CHAPTER IX

ON THE DEGREE OF APPROXIMATION TO A
FUNCTION BELONGING TO WEIGHTED $L^p, \mathcal{X}(t)$
CLAASS BY MEANS OF A CONJUGATE SERIES

9.1 DEFINITIONS AND NOTATIONS

Let $f$ be periodic with period $2\pi$ and integrable in the Lebesgue sense. Let its Fourier series be given by,

(9.1.1) $f(x) \sim \frac{3}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right).$

The conjugate series of the Fourier series (9.1.1) is given by,

(9.1.2) $\sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right).$

Let $\{p_n\}$ be a non-negative, non-increasing generating sequence for the $(N,p_n)$ method such that

(9.1.3) $P(n) \equiv P(n) = p_0 + p_1 + p_2 + \ldots + p_n \rightarrow \infty$ as $n \rightarrow \infty.$
We write
\[ p(y) = p[y] \quad \text{and} \quad P(y) = P[y] \]
where \([y]\) as usual denotes the greatest integer less than or equal to \(y\).

We define the norm \(|| \cdot ||_p\) by
\[ || f ||_p = \left[ \int_0^{2\pi} |f(x)|^p \, dx \right]^{1/p}, \quad p \geq 1, \]
and let the degree of approximation be given by\(^1\)
\[ \|f_n^*(f) = \min_{T_n} ||f - T_n||_p, \]
where \(T_n(x)\) is some \(n\)-th degree trigonometric polynomial.

Given a positive increasing function \(\psi_1(t)\) and an integer \(p > 1\), we say\(^2\) that \(f(x) \in \text{Lip}(\psi_1(t), p)\) if
\[ \left[ \int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right]^{1/p} = O(\psi_1(t)) \]
and that
\[ f(x) \in W(\mathbb{L}^p, \psi_1(t)) \]
\[ \left[ \int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{2p} x \, dx \right]^{1/p} = O(\psi_1(t)), \quad \mathcal{O} \]

\[ 1) \quad \text{Zygmund, A. (1).} \]
\[ 2) \quad \text{Siddiqui A. H. (1).} \]
In case $\beta = 0$, we notice that our newly defined class $W(L^p, \psi, (t))$ coincide with the class $\text{Lip}(\psi, (t), p)$.

9.2 INTRODUCTION: In chapter VIII, we proved the following theorem.

THEOREM A: If the sequence $\{p_n\}$ satisfies the following conditions:

\begin{equation}
\sum_{k=1}^{n} k |p_k - p_{k-1}| < C |p_n|
\end{equation}

then the degree of approximation of a function $f(x)$, conjugate to a periodic function $f$ with period $2\pi$ and belonging to the class of $\text{Lip} \alpha$, $0 < \alpha < 1$ by Nörlund means of its conjugate series is given by

\[
|f(x) - t_n(x)| = O \left[ \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}} \right]
\]

where $t_n(x)$ are the $(N, p_n)$ means of the series (9.1.2).

In this chapter, we generalize theorem A by proving the following:

*Compare Hille, E. and Tamarkin, J.D. (1)
3) Qureshi, K. (4).
THEOREM 4): \text{If a} 2\pi \text{ periodic function belongs to the class } W(L^p, \mathcal{C}_1(t)), \text{then its degree of approximation by Nörlund means of its conjugate series is given by}

\[ \| f(x) - \sum_n f_n(x) \|_p = O \left[ \mathcal{C}_1 \left( \frac{1}{n} \right)^{\beta + \frac{1}{p}} \right], \]

\text{provided } \mathcal{C}_1(t) \text{ satisfies the following:}

\begin{align*}
(1) & \quad \left[ \int_0^{\pi/n} \left[ \frac{t |\mathcal{C}_1(t)|}{\mathcal{C}_1(t)} \right]^p \sin^{\beta p} dt \right]^{1/p} = O \left( \frac{1}{n} \right) \\
(2) & \quad \left[ \int_{\pi/n}^{\pi} \left[ t^{-\delta} \frac{|\mathcal{C}_1(t)|}{\mathcal{C}_1(t)} \right]^p dt \right]^{1/p} = O \left( n^{\delta} \right)
\end{align*}

\text{where } \delta \text{ is an arbitrary number such that } q(1 - \delta) - 1 > 0, \text{ conditions (1) and (2) hold uniformly in } x \text{ and}

\[ \mathcal{C}_1(t) = f(x+t) - f(x-t), \]

\begin{align*}
(3) & \quad \left[ \int_0^{\pi/n} \left[ \frac{\mathcal{C}_1(t)}{t^{2+\beta}} \right]^q dt \right]^{1/q} = O \left[ \mathcal{C}_1 \left( \frac{1}{n} \right)^{\beta + \frac{1}{p}} \right],
\end{align*}

\text{where } \frac{1}{p} + \frac{1}{q} = 1 \text{ such that } 1 \leq p \leq \infty.

