CHAPTER III

SOME AXIOMS WEAKER THAN THE $R_0$-AXIOM

In his paper [2], A.S. Davis discussed a hierarchy of separation axioms. He called them 'Regularity Axioms'. He showed there that the $R_0$-axiom belonging to the said hierarchy is independent of the $T_0$-axiom but strictly weaker than $T_1$ and that $T_1 = R_0 + T_0$. In fact, any separation axiom of the type of 'Regularity Axioms' can be seen to be of the following character.

(1) It is independent of the $T_0$-axiom.
(2) It is strictly weaker than a standard $T_1$-axiom with which it coincides, in the presence of the $T_0$-axiom.

In the recent past (as we have seen in the preceding chapter II) the concept of $R_0$-axiom has established itself as a powerful and pervading separation axiom in various contexts of general topology. This inspired us to search for various axioms weaker than the $R_0$-axiom. We confine ourselves, here, only to those axioms which are weaker than the $R_0$-axiom. This provides an extension of the hierarchy of Regularity Axioms.
In [1], C.E. Aull and W.J. Thron introduced a number of separation axioms, like $T_\text{ys}$, $T_y$, $T_D$, $T_{UD}$ and other axioms lying between the axioms $T_D$ and $T_1$. In this chapter we introduce a set of axioms namely $R_\text{ys}$, $R_y$, $R_D$ and $R_{UD}$. These axioms are found to be of the type of 'Regularity Axioms', corresponding to the axioms $T_\text{ys}$, $T_y$, $T_D$ and $T_{UD}$, respectively. Each of these introduced axioms is independent of $T_0$ as well as strictly weaker than $R_0$. It may be mentioned, however, that in $T_0$-spaces these axioms $R_\text{ys}$, $R_y$, $R_D$ and $R_{UD}$ become equivalent to the corresponding axioms $T_\text{ys}$, $T_y$, $T_D$ and $T_{UD}$, respectively. We analyse their inter-relationships. In the last section of this chapter we show how one of our axioms is found to be of some interest in the context of the theory of proximity spaces.

1. Definitions and Notations.

Recall that $\text{cl}\{x\}$, $\{x\}'$ and $\text{ker}\{x\}$ stand for the closure, the derived set and the kernel of $\{x\}$; and that $\text{cl}\{x\} = \overline{x} = \{y : y \upharpoonright \overline{x} \}$, $\{x\} = \{y : y \upharpoonright \{x\}, y \neq x \}$, $\text{ker}\{x\} = \{y : x \upharpoonright \{y\} \}$; where by $\{x\} \upharpoonright \{y\}$ we shall mean that $\{x\}$ is not weakly separated from $\{y\}$.

The following classes of topological spaces were introduced by C.E. Aull and W.J. Thron [1].
Definition 1.1. : A topological space \((X, \tau)\) is said to be a \(T_D\)-space iff for every \(x \in X\), \([x]\) is closed.

Definition 1.2. : A topological space \((X, \tau)\) is said to be a \(T_{UD}\)-space iff for every \(x \in X\), \([x]\) is the union of disjoint closed sets.

Definition 1.3. : A topological space \((X, \tau)\) is said to be a \(T_y\)-space iff for all \(x, y \in X\), \(x \neq y\), \(\text{cl-}[x] \cap \text{cl-}[y]\) is degenerate.

Definition 1.4. : A topological space \((X, \tau)\) is said to be a \(T_{ys}\)-space iff for all \(x, y \in X\), \(x \neq y\), \(\text{cl-}[x] \cap \text{cl-}[y]\) is either \(\emptyset\) or \([x]\) or \([y]\).

Definition 1.5. [2] : A topological space \((X, \tau)\) is said to be an \(R_0\)-space iff any one of the following equivalent conditions holds:

(a) For any open set \(G\) in \(X\), \(x \in G \implies \text{cl-}[x] \subseteq G\).

(b) For any \(x, y\) in \(X\), either \(\text{cl-}[x] = \text{cl-}[y]\) or \(\text{cl-}[x] \cap \text{cl-}[y] = \emptyset\).

