CHAPTER III

ON $(J,P_n)$ SUMMABILITY OF FOURIER-SERIES
3.1 Let $P_n > 0$ be such that $\sum_{n=0}^{\infty} P_n$ diverges, and the radius of convergence of the power series.

$$(3.1.1) \quad P(X) = \sum_{n=0}^{\infty} P_n X^n$$

be unity. Given an infinite series $\sum a_n$ with the sequence of partial sums $\{s_n\}$

$$(3.1.2) \quad P_{s}(X) = \sum_{n=0}^{\infty} P_n s_n X^n$$

and

$$J_{s}(X) = \frac{P_{s}(X)}{P(X)}$$

If the series on the right of (3.1.2) is convergent in the right open interval $(0,1)$; and if

$$\lim_{x \to 1-0} J_{s}(x) = S,$$

We say that the series on the right of (3.1.2), or sequence $\{s_n\}$ is summable $(J, P_n)$ to $S$, where $(S)$ is finite.}

Particular cases of $(J, P_n)$ method of summability

(i) The $(A_K)$ method: when $P_n$ is given by

$$(1-x)^{-K-1} = \sum_{n=0}^{\infty} P_n x^n, \text{ for } K > -1, (|x| < 1)$$

1. Hardy, G.H. (1)
(ii) \( (L) \) method of summability:

\[
-x^{-1} \log (1-x)^{-1} = \sum_{n=0}^{\infty} p_n x^n
\]

when \( p_n = \frac{1}{n+1} \), \((J, P_n)\) method reduces to \((L)\) method of summability which was introduced for the first time by Borwein\(^2\).

(iii) The Abel method\(^3\):

when \( p_n = 1 \) for all \( n = 1, 2, \ldots \)

3.2\(^4\) Let \( f(x) \) be integrable \( L \) in \( (-\pi, \pi) \) and periodic with period \( 2\pi \), and

Let

\[
(3.2.1) \quad f(x) = \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

\[= \frac{1}{2} a_0 + \sum_{1}^{\infty} A_n(x)\]

Then the differentiated series of (3.2.1)

\[
(3.2.2) \quad \sum_{1}^{\infty} n (b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} n B_n(x)
\]

\[
(3.2.3) \quad \text{We write, fixing } x_0, \Psi(t) = f(x_0 + t) - f(x_0 - t),
\]

2. Borwein, D. (1)

3. Hsiang, F.C. (1)
\[(3.2.4) \quad g(t) = \frac{\Psi(t)}{4 \sin t/2} - C.\]

where $C$ is a function of $x$.

Let $S_n$, $\sigma_n$ and $t_n$ be respectively the $n^{th}$ partial sum, the first Cesaro mean, and the first logarithmic mean of the series (3.1.2), so

\[(3.2.5) \quad S_n = \sum_{r=1}^{n} r \cdot B_r(x)\]

\[(3.2.6) \quad \sigma_n = \left( S_1 + S_2 + \ldots + S_n \right)/n\]

\[(3.2.7) \quad t_n = \left( S_1 + \frac{1}{2} S_2 + \ldots + \frac{n-1}{n} S_n \right)/\log n\]

**DEFINITION**

The series (3.1.1) is said to be summable $(R, \log n, 1)$ to $C$ provided that $t_n \to C$ as $n \to \infty$.

3.3 Hsiang proved $(L)$ - Summability for Fourier series of a function $f(t)$ in the following form.

**Theorem A:** If

\[(3.3.1) \quad \int_{0}^{t} |\phi(u)| \, du = O(t \log 1/t) \quad (t \to 0)\]

\[(3.3.2) \quad \int_{t}^{\infty} |\phi(u)| \, du = O(\log 1/t), \quad (t \to + \infty)\]


5. Hsiang, F.C. (1)
for any arbitrary \( 0 < \xi < \kappa \), then the Fourier series of \( f(t) \) is summable \((L)\) to \( S \) at \( t = x \).

Mohanty and Nanda\(^6\) proved the following theorems for the first derived Fourier series.

**THEOREM B** : If

\[
(3.3.3) \quad \int_{t}^{\infty} \left| g(u) \right| \frac{du}{u} = O \left( \log \frac{1}{t} \right), \quad (t \to + \infty)
\]

Then

\[
\sum_{n=1}^{\infty} n B_n(x) \text{ is summable } (L) \text{ to } C.
\]

Improving the results of theorems A and B, Nanda\(^7\) proved the following theorems.

**THEOREM C** : If

\[
(3.3.4) \quad \phi(t) = \int_{t}^{\pi} \frac{\phi(u)}{u} du = O \left( \log \frac{1}{t} \right), \quad (t \to + \infty).
\]

Then the series (3.2.1), for \( t = x \) is summable \((L)\) to \( S \).

**THEOREM D** : If

\[
(3.3.5) \quad \mathcal{G}(t) = \int_{t}^{S} \frac{g(u)}{u} du = O \left( \log \frac{1}{t} \right), \quad (t \to + \infty)
\]

Then the series (3.2.2) is summable \((L)\) to the value \( C \).