4) Qureshi, K. (9).
The following lemmas are known:

**Lemma A**\(^5\): If the sequence \(\{p_n\}\) is non-negative and non-increasing, then for \(\alpha > 0\)

\[
\frac{1}{n^\alpha} \leq \frac{1}{P_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}}.
\]

**Lemma B**\(^6\): If \(\{p_n\}\) is non-negative and non-increasing, then, for \(0 \leq a \leq b \leq \infty; 0 \leq t \leq \pi\) and any \(n\), we have

\[
\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leq p_{\frac{n}{t}}.
\]

9.3 Proof of the Theorem: Since

\[
S_k(x) - f(x) = -\frac{1}{\pi} \int_{0}^{\pi} \psi(t) \frac{\cos(k+1/2)t}{2 \sin t/2} \, dt,
\]

we have

\[
t_n(x) - f(x) = -\frac{1}{\pi} \int_{0}^{\pi} \psi(t) \frac{1}{\sum_{k=0}^{n} p_{n-k}} \cos(k+1/2)t \, dt.
\]

5) Sahney, B.N. and Goel, D.S. (1).
6) McFadden, L. (1).
\[ = - \frac{1}{\pi^{p^2}} \left[ \int_{-\pi/n}^{\pi/n} \int_{-\pi/n}^{\pi/n} \frac{\psi(t)}{t} \right. \]

\[ + \sum_{k=0}^{m} \frac{p_{n-k}}{t} \cos(k+1/2)t \ dt + o(1) \]

\[ = I_1 + I_2 + o(1), \text{ say.} \]

Applying Hölder's inequality and the fact that

\[ \psi(t) \in W^{p, q}(t), \]

we get

\[ I_1 = - \frac{1}{\pi^{p^2}} \int_{0}^{\pi/n} \int_{0}^{\pi/n} \frac{\psi(t)}{t} \sum_{k=0}^{m} \frac{p_{n-k}}{t} \cos(k+1/2)t \ dt \]

\[ \leq o \left( \frac{1}{p_n} \right) \left[ \int_{0}^{\pi/n} \left( \frac{\psi(t)}{\psi_1(t)} \right) \sin^2 t \right]^{p^2} dt \]

\[ \cdot \left[ \int_{0}^{\pi/n} \left( \frac{\psi_1(t)}{t^2} \right) \left[ \frac{p_{n-k}}{\sin^2 t} \right]^{q^2} \ dt \right]^{1/q} \]

\[ = o \left( \frac{1}{p_n} \right) \left( \frac{1}{n} \right) \left[ \int_{0}^{\pi/n} \left( \frac{\psi_1(t)}{t^2} \right) \left[ \frac{1}{\sin^2 t} \right]^{q^2} \ dt \right]^{1/q} \]

\[ \cdot \left[ \frac{n}{\sum_{k=0}^{m} p_{n-k}} \frac{1}{\sin^2 t} \right]^{q} dt \right]^{1/q} \]
\[
= o\left(\frac{1}{n}\right)^p \left[ \int_0^{\pi/n} \left[ \frac{\psi_1(t)}{t^\beta + 2} \right]^q \, dt \right]^{\frac{1}{q}}
\]

\[
= o\left(\frac{1}{n}\right)^p \left( \gamma_1\left(\frac{1}{n}\right)^\beta + 1 + \frac{1}{p} \right)
\]

\[
= o\left(\gamma_1\left(\frac{1}{n}\right)^\beta + \frac{1}{p} \right).
\]

