 \geq N_2 \). Let \( N = \max\{N_1, N_2\} \). Then
\[ d(x, Fx) \leq d(x, x_n) + d(x_n, Fx) \]
\[ \leq \frac{\varepsilon}{3} + \frac{2}{3} \varepsilon \]
\[ = \varepsilon, \]
for all \( n \geq N \) and so
\[ x \in \overline{F}x, \text{ since } \varepsilon \text{ is arbitrary.} \]

If \( F \) is a point closed mapping, i.e. \( Fx \) is closed for each \( x \in X \), then \( x \in Fx \) and therefore \( F \) has a fixed point. This completes the proof of Theorem 2.

**REMARK 2.** If \( F \) is a single valued mapping \( T, r = 1 \) in Theorem 2 it reduces to Theorem D.

**4.2** In 1993, Ćirić [5] investigated a class of self-mappings \( T \) of a metric space \((X, d)\) which satisfy the following contractive condition:

\[(4.2.1) \quad d(Tx, Ty) \leq a \max \{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} [d(x, Ty) + d(y, Tx)]\} \]
\[+ b \max \{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)] \]
for all \( x, y \in X \), where \( a, b, c \) are non-negative real numbers such that \( a + b + 2c = 1 \).
He obtained the following interesting result.

**THEOREM E.** Let $T$ be a mapping of a complete metric space $(X, d)$ into itself satisfying the inequality (4.2.1), where $a \geq 0$, $b > 0$, $c > 0$ with $a + b + 2c = 1$. Then $T$ has a fixed point which is unique whenever $a + 2c < 1$.

Recently, Adiga and Giniswamy [1] have studied a new class of self-mappings $T$ of $X$ which satisfy the following contractive definition:

\[
(4.2.2) \quad d(Tx, Ty) \leq a \max \{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} [(x, Ty) + d(y, Tx)]\} \\
+ b \max \{d(x, Tx), d(y, Ty), \frac{1}{4} [d(x, Ty) + d(y, Tx)]\} \\
+ c \max \{d(x, Ty) + d(y, Tx), \frac{8}{\sqrt{3}} \sqrt{d(x, Ty)d(y, Tx)}\}
\]

for all $x, y \in X$, where $a, b, c$ are non-negative real numbers such that $a + b + 2c = 1$.

They have shown by an example ([1], example 1) that their definition (4.2.2) includes (4.2.1) as a special case. Their main results are:

**THEOREM F.** Let $(X, d)$ be a complete metric space and $T$ be a self-mapping of $X$ satisfying (4.2.2), where $a \geq 0$, $b > 0$, $c > 0$, $b > 4c$ with $a + b + 2c = 1$. Then $T$ has at least one fixed point in $X$.

**THEOREM G.** Let $(X, d)$ be a complete metric space and $T$ be a self-mapping of $X$ satisfying (4.2.2), where $a \geq 0$, $b > 0$, $c > 0$, $b > \frac{11c}{2}$ with $a + b + 2c = 1$. Then $T$ has a unique fixed point.
The aim of the present section is to generalize the above results for multivalued mappings. Before stating our results, we prove some Lemmas which will be required in the sequel.

**Lemma 1.** Let \( F \) be a point compact mapping of a metric space \((X, d)\) into \(CB(X)\) satisfying the inequality

\[
H(Fx, Fy) \leq \max \{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2} [d(x, Fy) + d(y, Fx)]\} \\
+ b \max \{d(x, Fx), d(y, Fy), \frac{1}{4} [d(x, Fy) + d(y, Fx)]\} \\
+ c \max \{d(x, Fy) + d(y, Fx), \frac{8}{\sqrt{3}} d(Fx, Fy)\}
\]

for all \( x, y \in X \), where \( a, b, c \) are non-negative real numbers such that \( a + b + 2c = 1 \). Then

\[
d(x_{n+1}, Fx_{n+1}) \leq d(x_n, Fx_n) \leq d(x_0, Fx_0).
\]

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). Since \( Fx_0 \) is a point compact mapping, there exists \( x_1 \in Fx_0 \) such that

\[
d(x_0, x_1) = d(x_0, Fx_0).
\]

Similarly, since \( Fx_1 \) is a point compact mapping, there exists \( x_2 \in Fx_1 \) such that

\[
d(x_1, x_2) = d(x_1, Fx_1).
\]

Thus, we obtain a sequence \( \{x_n\} \) such that \( x_{n+1} \in Fx_n \) and

\[
d(x_{n+1}, x_{n+2}) = d(x_{n+1}, Fx_{n+1}).
\]

