A RELATED FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS ON TWO L-SPACES

R.K. Jain, J.K. Verma and Brian Fisher

Abstract. A new related fixed point theorem for two pairs of mappings on two L-spaces is obtained.

Key words and phrases. Two L-spaces, fixed point.

2000 AMS Subject Classification : 54H25.

1. PRELIMINARIES

Definition 1. [2]. Let \( N \) denote the set of all non-negative integers and let \( C \) be a class consisting of pairs \( (\{x_n\}_{n \in N}, x) \), where \( \{x_n\}_{n \in N} \) is a sequence in a non-empty set \( X \) and \( x \) is a point in \( X \). We say that \( C \) is a convergence class (i.e. \( \{x_n\}_{n \in N} \) converges to \( x \)) if and only if it satisfies the following conditions:

(a) if \( x_n = x \in X \) for all \( n \in N \), then \( (\{x_n\}_{n \in N}, x) \in C \),

(b) if \( \{x_n\}_{n \in N} \) converges to \( x \), then so does each subsequence of \( \{x_n\} \).

Then the pair \( (X, C) \) is said to be an \( L \)-space.

Definition 2. [4]. Let \( d \) be a non negative real valued function on \( X \times X \), \( 0 \leq d(x, y) < \infty \) for all \( x, y \in X \). Then the \( L \)-space \( (X, \to) \) is said to be \( d \)-complete, if each sequence \( \{x_n\}_{n \in N} \) in \( X \) with \( \sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty \), converges to a unique point in \( X \).

Definition 3. [1]. Let \( f : X \to X \) \( (X \) is an \( L \)-space). If for each \( x \in X \), \( f^n (x) \to a \) in \( X \) implies \( f (f^n (x)) \to f(a) \), then \( f \) is called orbitally continuous.

Dealing with two \( L \)-spaces, Kanan [3] proved the following result.

Theorem 1. Let \( (X, d) \) and \( (Y, \rho) \) be \( d \)-complete \( L \)-spaces and suppose that \( T \) is an orbitally continuous mapping of \( X \) into \( Y \) and let \( S \) be a mapping of \( Y \) into \( X \) satisfying the inequalities

\[
d(STx, STx') \leq c \max \{d(x, x'), d(x, STx), d(x', STx'), d(x', STx), \rho(Tx, Tx')\},
\]

\[
\rho(TS,y, TSy') \leq c \max \{\rho(y, y'), \rho(y, TSy), \rho(y', TSy'), \rho(y', TSy), d(Sy, Sy')\}
\]

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \), where \( 0 \leq c < 1 \). Then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further, \( Tz = w \) and \( Sw = z \).
2. MAIN RESULT

We prove the following:

**Theorem 2.** Let \((X, d)\) and \((Y, \rho)\) be \(d\)-complete \(L\)-spaces, let \(A, B\) be mappings of \(X\) into \(Y\) and let \(S, T\) be mappings of \(Y\) into \(X\) satisfying the inequalities

\[
d(SAx\_x, TBx\'} \leq c \max \{d(x, x'), d(x, SAx), d(x', TBx'), \sqrt{d(x, TBx')d(x', SAx)}, \rho(Ax, Bx')\},
\]

\[
\rho(BSy, ATy') \leq c \max \{\rho(y, y'), \rho(y, BSy), \rho(y', ATy), \sqrt{\rho(y, ATy')\rho(y', BSy)}, d(Sy, Ty')\}
\]

(1)\)

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\), where \(0 \leq c < 1\). If one of the mappings \(A, B, S\) and \(T\) is orbitally continuous, then \(SA\) and \(TB\) have a unique common fixed point \(z\) in \(X\) and \(BS\) and \(AT\) have a unique common fixed point \(w\) in \(Y\). Further, \(Az = Bz = w\) and \(Sw = Tw = z\).

**Proof.** Let \(x = x_0\) be an arbitrary point in \(X\). We define the sequences \(\{x_n\}\) in \(X\) and \(\{y_n\}\) in \(Y\) by

\[
Ax_{2n} = y_{2n-1}, \quad Sy_{2n-1} = x_{2n-1}, \quad Bx_{2n-1} = y_{2n}, \quad Ty_{2n} = x_{2n}
\]

for \(n = 1, 2, \ldots\).

