CHAPTER VIII

ON THE DEGREE OF APPROXIMATION OF FUNCTIONS
BELONGING TO THE LIPSCHITZ CLASS BY MEANS
OF A CONJUGATE SERIES

8.1 DEFINITIONS AND NOTATIONS

Let

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

and

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)
\]

respectively denote the Fourier series and Conjugate Fourier series of a 2\pi periodic and Lebesgue integrable function \( f(x) \).

The corresponding sum function of (8.1.2) being

\[
-f(x) = -\frac{1}{2\pi} \lim_{t \to 0} \int_{-\infty}^{\infty} [f(x+t) - f(x-t)] \cot \frac{t}{2} \, dt
\]

which exists almost everywhere.

Let \( \{ p_n \} \) be a sequence of positive constants
such that

\[ p_n = p_0 + p_1 + p_2 + \cdots + p_n \xrightarrow{n \to \infty} \infty \text{ as } n \to \infty. \]

Suppose

\[ t_n = \frac{p_n s_0 + p_{n-1} s_1 + p_{n-2} s_2 + \cdots + p_0 s_n}{p_n}. \]  

(8.1.4)

If \( t_n \xrightarrow{n \to \infty} S \) as \( n \to \infty \), we say that

the sequence \( \{S_n\} \) is summable by Nörlund mean or

summable \( (N, p_n) \) to \( S \).

The class of all such series or functions given

by (8.1.1), (8.1.2) and (8.1.3) is denoted by \( \mathcal{I} \).

A point \( x \) for which \( f(x) \) has a finite definite value is said to

be \( (\mathcal{I}) \) regular if

\[ \gamma(t) = o(t) \]  

(8.1.5)

where

\[ \gamma(t) = f(x+t) - f(x-t), \]  

(8.1.6)

and \( f(x) \) exists and is finite.

We designate the set of \( (\mathcal{I}) \) regular points with

respect to a given function \( f(x) \) in the interval \((-\pi, \pi)\)

by \( E(\mathcal{I}; f) \).
The conjugate function \( f(x) \) is defined for almost every \( x \), in particular, for \( x \in \mathbb{E} (\tilde{I}, \tilde{f}) \), by

\[
(8.1.7) \quad f(x) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \gamma(t) \cot \frac{t}{2} dt.
\]

A function \( f \in \text{Lip}_\alpha \) if

\[
(8.1.8) \quad f(x+h) - f(x) = O(|h|^\alpha), \quad 0 < \alpha \leq 1
\]

8.2 INTRODUCTION

The well known theorem on the degree of approximation of a function \( f(x) \) belonging to the class \( \text{Lip}_\alpha \) by \( (C, \delta) \)-means of its Fourier series is given by Alexits\(^1\). Sahney and Goel\(^2\) have extended the above result to \( (N, p_n) \)-means. Later on Chandra\(^3\) has also studied similar problems for Riesz's \( (R, p_n) \)-means of its Fourier series.

Khan\(^4\) has proved the following theorem:

---

1) Alexits, G. (1).
2) Sahney, B.N. and Goel, D.S. (1).
3) Chandra, P. (1).
4) Khan, Huzoor H. (2).
THEOREM A: If \( \{ \Delta_{n,k}^1 \}_{k=0}^n \) is non-negative and non-decreasing sequence with respect to \( k \) and if

\[
\int_0^t |d \phi(u)| \leq A \psi(t) \text{ where } 0 \leq t \leq \delta
\]

then

\[
(8.2.1) \quad \sigma_n(x) - \tilde{f}(x) = O(\Psi(\frac{1}{n}))
\]

where \( \psi(t) \) is a positive increasing function such that

\[
\int_{\frac{1}{n}}^{\delta} \psi(t) \frac{dt}{t^2} = O(n \psi(\frac{1}{n})), \quad n \to \infty
\]

and

\[
\sigma_n(x) = \sum_{k=0}^n \Delta_{n,k}^1 S_k(x),
\]

where \( S_k(x) \) is the \( n \)-th partial sum of the series \((8.1.2)\).

In this chapter, we prove the following two similar theorems:

THEOREM B: If the sequence \( \{ p_n \} \) satisfies the following conditions:

5) Qureshi, K. (4),

*compare Hille, E. and Tamarkin, J. D. (1).*
\[(8.2.2) \quad n \mid p_n \mid < C \mid p_n \mid \]

\[(8.2.3) \quad \sum_{k=1}^{n} k \mid p_k - p_{k-1} \mid < C \mid p_n \mid \]

then the degree of approximation of a function \( \tilde{f}(x) \),
conjugate to a periodic function \( f \) with period \( 2\pi \)
and belonging to the class of \( \text{Lip}\alpha, \quad 0 < \alpha < 1 \) by
\((N,p_n)\)-means of its conjugate series is given by

\[(8.2.4) \quad \mid \tilde{f}(x) - \tilde{t}_n(x) \mid = O \left( \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}} \right) \]

where \( \tilde{t}_n(x) \) are the \((N,p_n)\)-means of the series \((8.1.2)\).