7. Nanda, M. (1)
On generalizing the results C & D of Nanda, Ved-Prakash proved the following theorem for \((J, P_n)\) summability of Fourier-series.

**THEOREM E:**

Let the sequence \(\{P_n\}\) be positive and decreasing steadily to zero, such that \((n, P_n)\) is bounded.

If

\[
(3.3.3) \quad \Phi(t) = \int_{t}^{\pi} \frac{\Phi(u)}{u} \, du = O(P(1-t)) \quad (t \to +0)
\]

then the series \((3.2.1)\) for \(t = X\) is summable \((J, P_n)\) to \(S\).

The aim of the present paper is to prove the above theorem for \((J, P_n)\) summability of conjugate series.

There we will prove the following theorem.

**Theorem 1:** If

\[
\Psi(t) = \int_{t}^{\pi} \frac{\Psi(u)}{u} \, du = O[P(1-t)] \quad [\text{as } t \to +0]
\]

then the conjugate series \((3.2.2)\) for \(t = X\) is summable \((J, P_n)\) to \(S\).

**3.4 PROOF OF THEOREM**

Denoting the \(n\)-th partial sum of the conjugate series by \(S_n\), we have

8. Ved Prakash (1) (Chapter V)
\[ \overline{S}_n(X) = \frac{1}{n} \pi \int_0^{\pi} \psi(t) \frac{\cos \frac{1}{2} t - \cos (n+\frac{1}{2}) t}{\sin \frac{1}{2} t} dt \]

Therefore,

\[ (3.4.1) \quad \overline{S}_n(X) = \frac{1}{n} \pi \int_0^{\pi} \psi(t) \cot \frac{1}{2} t \ dt \]

\[ = \frac{1}{n} \pi \int_0^{\pi} \psi(t) \frac{\cos(n+\frac{1}{2}) t}{\sin \frac{1}{2} t} \ dt \]

Then the \((J,P_n)\) transform of \((3.4.1)\) is

\[ = \frac{1}{n} \pi \sum_{n=0}^{\infty} P_n x^n \int_0^{\pi} \psi(t) \cos(n+\frac{1}{2}) t \left(1-\frac{1}{2} \cos t\right) dt \]

\[ = \frac{1}{n} \left[ \int_0^{\pi} \psi(t) \left(1-x\right) \cos \frac{1}{2} t \ dt \right] \]

\[ - \int_0^{\pi} \psi(t) \cos \frac{1}{2} t \ dt \]

\[ = I_2 - I_1, \text{ Say} \]

Obviously

\[ (3.4.3) \quad I_2 = O(P(x)) \]

\[ (3.4.4) \quad I_1 = \left( \int_0^{\pi} + \int_{1-x}^{\pi} \right) \psi(t) \left(1-x\right) \cos \frac{1}{2} t \left[1-2x \cos t + x^2\right] dt \]

\[ = I_{1.1} + I_{1.2}, \text{ Say} \]

Hence
\[ I_{1.1} = \int_{0}^{1-x} \frac{\psi(t)}{\sin \frac{1}{2} t} (1-X) t \cos \frac{1}{2} t \frac{(1-X) t \cos \frac{1}{2} t}{1-2 X \cos t + x^2} \, dt \]

Where \( g(t) = \frac{\psi(t)}{\sin \frac{1}{2} t} \)

\[ = \left[ \frac{(1-X) t \cos \frac{1}{2} t}{1-2 X \cos t + x^2} g'(t) \right]_{0}^{1-x} \]

\[ + (1-X) \int_{0}^{1-x} g(t) \frac{d}{dt} \left[ \frac{t \cos \frac{1}{2} t}{1-2 X \cos t + x^2} \right] \, dt \]

\[ = O(P(X)) + (1-X) O(P(X)) \]

\[ = 0 \left[ P(x) \right] \]

(3.4.5) \[ = 0 \left[ P(x) \right] \]

Also,

\[ I_{1.2} = \int_{1-x}^{1} \frac{g'(t)}{(1-X) t \cos \frac{1}{2} t} \frac{(1-X) t \cos \frac{1}{2} t}{1-2 X \cos t + x^2} \, dt \]

\[ = [g(t) \frac{(1-X) t \cdot \cos \frac{1}{2} t}{1-2 X \cos t + x^2} \left. \right|_{1-x}^{1-x}] \]

\[ + (1-X) \int_{1-x}^{1} g(t) \frac{d}{dt} \left[ \frac{t \cdot \cos \frac{1}{2} t}{1-2 X \cos t + x^2} \right] \, dt \]
(3.4.6) \[ = O(P(X)) \]

as in \( I_{1.1} \)

Hence \( I_1 = O(P(X)) \)

So collection of (3.4.1), (3.4.2), (3.4.3) & (3.4.5), (3.4.6) completes proof of theorem.