Applying inequality (1), we get

\[
d(x_{2n+1}, x_{2n}) = d(SAx_{2n}, TBx_{2n-1})
\]

\[
\leq c \max \{d(x_{2n}, x_{2n-1}), d(x_{2n}, SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), \sqrt{d(x_{2n}, TBx_{2n-1})d(x_{2n-1}, SAx_{2n})}, \rho(Ax_{2n}, Bx_{2n-1})\}
\]

\[
= c \max \{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \sqrt{d(x_{2n}, x_{2n})d(x_{2n-1}, x_{2n+1})}, \rho(y_{2n}, y_{2n+1})\}
\]

\[
= c \max \{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \rho(y_{2n}, y_{2n+1})\}
\]

\[
= c \max \{d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1})\},
\]

(3)

since \(c < 1\).

Using inequality (1) again, it follows that

\[
d(x_{2n}, x_{2n-1}) \leq c \max \{d(x_{2n-1}, x_{2n-2}), \rho(y_{2n}, y_{2n-1})\}.
\]

(4)

Similarly, using inequality (2), we have

\[
\rho(y_{2n}, y_{2n+1}) \leq c \max \{\rho(y_{2n-1}, y_{2n}), d(x_{2n-1}, x_{2n})\}
\]

(5)
and
\[ \rho(y_{2n-1}, y_{2n}) \leq c \max\{\rho(y_{2n-2}, y_{2n-1}), d(x_{2n-2}, x_{2n-1})\}. \] (6)

It now follows from inequalities (3) and (5) that
\[ d(x_{2n+1}, x_{2n}) \leq c \max\{d(x_{2n}, x_{2n-1}), \rho(y_{2n-1}, y_{2n})\} \] (7)

and from inequalities (4) and (6) that
\[ d(x_{2n}, x_{2n-1}) \leq c \max\{d(x_{2n-1}, x_{2n-2}), \rho(y_{2n-2}, y_{2n-1})\} \] (8)

It now follows from inequalities (5) to (8) that
\[ d(x_{n+1}, x_n) \leq c \max\{d(x_n, x_{n-1}), \rho(y_n, y_{n-1})\}, \]
\[ \rho(y_{n+1}, y_n) \leq c \max\{d(x_n, x_{n-1}), \rho(y_n, y_{n-1})\} \]

and by induction, we get
\[ d(x_{n+1}, x_n) \leq c^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\}, \]
\[ \rho(y_{n+1}, y_n) \leq c^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\} \]

for \( n = 1, 2 \ldots \).

Therefore
\[ \sum_{n=1}^{\infty} d(x_{n+1}, x_n) \leq (1 - c)^{-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\}, \]
\[ \sum_{n=1}^{\infty} \rho(y_{n+1}, y_n) \leq (1 - c)^{-1} \max\{d(x_1, x_2), \rho(y_1, y_2)\} \]

and since \( X \) and \( Y \) are \( d \)-complete \( L \)-spaces, it follows that the sequence \( \{x_n\} \) has a limit \( z \) in \( X \) and the sequence \( \{y_n\} \) has a limit \( w \) in \( Y \).

Now suppose that \( A \) is orbitally continuous. Then
\[ w = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Ax_{2n} = Az. \]

Applying inequality (1), we have
\[ d(SAz, x_{2n}) = d(SAz, TBx_{2n-1}) \]
\[ \leq c \max\{d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1}, TBx_{2n-1}), \sqrt{d(z, TBx_{2n-1}) d(x_{2n-1}, SAz)}, \rho(Az, y_{2n})\}. \]

Letting \( n \) to infinity, it follows that
\[ d(SAz, z) \leq cd(SAz, z) \]

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and since $c < 1$, we have

$$S_w = SAz = z.$$  \hspace{1cm} (9)

