CHAPTER X

ON THE DEGREE OF APPROXIMATION TO A
FUNCTION BY TRIANGULAR MATRIX OF ITS
CONJUGATE SERIES

10.1 DEFINITIONS AND NOTATIONS:

Let \( f(x) \) be a periodic function with period \( 2\pi \) and integrable in the Lebesgue sense. Let its Fourier series be given by

\[
(10.1.1) \quad f(x) \sim \sum_{n=\infty}^{\infty} f_n e^{inx}, \quad f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.
\]

The conjugate series of the Fourier series of \( f(x) \) is given by

\[
(10.1.2) \quad \sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right),
\]

the corresponding sum-function being

\[
(10.1.3) \quad f(x) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \left[ f(x+t) - f(x-t) \right] \cot \frac{t}{2} dt
\]

which exists almost everywhere.
Let \((\delta_{n,k})\) \((n=0,1,2 \ldots, k=0,1, \ldots n)\), \(\delta_{n,0} = 1\) be a triangular matrix with real or complex elements.

The transformation of \(\{S_n\}\) given by

\[(10.1.4) \quad t_n = \sum_{k=0}^{n} \Delta \delta_{n,k} S_k \quad \text{where} \quad \Delta \delta_{n,k} = \delta_{n,k} - \delta_{n,k+1}\]

defines the matrix transform of sequence \(\{S_n\}\).

A series \(\sum u_n\) with sequence of partial sums \(\{S_n\}\) is said to be summable \((\sigma)\) to \(S\) if the sequence \(\{t_n\}\) tends to a finite limit \(S\) as \(n \to \infty\).

The necessary and sufficient condition for the \((\sigma)\) to be regular are that there is a constant \(C\) such that

\[\text{(a)} \quad \sum_{k=0}^{n} |\Delta \delta_{n,k}| < C \quad \text{for every} \quad n.\]

\[\text{(b)} \quad \text{For every} \quad k\]

\[\lim_{n \to \infty} \Delta \delta_{n,k} = 0,\]

\[\text{(c)} \quad \lim_{n \to \infty} \sum_{k=0}^{n} \Delta \delta_{n,k} = 1.\]
In particular if

$$\Delta_{n,k} = \frac{p_{n-k}}{p_n} \quad (k \leq n)$$

$$= 0 \quad (k \leq n),$$

then \( t_n \) defined by (10.1.4) is the same as

Nörlund mean generated by the sequence of coefficients \( \{p_n\} \).

Similarly if

$$\Delta_{n,k} = \frac{(n-k+\alpha-1)}{\alpha-1}, \quad \alpha > 0 \quad \text{for} \quad k < n$$

$$= 0 \quad \text{for} \quad k \geq n$$

then \( t_n \) means is the same as \((C,\alpha)\) mean, the familiar

Cesàro mean of order \( \alpha > 0 \).

10.2 INTRODUCTION :

The well known theorem on the degree of approximation of a function \( f(x) \) belonging to the class \( \text{Lip}_\alpha \) by
(C, δ) means of its Fourier series is given by Alexits\(^1\). Sahney and Goel\(^2\) have extended the above result to \((N, p_n)\) means of its Fourier series. Later on Chandra\(^3\) has also studied similar problem for \((R, p_n)\)-means of its Fourier series.

We proved the following theorem\(^4\):

**Theorem A**: If the sequence \([p_n]\) is positive and non-increasing then the degree of approximation of a function \(f(x)\), conjugate to a periodic function \(f\) with period \(2\pi\) and belonging to the class of \(\text{Lip}_\alpha\), \(0 < \alpha < 1\) by \((N, p_n)\) means of its conjugate series, is given by

\[
| f(x) - \overline{t}_n(x) | = O \left( \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}} \right)
\]

where \(\overline{t}_n(x)\) are the \((N, p_n)\) means of the series \((10.1.2)\).

Our object of the present Chapter is to prove the following theorem:

1) Alexits, G. (1),
2) Sahney, B.N. and Goel, D.S. (1),
3) Chandra, P. (1),
4) Qureshi, K. (10).
Theorem 5) The degree of approximation of a periodic function \( f \) with period \( 2\pi \) and belonging to the class \( \text{Lip}_\alpha, \ 0 < \alpha < 1 \), is given by,

\[
|f(x) - t_n(x)| = O\left[ \sum_{k=1}^{n} \frac{\Delta^{r} n, n-k}{k^{\alpha}} \right],
\]

where \( t_n(x) \) are the matrix means of the series (10.1.2) and \( [\Delta^{r} n, k]_{k=0}^{n} \) is non-negative and non-decreasing sequence with respect to \( k \).

In order to prove the theorem, we shall use the following lemma.