2. Some New Axioms.

We know that in an \(R_0\)-space \((X, \tau)\) for any \(x\) and \(y\), \(\text{cl-}[x] \neq \text{cl-}[y] \implies \text{cl-}[x] \cap \text{cl-}[y] = \emptyset\) (refer to definition
1.5 (b) above). This suggests the introduction of the following axioms.

**Definition 2.1.** A topological space \((X, \mathcal{T})\) will be called an \(R_\text{ys}\)-space iff for all \(x\) and \(y\) in \(X\),
\[
\text{cl-} \{x\} \neq \text{cl-} \{y\} \implies \text{cl-} \{x\} \cap \text{cl-} \{y\} \text{ is either } \emptyset \text{ or } \{x\}\text{or}\{y\}.
\]

**Definition 2.2.** A topological space \((X, \mathcal{T})\) will be called an \(R_y\)-space iff for all \(x\) and \(y\) in \(X\),
\[
\text{cl-} \{x\} \neq \text{cl-} \{y\} \implies \text{cl-} \{x\} \cap \text{cl-} \{y\} \text{ is degenerate.}
\]

We have seen the following result in theorem 2.2 of chapter II.

"A topological space \((X, \mathcal{T})\) is an \(R_\text{o}\)-space iff for any \(x\) in \(X\), \(\text{cl-} \{x\} \subseteq \text{ker-} \{x\}\)".

As an immediate consequence of the above result we have the following lemma which will be useful in the next study.

**Lemma 2.1.** In an \(R_\text{o}\)-space \((X, \mathcal{T})\) if for any \(x \in X\),
\[
\text{cl-} \{x\} \cap \text{ker-} \{x\} = \{x\}, \text{ then } \text{cl-} \{x\} = \{x\}.
\]

Now, the above lemma (2.1) suggests the introduction of the following axioms.
Definition 2.3. : A topological space \((X, \mathcal{T})\) will be called an \(R_D\)-space iff for all \(x\) in \(X\),
\[
\overline{x} \cap \ker\{x\} = \{x\} \implies \{x\} \text{ is closed.}
\]

Definition 2.4. : A topological space \((X, \mathcal{T})\) will be called an \(R_{UD}\)-space iff for all \(x\) in \(X\),
\[
\overline{x} \cap \ker\{x\} = \{x\} \implies \{x\} \text{ is the union of disjoint closed sets.}
\]

We, now, proceed to examine the inter-relationships between these new concepts. Infact, we have the following theorems.

Theorem 2.1. : (a) \(R_o \implies R_{ys} \implies R_y\).

(b) \(R_o \implies R_D \implies R_{UD}\).

The above implications are the consequences of the definitions themselves and lemma 2.1. The easy proofs are omitted.

Theorem 2.2. : \(R_y \implies R_{UD}\).

Proof : Let \((X, \mathcal{T})\) be an \(R_y\)-space and \(x \in X\) be such that \(\overline{x} \cap \ker\{x\} = \{x\}\). This means for any \(z \neq x\) in \(X\),
\[
z \in \overline{x} \implies z \notin \ker\{x\} \implies x \notin \overline{z} \implies \overline{\{z\}} = \{z\} \implies \overline{\{x\}} \neq \overline{\{z\}}.
\]
Also, \(z \in \overline{x} \implies \overline{z} \subseteq \overline{x}\). But, in the \(R_y\)-space
(X, \mathcal{T}), \text{cl}\{-x\} \neq \text{cl}\{-z\} \implies \text{cl}\{-x\} \cap \text{cl}\{-z\} \text{ must be a degenerate set. It follows that we must have cl}\{-z\} = \{z\}\). Thus, \(z \in \text{cl}\{-x\} \implies \text{cl}\{-z\} = \{z\}, \text{if } z \neq x\). We can therefore write as follows:

\[
\{x\}' = \bigcup \{\{z\}: z \neq x, z \in \text{cl}\{-x\}\}.
\]
\[
= \bigcup \{\text{cl}\{-z\}: z \neq x, z \in \text{cl}\{-x\}\}.
\]
\[
= \text{Union of disjoint closed sets.}
\]

Hence, \(R^Y \implies R^U_D\).

Next, we shall give a few examples which further analyse the implications between our new axioms.

**Example 2.1.** Let \(X = \{a, b, c\}\),
and \(\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}\).
Then, the space \((X, \mathcal{T})\) is \(R_D\) and \(R^y_s\) but not \(R^D_0\).