Then by (4.2.3),
\[ d(x_{n+1}, x_{n+2}) = d(x_{n+1}, Fx_{n+1}) \]

\[ \leq H(Fx_n, Fx_{n+1}) \]

\[ \leq a \max \{ d(x_n, x_{n+1}), d(x_n, Fx_n), d(x_{n+1}, Fx_{n+1}), \]

\[ \frac{1}{2} [d(x_n, Fx_{n+1}) + d(x_{n+1}, Fx_n)] \} \]

\[ + b \max \{ d(x_n, Fx_n), d(x_{n+1}, Fx_{n+1}), \]

\[ \frac{1}{4} [d(x_n, Fx_{n+1}) + d(x_{n+1}, Fx_n)] \} \]

\[ + c \max \{ d(x_n, Fx_{n+1}) + d(x_{n+1}, Fx_n), \]

\[ \frac{8}{\sqrt{3}} \sqrt{d(x_n, Fx_{n+1}) d(x_{n+1}, Fx_n)} \} \]

\[ = a \max \{ d(x_n, Fx_n), d(x_{n+1}, Fx_{n+1}), \frac{1}{2} d(x_n, Fx_{n+1}) \} \]

\[ + b \max \{ d(x_n, Fx_n), d(x_{n+1}, Fx_{n+1}), \frac{1}{4} d(x_n, Fx_{n+1}) \} \]

\[ + c \ d(x_n, Fx_{n+1}). \]

Thus, it follows that

\[ d(x_{n+1}, Fx_{n+1}) \leq H(Fx_n, Fx_{n+1}) \]

\[ \leq a \max \{ d(x_n, Fx_n), d(x_{n+1}, Fx_{n+1}), \]

\[ \frac{1}{2} [d(x_n, Fx_{n+1}) + d(x_{n+1}, Fx_{n+1})] \} \]

\[ + b \max \{ d(x_n, Fx_n), d(x_{n+1}, Fx_{n+1}), \]

\[ \frac{1}{4} [d(x_n, Fx_{n+1}) + d(x_{n+1}, Fx_{n+1})] \} \]

\[ + c \max \{ d(x_n, Fx_n) + d(x_{n+1}, Fx_{n+1}) \}. \]
If \( d(x_{n+1}, Fx_{n+1}) > d(x_n, Fx_n) \) for some \( n \), then we have

\[
d(x_{n+1}, Fx_{n+1}) < (a + b + 2c) \cdot d(x_{n+1}, Fx_{n+1}) = d(x_{n+1}, Fx_{n+1})
\]

a contradiction. Hence

\[
d(x_{n+1}, Fx_{n+1}) \leq d(x_n, Fx_n) \leq d(x_0, Fx_0)
\]

for all positive integer \( n \).

**LEMMA 2.** Let \( F \) be a point compact mapping of a metric space \( (X, d) \) into \( CB(X) \) satisfying the inequality (4.2.3), where \( a \geq 0, b > 0, c > 0, b > 4c \) with \( a + b + 2c = 1 \). Then

\[
(4.2.4) \quad \inf \{d(x, Fx) : x \in X\} = 0
\]

**PROOF.** Applying (4.2.3) and lemma 1, we have

\[
d(x_{2k+1}, Fx_{2k+2}) \leq H(Fx_{2k}, Fx_{2k+2})
\]

\[
\leq a \max \{d(x_{2k}, x_{2k+2}), d(x_{2k}, Fx_{2k}), (x_{2k+2}, Fx_{2k+2}),
\]

\[
\frac{1}{2} \left[ d(x_{2k}, Fx_{2k+2}) + d(x_{2k+2}, Fx_{2k}) \right]
\]

\[
+ b \max \{d(x_{2k}, Fx_{2k}), d(x_{2k+2}, Fx_{2k+2}),
\]

\[
\frac{1}{4} \left[ d(x_{2k}, Fx_{2k+2}) + d(x_{2k+2}, Fx_{2k}) \right]
\]

\[
+ c \max \{d(x_{2k}, Fx_{2k+2}) + d(x_{2k+2}, Fx_{2k}),
\]

\[
\frac{8}{\sqrt{3}} \sqrt{d(x_{2k}, Fx_{2k+2}) d(x_{2k+2}, Fx_{2k})}
\]
\[(4.2.5)\]
\[
= \text{a max } \{d(x_{2k}, F_{x_{2k+1}}), d(x_{2k}, F_{x_{2k}}),
\]
\[
\frac{1}{2} [d(x_{2k}, F_{x_{2k+2}}) + d(F_{x_{2k+1}}, x_{2k+1})]
\]
\[
+ b \text{ max } \{d(x_{2k}, F_{x_{2k}}), \frac{1}{4} [d(x_{2k}, F_{x_{2k+2}}) + d(F_{x_{2k+1}}, x_{2k+1})]
\]
\[
+ c \text{ max } \{d(x_{2k}, F_{x_{2k+2}}) + d(F_{x_{2k+1}}, x_{2k+1}),
\]
\[
\frac{8}{\sqrt{3}} \frac{d(x_{2k}, F_{x_{2k+2}})}{d(F_{x_{2k+1}}, x_{2k+1})}
\}\}