Now, using inequality (2), we have

$$\rho(BSw, y_{2n+1}) = \rho(BSw, ATy_{2n})$$

$$\leq c \max \{ \rho(w, y_{2n}), \rho(w, BSw), \rho(y_{2n}, ATy_{2n}),$$

$$\sqrt{\rho(w, ATy_{2n})\rho(y_{2n}, BSw), d(Sw, Ty_{2n})} \}.$$

Letting $n$ to infinity, we obtain

$$\rho(BSw, w) \leq c \rho(w, BSw)$$

and since $c < 1$ we have

$$Bz = BSw = w.$$ \hspace{1cm} (10)

Using inequality (1) again, we have

$$d(z, Tw) = d(SAz, TBz)$$

$$\leq c \max \{ d(z, z), d(z, SAz), d(z, TBz),$$

$$\sqrt{d(z, TBz) d(z, SAz), \rho(Az, Bz)} \}$$

$$= cd(z, Tw)$$

and since $c < 1$ we, have

$$z = Tw = TBz.$$

The same results of course hold if one the mapping $B, S, T$ is orbitally continuous instead of $A$.

To prove uniqueness, we suppose that $SA$ and $TB$ have a second fixed point $z'$. Then using inequality (2), we have

$$\rho(Bz', Az') = \rho(BSAz', ATBz'),$$

$$\leq c \max \{ \rho(Az', Bz'), \rho(Az', Bz'), \rho(Bz', Az'),$$

$$\sqrt{\rho(Az', Az') \rho(Bz', Bz'), d(SAz', TBz')} \}$$

$$= c \rho(Bz', Az')$$

and so $Bz' = Az'$ since $c < 1$.

Now using inequalities (1) and (2), we have

$$d(z, z') = d(SAz, TBz')$$

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\[ \leq c \max \{ d(z, z'), d(z, SAz), d(z', TBz'), \]
\[ \sqrt{d(z, TBz')d(z', SAz), \rho(Az, Bz')} \}\]
\[ = c \max \{ d(z, z'), d(z, z), d(z', z'), \]
\[ \sqrt{d(z, z')d(z', z), \rho(Az, Bz')} \}\]
\[ = c \rho(Az, Bz') \]
\[ = c \rho(BSAz', ATBz) \]
\[ \leq c^2 \max \{ \rho(Az', Bz), \rho(Az', Bz'), \rho(Bz, Az), \]
\[ \sqrt{\rho(Az', Az)\rho(Bz, Bz'), d(SAz', TBz')} \}\]
\[ = c^2 \max \{ \rho(Az, Bz'), d(z, z') \}\]
\[ = c^2 d(z, z') \]

since \( c < 1 \), and so \( z = z' \).

It follows similarly that \( z \) is the unique fixed point of \( SA \) and that \( w \) is the unique fixed point of \( AT \) and of \( BS \). This completes the proof of the theorem.

On taking \( X = Y \) in Theorem 2, we obtain the following:

**Corollary.** Let \((X, d)\) be a complete \( L \)-space and let \( A, B, S, T \) be mappings of \( X \) into itself satisfying the inequalities
\[ d(SAx, TBy) \leq c \max \{ d(x, y), d(x, SAx), d(y, TBy), \]
\[ \sqrt{d(x, TBy)d(y, SAx), d(Ax, By)} \}, \quad (11) \]
\[ d(BSx, ATy) \leq c \max \{ d(x, y), d(x, BSx), d(y, ATy), \]
\[ \sqrt{d(x, ATy)d(y, BSx), d(Sx, Ty)} \} \quad (12) \]
for all \( x, y \) in \( X \), where \( 0 \leq c < 1 \). If one of the mappings \( A, B, S \) and \( T \) is orbitally continuous, then \( SA \) and \( TB \) have a unique common fixed point \( z \) in \( X \) and \( BS \) and \( AT \) have a unique common fixed point \( w \) in \( Y \). Further, \( Az = Bz = w \) and \( Sw = Tw = z \).
References


R.K. Jain
C-62, University Campus, Dr.H.S.Gour University,
Sagar-470003 (M.P.), India

J.K. Verma
Govt. Chhatrasal P.G. College, Panna- 488001 (M.P.), India

B. Fisher
Department of Mathematics
University of Leicester, Leicester, LE1 7RH, England
e-mail: fbr@le.ac.uk
ULTRA SCIENTIST OF PHYSICAL SCIENCES