Lemma 6) If \( [\Delta^{r} n, k]_{k=0}^{n} \) is non-negative and non-decreasing sequence with respect to \( k \), then, for \( 0 \leq a \leq b \leq \pi \), \( 0 \leq t \leq \pi \) and for every \( n \)

\[
\sum_{k=a}^{b} \Delta^{r} n, n-k \delta(n-k)t < B \frac{\Delta^{r} n, n-\bar{t}}{t}
\]

where \( B \) is a constant and \( \bar{t} \) is the integral part of \( \frac{1}{t} \).

5) Qureshi, K. (1),
10.3 PROOF OF THE THEOREM: Since

\[- \tilde{S}_k(x) - \tilde{f}(x) = - \frac{1}{\pi} \int_0^\pi \psi(t) \frac{\cos(k+1/2)t}{2\sin t/2} \, dt\]

where \(\psi(t) = f(x+t) - f(x-t)\),

we have

\[- \tilde{t}_n(x) - \tilde{f}(x) = \sum_{k=0}^n \Delta d_{n,k} \left[ - \tilde{S}_k(x) - \tilde{f}(x) \right]\]

\[= - \frac{1}{2\pi} \int_0^\pi \psi(t) \sum_{k=0}^n \Delta d_{n,k} \cos(k+1/2)t \, dt \sin t/2\]

Therefore

\[| \tilde{f}(x) - \tilde{t}_n(x) | \leq \frac{1}{2\pi} \int_0^\pi t^\alpha \left| \sum_{k=0}^n \Delta d_{n,k} \cos(k+1/2)t \right| \sin t/2 \, dt \]

\[+ \frac{1}{2\pi} \int_{\pi/n}^\pi \frac{t^\alpha}{\tan t/2} \left| \sum_{k=0}^n \Delta d_{n,k} \cos kt \right| \, dt\]

\[+ \frac{1}{2\pi} \int_{\pi/n}^\pi t^\alpha \left| \sum_{k=0}^n \Delta d_{n,k} \sin kt \right| \, dt\]
Now

\[ I_1 = 0 \left[ \int_0^{\pi/n} \frac{t^\alpha}{t} \right] \left[ \sum_{k=0}^{n} \Delta_{n,k} \cos(k+1/2)t \right] dt \]

\[ = 0 \left[ \int_0^{\pi/n} t^{\alpha-1} \right] \left[ \sum_{k=0}^{n} \Delta_{n,k} dt \right] \]

\[ = 0 \left[ \int_0^{\pi/n} t^{\alpha-1} dt \right] \]

\[ = 0 \left[ \left( \frac{1}{n} \right)^\alpha \right] \]

\[ = 0 \left[ \sum_{k=1}^{n} \frac{\Delta_{n,n-k}}{k^\alpha} \right] \]

since

\[ \sum_{k=1}^{n} \frac{\Delta_{n,n-k}}{k^\alpha} > \sum_{k=1}^{n} \frac{\Delta_{n,n-k}}{n^\alpha} = C \frac{1}{n^\alpha}. \]

Also

\[ I_2 = 0 \left[ \int_{\pi/n}^{\pi} \frac{t^\alpha}{\tan t/2} \right] \left[ \sum_{k=0}^{n} \Delta_{n,k} \cos kt \right] dt \]

\[ = 0 \left[ \int_{\pi/n}^{\pi} t^{\alpha-1} \right] \left[ \sum_{k=0}^{n} \Delta_{n,n-k} \cos(n-k)t \right] dt \]
\[
= 0 \left[ \int_{\pi/n}^{\pi} t^{\alpha-1} \frac{\Delta c_{n, n-1}}{t} \, dt \right] \quad \text{(by Lemma)}
\]

\[
= 0 \left[ \int_{\pi/n}^{\pi} t^{\alpha-2} \Delta c_{n, n-1} \, dt \right]
\]

\[
= 0 \left[ \int_{n/\pi}^{\pi} \left( \frac{1}{t} \right)^{\alpha-2} \Delta c_{n, n-t} \left( -\frac{dt}{t^2} \right) \right]
\]

\[
= 0 \left[ \sum_{k=1}^{n} \frac{\Delta c_{n, n-k}}{k^\alpha} \right].
\]

Similarly as in the proof of \( I_2 \), we have

\[
I_3 = 0 \left[ \int_{\pi/n}^{\pi} t^{\alpha-1} \Delta c_{n, n-1} \, dt \right]
\]

\[
= 0 \left[ \sum_{k=1}^{n} \frac{\Delta c_{n, n-k}}{k^{\alpha+1}} \right]
\]

which is dominated by the bound for \( I_2 \).

Hence, combining \( I_i \) (\( i = 1, 2, 3 \)) we follow the proof of the theorem.

10.4 \textbf{REMARK} : It is to be noted that for

\[
\Delta c_{n, k} = \frac{p_{n-k}}{p_n} \quad (k \leq n)
\]

\[
= 0 \quad (k > n)
\]

our theorem B is equivalent to theorem A for regular Nörlund method.