Further, by triangle inequality and Lemma 1,
\[
d(x_{2k}, F_{x_{2k+1}}) \leq d(x_{2k}, F_{x_{2k}}) + d(x_{2k+1}, F_{x_{2k+1}})
\]
\[(4.2.6)\]
\[
\leq 2 d(x_{2k}, F_{x_{2k}}),
\]
\[
d(x_{2k}, F_{x_{2k+2}}) \leq d(x_{2k}, F_{x_{2k}}) + d(x_{2k+1}, F_{x_{2k+1}}) + d(x_{2k+2}, F_{x_{2k+2}})
\]
\[(4.2.7)\]
\[
\leq 3 d(x_{2k}, F_{x_{2k}}),
\]
and
\[(4.2.8)\]
\[
d(x_{2k}, F_{x_{2k+2}}) + d(F_{x_{2k+1}}, x_{2k+1}) \leq 4 d(x_{2k}, F_{x_{2k}}).
\]

It now follows from (4.2.5) to (4.2.8) that
\[
d(x_{2k+1}, F_{x_{2k+2}}) \leq (2a + b) d(x_{2k}, F_{x_{2k}}) + c \text{ max } \{4d(x_{2k}, F_{x_{2k}}), 8d(x_{2k}, F_{x_{2k}})\}
\]
\[
= (2a + b + 8c) d(x_{2k}, F_{x_{2k}})
\]
\[(4.2.9)\]
\[
= [2 - (b - 4c)] d(x_{2k}, F_{x_{2k}})
\]

Using (4.2.3), Lemma 1 and (4.2.9), we have
\[
d(x_{2k+2}, F_{x_{2k+2}}) \leq H(F_{x_{2k+1}}, F_{x_{2k+2}})
\]

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\[ \leq a \max \{ d(x_{2k+1}, x_{2k+2}), d(x_{2k+1}, Fx_{2k+1}), d(x_{2k+2}, Fx_{2k+2}), \]
\[ \quad \frac{1}{2} \left[ d\left( (x_{2k+1}, Fx_{2k+2}) + d(x_{2k+2}, Fx_{2k+1}) \right) \right] \]
\[ + b \max \{ d(x_{2k+1}, Fx_{2k+1}), d(x_{2k+2}, Fx_{2k+2}), \]
\[ \quad \frac{1}{4} \left[ d\left( (x_{2k+1}, Fx_{2k+2}) + d(x_{2k+2}, Fx_{2k+1}) \right) \right] \]
\[ + c \max \{ d(x_{2k+1}, Fx_{2k+2}) + d(x_{2k+2}, Fx_{2k+1}), \]
\[ \quad \frac{8}{3} \sqrt{d(x_{2k+1}, Fx_{2k+2}) d(x_{2k+2}, Fx_{2k+1})} \} \]
\[ = a \max \{ d(x_{2k+1}, Fx_{2k+1}), \frac{1}{2} d(x_{2k+1}, Fx_{2k+2}) \}
\[ + b \max \{ d(x_{2k+1}, Fx_{2k+1}), \frac{1}{4} d(x_{2k+1}, Fx_{2k+2}) \}
\[ + c d(x_{2k+1}, Fx_{2k+2}) \]
\[ \leq (a + b) d(x_{2k}, Fx_{2k}) + c[2 - (b - 4c)] d(x_{2k}, Fx_{2k}) \]
\[ = [1 - c(b - 4c)] d(x_{2k}, Fx_{2k}). \]

(4.2.10)

Therefore, we have

\[ d(x_{2k+2}, Fx_{2k+2}) \leq [1 - c(b - 4c)]^{k+1} d(x_0, Fx_0), \]

which, in view of Lemma 1, simply gives

(4.2.11) \[ d(x_{2k+1}, Fx_{2k+1}) \leq d(x_{2k}, Fx_{2k}) \leq [1 - c(b - 4c)]^k d(x_0, Fx_0). \]

Thus, (4.2.10) and (4.2.11) together implies that

(4.2.12) \[ d(x_n, Fx_n) \leq [1 - c(b - 4c)]^{[n/2]} d(x_0, Fx_0), \]

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where \( \left\lfloor \frac{n}{2} \right\rfloor \) denotes the greatest integer not exceeding \( \frac{n}{2} \). Since by hypothesis \( 0 < c (b - 4c) < 1 \), from (4.2.12) we conclude that (4.2.4) holds. The proof is complete.