REPRINT
A fixed point theorem for multivalued mappings

J. K. VERMA

Department of Mathematics
Govt. P.G. College, Panna, PANNA (M.P.)-488001 (INDIA)

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Abstract

A fixed point theorem for multivalued mapping on an orbitally complete metric space is proved in the present paper which extends a results of Jain & Bohre.\textsuperscript{1,2}

Key words: Multivalued Map, orbitally complete metric space.
AMS mathematics subject classification (2001): 54H25,47H10

1. Notations and Definitions:

Let $(X,d)$ be a metric space and $B(X)$ be the set of all bounded subsets of $X$.

**Definition 1.1:** A multivalued function (or set valued mapping) $F$ on $X$ into $X$ is a point to set correspondence $x \mapsto Fx$ such that $Fx$ is a nonempty bounded subset of $X$ for each $x \in X$.

Such a mapping will be denoted by $F : X \rightarrow B(X)$.

For any $x \in X$, $A,B \in B(X)$, we write

\[
\delta(A,B) = \inf \{d(x,a) / a \in A\},
\]

\[
\delta(A,B) = \sup \{d(a,b) / a \in A, b \in B\}.
\]

The function $\delta$ satisfies:

(i) $\delta(A,B) = \delta(B,A) \geq 0$,

(ii) $\delta(A,B) = 0 \Rightarrow A = B = \{a\}$

If $A = \{a\}$, we write $\delta(A,B) = \delta(a,B)$ and furthermore, if $B = \{b\}$, we write $\delta(A,B) = \delta(a,b) = d(a,b)$.

**Definition 1.2:** A sequence $\{A_n\}$ of sets in $B(X)$ is said to converge to the subject $A$ of $X$ if the following conditions are satisfied:

(i) for each $a$ in $A$, there is a sequence $\{a_n\}$ such that $a_n \in A_n$ for all $n$ and $a_n \rightarrow a$

(ii) for every $\varepsilon > 0$, there is an integer $N$ such that $A_n \subset A_{\varepsilon}$ for all $n \geq N$, where $A_{\varepsilon}$ is the union of all open spheres with centres in $A$ and radius $\varepsilon$. The set $A$ is then said to be the limit of the sequence $\{A_n\}$ and we write $\lim A_n = A$

$n \rightarrow \infty$

**Definition 1.3:** A multivalued map $X \rightarrow B(X)$ is said to be continuous at $x \in X$ if $x_n \rightarrow x$ in $X$ implies $F(x_n) \rightarrow F(x)$ in $B(X)$.

\[x_n \rightarrow x\text{ in } X \text{ implies } F(x_n) \rightarrow F(x) \text{ in } B(X)\]
continuous on $X$ if $F$ is continuous at every point of $X$.

An orbit of $F$ at a point $x_0 \in X$ is a sequence 
\{$x_n\} \text{ in } X \text{ given by}

\[0 \text{ (F,x_0) = \{x_n/x_n \in Fx_{n-1}, n = 1,2,3,\ldots\}}\]

**Definition 1.4:** A metric space $X$ is said to be $F$-orbitally complete if every Cauchy sequence which is a subsequence of an orbit of $F$ at each point $x \in X$ converges to a point of $X$.

**Definition 1.5:** A single valued mapping $T$ of $X$ into $X$ is orbitally continuous on $X$ if for each $x \in X$, $\lim_{n \to \infty} T^n x = u \Rightarrow \lim_{n \to \infty} T(T^n x) = Tu$.

**Definition 1.6:** A point $z \in X$ is said to be a fixed point of a multivalued mapping $F: X \to B(X)$ if $z \in Fz$.

Finally, $\overline{A}$ denotes the closure of $A$.

**Main Results:**

We prove the following.