Also, similarly, as above

\[
I_2 \leq o\left(\frac{1}{p}\right)^p \left[ \int_{\pi/n}^\pi \left| \frac{t^{-\delta \sin^2 t} \psi(t)}{(\gamma_1(t))} \right|^p \, dt \right]^{\frac{1}{p}}
\]

\[
\cdot \left[ \int_{\pi/n}^\pi \left| \sum_{k=0}^{n \, p_n - k} \frac{\cos(k+1/2)t \gamma_1(t)}{\sin^2 t \, t^{-\delta + 1}} \right|^q \, dt \right]^{\frac{1}{q}}
\]

\[
= o\left(\frac{1}{p}\right)^p \left[ \int_{\pi/n}^\pi \left| \frac{\psi(t)}{(\gamma_1(t))} \right|^p \, dt \right]^{\frac{1}{p}}
\]

\[
\cdot o\left[ \int_{\pi/n}^\pi \left| \frac{\gamma_1(t) P\left(\frac{1}{x}\right)}{t^{-\delta + 1} \sin^2 t} \right|^q \, dt \right]^{\frac{1}{q}} \text{(by Lemma B)}
\]

\[
= o\left(\frac{1}{p}\right)^p \left( n^\delta \right) \frac{1}{\sin^8(\pi/n)} o\left[ \int_1^n \left[ \frac{\psi_1\left(\frac{1}{y}\right) P(y)}{y^{\delta - 1}} \right] \, dy \right]^{\frac{1}{q}}
\]
\[
= o \left( \frac{1}{P_n} \right) o \left( n^{\delta} \right) o \left( \frac{P(n) \gamma_1(\frac{1}{n})}{\sin^{\beta}(\frac{1}{n})} \right) o \left[ \int_1^n \frac{1}{y^{\delta_q-q+2}} \, dy \right]^{\frac{1}{q}}
\]

\[
= o \left( n^{\delta} \right) o \left( \gamma_1(\frac{1}{n}) \frac{1}{\beta} \right) o \left( n^{\delta_q-q+1-\frac{1}{q}} \right)
\]

\[
= o \left( \gamma_1(\frac{1}{n}) n^{\beta + 1 - \frac{1}{q}} \right)
\]

\[
= o \left( \gamma_1(\frac{1}{n}) n^{\beta + \frac{1}{p}} \right).
\]

\[
= o \left[ \gamma_1(\frac{1}{n}) n^{\beta + \frac{1}{p}} \right].
\]

Hence

\[
| t_n(x) - f(x) | = o \left[ \gamma_1(\frac{1}{n}) n^{\beta + \frac{1}{p}} \right].
\]

Therefore

\[
\| f(x) - t_n(x) \|_p = o \left[ \left( \int_0^{2\pi} (\gamma_1(\frac{1}{n}) n^{\beta + \frac{1}{p}})^\frac{1}{p} \right)^p \right]
\]

\[
= o \left[ \left( \gamma_1(\frac{1}{n}) n^{\beta + \frac{1}{p}} \right) \left( \int_0^{2\pi} dx \right)^\frac{1}{p} \right]
\]
\[ \begin{align*}
&\quad = 0 \left[ (\frac{1}{n}^2 + \frac{1}{p}) \right] \left[ x^2 \right]_0^1 \]
&\quad = 0 \left[ \Psi \left( \frac{1}{n} \right) n^\beta + \frac{1}{p} \right].
\end{align*} \]

This completes the proof of the theorem.

9.4 COROLLARY: If \( p \to \pm, \beta = 0, \Psi(t) = t^\alpha \)

and using lemma B, we have theorem A.

PROOF:

\[ \| f(x) - t_n(x) \| = \left[ \int_0^{2\pi} |f(x) - t_n(x)|^p \, dx \right]^{1/p} \]
\[ = 0 \left[ \frac{1}{\Psi \left( \frac{1}{n} \right) n^\beta + \frac{1}{p}} \right] \]
\[ = 0 \left[ \frac{1}{\Psi \left( \frac{1}{n} \right) n^\beta + \frac{1}{p}} \right] .\]

Hence
\[ |f(x) - t_n(x)| = 0 \left[ \Psi \left( \frac{1}{n} \right) n^\beta + \frac{1}{p} \right] .\]

for if not the left hand side will not be \( O(1) \).
Taking $p \to \infty$, $\beta = 0$ and $\psi(t) = t^\alpha$, we have

$$| \bar{f}(x) - \bar{t}_n(x) | = O \left[ \frac{1}{n} \alpha \right]$$

$$= O \left[ \frac{1}{n} \sum_{k=1}^{p_n} \frac{p_k}{k^{\alpha+1}} \right]$$

(by lemma A).