Now we have the following fixed point theorems

**THEOREM 3.** Let \( (X, d) \) be complete metric space and \( F : X \to CB(X) \) be a point compact mapping satisfying (4.2.3), where \( a \geq 0, b > 0, c > 0, b > 4c \) and \( a + b + 2c = 1 \) holds. Then \( F \) has at least one fixed point in \( X \).

**PROOF.** Let \( x = x_0 \) be an arbitrary point in \( X \) and define \( \{x_n\} \) as in Lemma 1. Then by (4.2.12), since \( 0 < c(b - 4c) < 1 \), we have for arbitrary \( \varepsilon > 0 \)

\[
D(x_m, x_n) < \varepsilon
\]

for \( m, n \) greater than some \( N \), it follows that \( \{x_n\} \) is a Cauchy sequence in the complete metric space \( X \) and so has a limit \( p \) in \( X \). Applying (4.2.3), we have

\[
d(p, Fp) \leq d(p, x_{n+1}) + d(x_{n+1}, Fp)
\]

\[
\leq d(p, x_{n+1}) + H(Fx_n, Fp)
\]

\[
\leq d(p, x_{n+1}) + a \max \{ d(x_n, p), d(x_n, Fx_n), d(p, Fp), \frac{1}{2} (d(x_n, Fp) + d(p, Fx_n)) \}
\]

\[
+ b \max \{ d(x_n, Fx_n), d(p, Fp), \frac{1}{4} (d(x_n, Fp) + d(p, Fx_n)) \}
\]
\[ + c \max \{ d(x_n, Fp) + d(p, Fx_n), \]
\[ \frac{8}{\sqrt{3}} \sqrt{d(x_n, Fp) d(p, Fx_n)} \]  

Taking the limit as \( n \) tends to infinity, we obtain

\[ d(p, Fp) \leq a d(p, Fp) + b d(p, Fp) + c d(p, Fp) \]
\[ = (1 - c) d(p, Fp) \]

which implies \( d(p, Fp) = 0 \) as \( c > 0 \) and hence \( p \in Fp \). This completes the proof of Theorem 3.

**THEOREM 4.** Let \((X, d)\) be a complete metric space and \( F : X \to CB(X) \) be a point compact mapping satisfying (4.2.3) where \( a \geq 0, \ b > 0, \ c > 0, \ b > \frac{11}{2} c \) and \( a + b + 2c = 1 \) holds. Then \( F \) has a unique fixed point in \( X \).

**PROOF.** In view of Theorem 3, it is sufficient to show that \( F \) has a unique fixed point. If \( p \) and \( q \) are two distinct fixed points of \( F \), then using (4.2.3), we have

\[ d(p, q) \leq H(Fp, Fq) \]
\[ \leq a \max \{ d(p, q), d(p, Fp), d(q, Fq), \frac{1}{2} [d(p, Fq) + d(q, Fp)] \} \]
\[ + b \max \{ d(p, Fp), d(q, Fq), \frac{1}{4} [d(p, Fq) + d(q, Fp)] \} \]
\[ + c \max \{ d(p, Fq) + d(q, Fp), \frac{8}{\sqrt{3}} \sqrt{d(p, Fq) d(q, Fp)} \} \]
\[ = a d(p, q) + \frac{b}{2} d(p, q) + \frac{8}{\sqrt{3}} c d(p, q) \]
= \left( a + \frac{b}{2} + \frac{8}{\sqrt{3}} c \right) \ d(p, q)

Hence, we have

\begin{equation}
1 - \left( a + \frac{b}{2} + \frac{8}{\sqrt{3}} c \right) \ d(p, q) \leq 0.
\end{equation}

If \( b > \frac{11}{2} c \) and \( a + b + 2c = 1 \), then it follows that \( 1 - \left( a + \frac{b}{2} + \frac{8}{\sqrt{3}} c \right) \) > 0. Hence (4.2.13) implies \( p = q \) and this completes the proof of Theorem 4.

**REMARK 3.** In Theorem 3 and 4, if \( F \) is a single valued mapping \( T \), then we obtain Theorem \( F \) and \( G \) respectively.