**Theorem 2.1:** Let $X$ be an $F$-orbitally complete metric space and $F: X \to B(X)$ be continuous mapping satisfying

\[
\alpha_1 \delta (Fx, Fy)' + \alpha_2 \delta (x, Fx) \delta (y, Fy)' - \min \left\{ \frac{d(x, Fx)d(y, Fy)}{d(x, y)} \right\},
\]

\[
\leq \beta d(x, y) \cdot d(y, Fy)'^{-1},
\]

for all distinct $x, y \in X$, where $r \geq 1$ is an integer, $\alpha_i (i = 1, 2, 3)$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \beta$ and $\beta - \alpha_3 > 0$, then there exists $x \in X$ such that $x \in \overline{Fx}$. If $F$ is a point closed mapping, then $F$ has fixed point.

**Proof:** Let $x_0 \in X$ be an arbitrary point in $X$, define sequence \{$x_n\}$ in $X$ by

\[x_1 \in \overline{Fx}_0, x_2 \in \overline{Fx}_1, \ldots, x_n \in \overline{Fx}_{n-1}.\]

Let us suppose that $d(x_n, Fx_n) > 0$ for all $n = 0, 1, 2, 3, \ldots$ (otherwise for some positive integer $n$, $x_n \in Fx_n$) applying the condition (2.1) for $x = x_{n-1}$ and $y = x_n$ we have

\[
\alpha_1 \delta (Fx_{n-1}, Fx_{n-1})' + \alpha_2 \delta (x_{n-1}, Fx_{n-1})',
\]

\[
\delta (x_n, Fx_n)'^{-1} + \alpha_3 \delta (x_n, Fx_n)',
\]

\[+ \alpha_4 \frac{\delta (x_{n-1}, Fx_{n-1})'}{d(x_{n-1}, x_n)} - \min \left\{ \frac{d(x_{n-1}, Fx_{n-1})d(x_n, Fx_n)}{d(x_{n-1}, x_n)} \right\},
\]

\[
\leq \beta d(x_{n-1}, x_n) \cdot d(x_n, Fx_n)'^{-1}.
\]

\[
\Rightarrow \alpha_1 d(x_n, x_{n+1})' + \alpha_2 d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1})^{-1},
\]

\[+ \alpha_3 d(x_{n-1}, x_n)',
\]
A fixed point theorem for multivalued mappings.

\[
\begin{align*}
&+ \alpha_4 \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \cdot \text{min.} \\
&\frac{d(x_{n-1}, x_n) d(x_n, x_n)}{d(x_{n-1}, x_n)} + \frac{d(x_{n-1}, x_n) d(x_n, Fx_n)}{d(x_{n-1}, x_n)} \\
&\leq \beta \frac{d(x_{n-1}, x_n) d(x_n, Fx_n)}{d(x_{n-1}, x_n)}^{\gamma-1} \\
&\Rightarrow \{ \alpha_1 + \alpha_3 + \alpha_4 \} d(x_n, x_{n+1})^\gamma + \alpha_2 d(x_{n-1}, x_n) d(x_n, x_{n+1})^{\gamma-1} \\
&\geq \{ \alpha_1 + \alpha_3 + \alpha_4 \} d(x_n, x_{n+1})^\gamma \leq (\beta - \alpha_2) d(x_{n-1}, x_n) d(x_n, x_{n+1})^{\gamma-1} \\
&\Rightarrow d(x_n, x_{n+1}) \leq \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3 + \alpha_4} d(x_{n-1}, x_n) \\
&\text{Proceeding in this manner we obtain} \\
&d(x_n, x_{n+1}) \leq k' d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_n) \leq \cdots \leq k^n d(x_0, x_1).
\end{align*}
\]

Where

\[k' = \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3 + \alpha_4} < 1\]

Since \(0 < k' < 1\), it follows that \(\{x_n\}\) is a Cauchy sequence in \(X\) and since \(X\) is orbitally complete, there is a point \(x \in X\) such that \(x_n \to x\). Now the continuity of \(F\) implies that \(Fx_n \to Fx\) in \(B(X)\).

It remains to show that \(d(x, Fx) = 0\) that is \(x \in Fx\).