**Theorem 6.** The degree of approximation of a periodic
function \( f(x) \), conjugate to a periodic function \( f \) with
period \( 2\pi \) and belonging to the class \( \text{Lip}\alpha, \quad 0 < \alpha < 1 \) by
the Nörlund means of the conjugate series for \( f \) is
given by

\[(8.2.5) \quad \mid \tilde{t}_n[f(x),p_v] - \tilde{f}(x) \mid = O \left( \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}} \right) \]

where \( \tilde{t}_n[f(x),p_v] \) denotes the Nörlund mean of the series \((8.1.2)\)

6) Qureshi, K. (5).
In order to prove the above theorems, we need the following lemmas.

**Lemma A**\(^7\) : If the sequence \(\{p_n\}\) is positive and non-increasing then, for \(\alpha > 0\),

\[
\frac{1}{n^\alpha} \leq \frac{1}{P_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}}.
\]

**Lemma B**\(^8\) : If \(\{p_n\}\) is non-negative and non-increasing, then, for \(0 \leq a \leq b \leq \infty\), \(0 \leq t \leq \pi\) and any \(n\), we have

\[
\frac{b}{\sum_{k=a}^{b} p_k e^{i(n-k)t}} \leq P\left(\frac{1}{t}\right).
\]

**Lemma C**\(^9\) : If \(x \in E (\bar{a} ; \bar{b})\), then

\[
\lim_{C \to 0} \frac{1}{n} \int_{E} N_n(t) \psi(t) \, dt = o(1)
\]

where

\[
N_n(t) = \left(4\pi p_n \sin \frac{t}{2}\right)^{-1} \left(e^{-(n+\frac{1}{2})it} \sum_{k=0}^{n} p_ke^{ikt} + e^{\frac{1}{2}it} \sum_{k=0}^{n} p_ke^{-ikt}\right)
\]

---

\(^7\) Sahney, B.N. and Goel, D.S. (1).

\(^8\) McFadden, L. (1).

\(^9\) Hille, E. and Tamarkin, J.D. (1).
8.3 PROOF OF THEOREM B : Since

\[ -S_k(x) - f(x) = - \frac{1}{\pi} \int_0^\pi \psi(t) \frac{\cos \left( \frac{k+1}{2}t \right)}{2 \sin \frac{t}{2}} \, dt \]

where

\[ \psi(t) = f(x+t) - f(x-t), \]

we have

\[ -t_n(x) - f(x) = - \frac{1}{\pi} \int_0^\pi \psi(t) \frac{1}{p_n} \sum_{k=0}^{n-1} p_{n-k} \cos \left( \frac{k+1}{2}t \right) \frac{t}{2 \sin \frac{t}{2}} \, dt. \]

Therefore

\[ |f(x) - t_n(x)| \leq \frac{1}{\pi} \int_0^\pi \left| \psi(t) \right| \frac{1}{2 \sin \frac{t}{2}} \frac{1}{p_n} \sum_{k=0}^{n-1} p_{n-k} \cos \left( \frac{k+1}{2}t \right) \, dt \]

\[ = \frac{1}{\pi} \left[ \int_0^{\pi/n} + \int_{\pi/n}^\pi \right] \left| \psi(t) \right| \frac{1}{2 \sin \frac{t}{2}} \]

\[ \cdot \left| \frac{1}{p_n} \sum_{k=0}^{n-1} p_{n-k} \cos \left( \frac{k+1}{2}t \right) \right| \, dt \]

\[ = I_1 + I_2, \text{ say.} \]
Now
\[ I_1 = \frac{1}{\pi} \int_0^{\pi/n} \frac{|\psi(t)|}{2 \sin t/2} \left| \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} \cos \left( k+\frac{1}{2} \right) t \right| \, dt. \]