**Example 2.2.** Let \(X = \{a, b, c\}\),
and \(\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}\).
Then, the space \((X, \mathcal{T})\) is \(R_D\) and so is \(R^U_D\) but not \(R^y\).

**Example 2.3.** Let \(X = \{a, b, c\}\),
and \(\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\).
Then, the space \((X, \mathcal{T})\) is \(R_D\) and \(R^y\) but not \(R^y_s\).
Example 2.4. : Let $X$ be the set of all real numbers. Let $\emptyset$, $X$, and the sets $C_\alpha$ defined by $x \geq \alpha$ be closed for each real $\alpha$. This is a $T_0$-space but not an $R_{UD}$-space.

Example 2.5. : Let $X$ be the set of all reals. In addition to the whole set $X$ and null set $\emptyset$, let all sets $\{x\}$ for $x \neq 0$ and all finite unions of $\{x\}$ for $x \neq 0$ be the closed sets. Then the space is $R_{ys}$ but not an $R_D$-space.

Proof. We note that, for any point $x$ other than 0 in $X$, $\text{cl-} \{x\} = \{x\}$ and also $\text{cl-} \{0\} = X$ and so $\{0\}' = X \setminus \{0\}$. Again, for any distinct $x$ and $y$ other than 0 in $X$, we have $\text{cl-} \{x\} \cap \text{cl-} \{y\} = \emptyset$, and $\text{cl-} \{x\} \cap \text{cl-} \{0\} = \{x\}$. Therefore, for all $x, y$ in $X$, $\text{cl-} \{x\} \neq \text{cl-} \{y\}$ implies $\text{cl-} \{x\} \cap \text{cl-} \{y\}$ is either $\emptyset$ or $\{x\}$ or $\{y\}$. That is, the space is $R_{ys}$.

Finally, since $\text{cl-} \{0\} \cap \text{ker-} \{0\} = \{0\}$ and $\{0\}' = X \setminus \{0\}$, which being infinite is not closed, the space $(X, \tau)$ is not $R_D$.

Remark 2.1. : The following diagram shows the implications between $R_0$, $R_{ys}$, $R_D$, $R_Y$ and $R_{UD}$ axioms. We have seen in earlier examples that in general, no other implications will be true.
Remark 2.2. : The axioms, introduced in this chapter, are strictly weaker than the $R_o$-axiom and the example (2.4) shows that the $T_0$-axiom may fail to imply the $R_{UD}$-axiom. However, analogous to the result $R_o + T_0 = T_1$, we have the following results:

1. $R_{ys} + T_0 = T_{ys}$.
2. $R_y + T_0 = T_y$.
3. $R_D + T_0 = T_D$.
4. $R_{UD} + T_0 = T_{UD}$.

These follow in virtue of the fact that the following statements hold in a $T_0$-space:

(a) For all $x, y$ in $X$, $x \neq y \implies cl-\{x\} \neq cl-\{y\}$.
(b) For all $x$ in $X$, $cl-\{x\} \cap ker-\{x\} = \{x\}$.

Remark 2.3. : It is known that normality, in general, does not imply the regularity, and the converse may also not be true. However, we have already noted that, normality together with $R_o$-axiom implies regularity.
Naturally, a question arises whether there exists an $R_1$-axiom strictly weaker than the $R_0$-axiom such that normality together with $R_1$-axiom implies the $R_0$-axiom and so implies the axiom of regularity. Unfortunately, none of the axioms introduced in this chapter can serve this purpose.

Remark 2.4. : We have shown that the $R_D$-axiom implies the $R_{UD}$-axiom but, the converse may not be true, as seen in example (2.5). But in a normal space both $R_D$-axiom and $R_{UD}$-axiom are equivalent. This is so, because in a normal space, $\{x\}$ can not contain two disjoint closed sets.

3. Some Further Results on the $R_D$-Space.

It is known that every topological space $(X, \mathcal{T})$ has a compatible LE-proximity [4] and every $R_0$-space has a compatible LO-proximity [5]. On this line it is shown here that an LE-proximity structure on a nonempty set $X$ and satisfying an additional Hayashi-type of axiom induces an $R_D$-topology on $X$.