Suppose \(y \in Fx\), then for any \(n\)

\[d(x, y) \leq d(x, x_n) + d(x_n, y)\]

and therefore

\[d(x, Fx) \leq d(x, x_n) + d(x_n, Fx)\]

Since \(x_n \to x\), for given \(\varepsilon > 0\) we can choose an \(N_1\) such that \(d(x_n, x) < \varepsilon / 3\) for all \(n \geq N_1\). On the other hand, since \(Fx_n \to Fx\), for the same \(\varepsilon\) we can choose an \(N_2\) such that

\[Fx_{n-1} \subset A_{\varepsilon/3} = U S (a, \varepsilon / 3)\]

for all \(n-1 \geq N_2\). Further, since \(x_n \in Fx_{n-1}\), there exists a \(y \in Fx_{n-1}\) such that

\[d(x, y) < \varepsilon / 3\]

and \(y \in Fx_{n-1} \subset U S (a, \varepsilon / 3)\)

implies that there exists an \(a \in Fx\) such that

\[d(a, y) < \varepsilon / 3\]

thus

\[d(x, Fx) \leq d(x, a) \leq d(x, y) + d(y, a) \leq \varepsilon / 3 + \varepsilon / 3 = 2 \varepsilon / 3\]

for all \(n-1 \geq N_2\). Let \(N = \max \{N_1, N_2\}\) then

\[d(x, Fx) \leq d(x, x_n) + d(x_n, Fx) < \varepsilon / 3 + 2 \varepsilon / 3 = \varepsilon\]

for all \(n \geq N\) and so \(x \in Fx\). Since \(\varepsilon\) is arbitrary.

If \(F\) is a point closed mapping i.e. \(Fx\) is closed for each \(x \in X\), then \(x \in Fx\), and therefore \(F\) has a fixed point. This completes the proof of the theorem.

If \(F\) is a single valued mapping \(T, r = 1\) in theorem 2.1, the following corollary is immediate:
Corollary 2.1:

Let $X$ be orbitally complete and $T$ be an orbitally continuous self-mapping of $X$ satisfying

$$\alpha_1, d(Tx,Ty) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty) + \alpha_4 d(x,Tx)d(y,Ty)(d(x,y))^{-1}$$

- $\min \{d(x,Tx) d(y,Ty) (d(x,y))^{-1}, d(x,Tx)d(y,Tx)(d(x,y))^{-1}\} \leq \beta d(x,y)$

for all distinct $x,y \in X$, where $\alpha_i (i = 1,2,3)$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \beta$ and $\beta - \alpha_2 \geq 0$. Then for each $x \in X$, the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to a fixed point of $T$.

Theorem 2.2: Let $X$ be an $F$-orbitally complete metric space and $F: X \to B(X)$ be continuous mapping satisfying

$$\alpha_1 \delta(Fx,Fy) + \alpha_2 \delta(x,Fx) \delta(y,Fy)$$

$$\min \{d(x,Fx)d(x,Fy) d(y,Fy) d(y,Fx)\}$$

$$\leq \beta d(x,y) d(y,Fy) (d(x,y))^{-1} \min \{d(x,Fx), d(y,Fy)\}$$

for all $x,y \in X$ where $r \geq 1$ is an integer, $d(x,Fx) \neq 0$ and $d(y,Fy) \neq 0$, where $\alpha_1, \alpha_2$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 > \beta$ and $\beta - \alpha_2 \geq 0$, then there exists $x \in X$ such that $x \in Fx$. If $F$ is a point closed mapping then $F$ has fixed point.

Proof: It is omitted as it is similar to that of theorem 2.1

Corollary 2.2:

Let $X$ be orbitally complete metric space and $T$ be an orbitally continuous self-mapping of $X$ satisfying

$$\alpha_1 d(Tx,Ty) d(x,y) + \alpha_2 d(x,Tx) d(y,Ty)$$

$$\min \{d(x,Ty) d(y,Ty) d(y,Tx)\} \leq \beta d(x,y)$$

for all $x,y \in X$, $d(x,Tx) \neq 0$, and $d(y,Ty) \neq 0$, where $\alpha_1, \alpha_2$ and $\beta$ are real numbers with $\alpha_1 + \alpha_2 > \beta$ and $\beta - \alpha_2 \geq 0$,

then for each $x \in X$ the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to a fixed point of $T$.

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