Since
\[ |\psi(t)| = O \left( t^\alpha \right), \]

\[ \frac{1}{\sin t/2} = O \left( \frac{1}{t} \right) \]

and
\[ \left| \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} \cos \left( k+\frac{1}{2} \right) t \right| \leq 1, \]

we have
\[ I_1 = O \left[ \int_0^{\pi/n} \frac{t^\alpha}{t} \, dt \right] \]
\[ = O \left[ \int_0^{\pi/n} t^{\alpha-1} \, dt \right] \]
\[ = O \left( \frac{1}{n^\alpha} \right) \]
\[ = O \left( \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}} \right) \text{ (by lemma A).} \]
Also

\[ I_2 = \frac{1}{\pi} \int_{\pi/n}^{\pi} \left| \frac{\Psi(t)}{2 \sin t/2} \right| \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} \cos(k+\frac{1}{2})t \, dt \]

\[ = 0 \left[ \frac{1}{p_n} \int_{\pi/n}^{\pi} \frac{t^\alpha}{n} \left| \sum_{k=0}^{n} p_{n-k} \cos(k+\frac{1}{2})t \right| \, dt \right] \]

\[ = 0 \left[ \frac{1}{p_n} \int_{\pi/n}^{\pi} \frac{t^\alpha-1}{n} \frac{1}{t} \, dt \right] \text{ (by lemma B)} \]

\[ = 0 \left[ \frac{1}{p_n} \int_{n/\pi}^{1} \left( \frac{1}{y} \right)^{\alpha-1} p(y) \left( -\frac{dy}{y^2} \right) \right] \]

\[ = 0 \left[ \frac{1}{p_n} \int_{n/\pi}^{1} \left( \frac{1}{y} \right)^{\alpha+1} p(y) \, dy \right] \]

\[ = 0 \left[ \frac{1}{p_n} \sum_{k=1}^{n} \frac{p_k}{k^{\alpha+1}} \right]. \]

This completes the proof of theorem B.

8.4 PROOF OF THEOREM C: Following Hille and Tamarkin\(^{10}\), we have for \( x \in E \left( \bar{I} ; \bar{f} \right) \)

\(^{10}\) Hille, E. and Tamarkin, J.D. (1), see equation 4.13.
\[
\begin{align*}
  t_n[f(x), p_v] - f(x) &= \lim_{C \to 0} \int_{\frac{1}{n}}^{\infty} \psi(t) N_n(t) \, dt \\
  &= \lim_{C \to 0} \int_{\frac{1}{n}}^{\infty} \psi(t) N_n(t) \, dt \\
  &\quad + \int_{\frac{1}{n}}^{\infty} \psi(t) N_n(t) \, dt.
\end{align*}
\]

Therefore
\[
\begin{align*}
  t_n[f(x), p_v] - f(x) &= o(1) + \int_{\frac{1}{n}}^{\infty} \psi(t) N_n(t) \, dt \\
  &\quad (\text{by lemma C}).
\end{align*}
\]

Since
\[
N_n(t) = 2 \left[ 4\pi P_n \sin \frac{t}{2} \right]^{-1} \left[ \sum_{k=0}^{n} p_k \cos \left( k-(n+\frac{1}{2})t \right) \right],
\]

we have
\[
\begin{align*}
  \left| t_n[f(x), p_v] - f(x) \right| &= 2 \left[ \left| \frac{1}{4\pi P_n} \int_{\frac{1}{n}}^{\infty} \frac{\psi(t)}{\sin \frac{t}{2}} \right| \sum_{k=0}^{n} p_k \cos[k-(n+\frac{1}{2})t] \right] dt.
\end{align*}
\]

Hence
\[
\begin{align*}
  \left| t_n[f(x), p_v] - f(x) \right| &= o \left[ \frac{1}{P_n} \int_{\frac{1}{n}}^{\infty} \frac{\left| \psi(t) \right|}{t} P(\frac{1}{t}) \, dt \right] \\
  &\quad (\text{by Lemma B}).
\end{align*}
\]
\[= 0 \left[ \frac{1}{p_n} \int_1^N \frac{t^\alpha}{t} \ p\left( \frac{1}{t} \right) \ dt \right]\]

\[= 0 \left[ \frac{1}{p_n} \int_1^N t^{\alpha-1} \ p\left( \frac{1}{t} \right) \ dt \right]\]

\[= 0 \left[ \frac{1}{p_n} \int_1^N \left( \frac{1}{y} \right)^{\alpha-1} \ p\left( \frac{1}{y} \right) \left( -\frac{dy}{y^2} \right) \right]\]

\[= 0 \left[ \frac{1}{p_n} \int_1^N \left( \frac{1}{y} \right)^{\alpha+1} \ p\left( \frac{1}{y} \right) \ dy \right]\]

\[= 0 \left[ \frac{1}{p_n} \sum_{k=1}^n \frac{p_k}{k^{\alpha+1}} \right].\]

This completes the proof of theorem C.

8.5 \underline{REMARK}: Theorem similar to theorem 1 and theorem 2 have been proved in chapter V (theorem 1) but slight change in the proof had improved the result.