Definition 3.1. : A binary relation $\delta$ defined on the power set of a nonempty set $X$ is called an LE Leader or LE-proximity on $X$ iff it satisfies the following axioms:
Axiom \( (P_1) \) : \( A \delta (B \cup C) \) iff \( A \delta B \) or \( A \delta C \).

Axiom \( (P_2) \) : \( (A \cup B) \delta C \) iff \( A \delta C \) or \( B \delta C \).

Axiom \( (P_3) \) : \( A \delta B \) implies \( A \neq \emptyset \), \( B \neq \emptyset \).

Axiom \( (P_4) \) : \( A \delta B \) and \( b \delta C \) for each \( b \in B \) together implies \( A \delta C \).

Axiom \( (P_5) \) : \( A \cap B \neq \emptyset \) implies \( A \delta B \).

The pair \((X, \delta)\) is called an LE-proximity space.

The above terminology is due to S.A.Naimpally [6]. He used the terms LE-proximity and LO-proximity spaces in place of 'Topological d-space' and 'Symmetric generalized' proximity spaces, respectively. It may be mentioned that the definition of LE-proximity was first given by S.Leader [4] and that of LO-proximity by M.W. Lodato [5].

We prove the following theorem.

**Theorem 3.1.** If an LE-proximity \( \delta \), defined on a nonempty set \( X \), satisfies the axiom \( (PH_\_1) \) given below and if \( \mathcal{T} = \mathcal{T}(\delta) \) is the induced topology on \( X \), then \((X, \mathcal{T})\) is an \( R_D \)-space.

Axiom \( (PH_\_1) \) : \( x \delta \bigcup_{\lambda \in \Lambda} B_\lambda \) iff there exists an index \( \mathcal{N} \) such that \( x \delta B_\mathcal{N} \), \( \mathcal{N} \in \Lambda \).
Proof : Let us first note that,

$$\text{cl-}\{x\} = \{y : y \in x\}.$$  
$$\text{ker-}\{x\} = \{y : x \in y\}.$$  
$$\{x\}' = \{y : y \neq x, y \in x\}.$$ 

Now, we have to prove that for any $x$ in $X$, if $\text{cl-}\{x\} \cap \text{ker-}\{x\} = \{x\}$, then $\{x\}'$ is closed. If $\{x\}'$ is empty, then there is nothing to prove. Hence, it is no restriction to let $\{x\}' \neq \emptyset$. Now, it will suffice to show that, then, $\text{cl-}\{x\}' \subseteq \{x\}'$; that is to show that for such an $x$ and any $z \in X$, $z \in \{x\}'$ implies that $z \in \{x\}'$. This shall be done by proving the following two propositions separately.

(A) : If $z \neq x$, then $z \in \{x\}' \implies z \in \{x\}'$.

(B) : If $z = x$, then $x \notin \{x\}'$.

Proof of (A): Let $z \neq x$. Since $y \in \{x\}'$ implies that $y \in x$, therefore, for any $z$ in $X$, such that $z \in \{x\}'$, we have by the axiom $(P_4)$, $z \in x$. This implies that $z \in \{x\}'$.

Proof of (B): Let $z = x$.

If $x \in \{x\}'$, that is equivalently if $x \in \bigcup_{y \in \{x\}'} \{y\}$, then $\{x\}'$ being nonempty there exists a point $y$, by the axiom $(P_{H_1})$ such that $y \in \{x\}'$ and $x \in y$ (where $y \neq x$). Now,
since \( x \notin y \) implies that \( y \in \ker\{x\} \) and \( y \notin \{x\}' \) implies that \( y \in \text{cl}\{x\} \), therefore, \( y \in \text{cl}\{x\} \cap \ker\{x\} \) which is against the assumption that \( \text{cl}\{x\} \cap \ker\{x\} = \{x\} \). Hence, \( x \notin \{x\}' \).

Combining (A) and (B) it follows that for any \( x \) in \( X \) such that \( \text{cl}\{x\} \cap \ker\{x\} = \{x\}', \{x\}' \) is closed. Hence the theorem follows.

**Remark 3.1.** : Let us consider the following axiom:

Axiom \((\text{PH}_2)\) : \( A \delta \cup B_\chi \) if there exists an index \( \chi \) such that \( A \delta B_\chi \), \( \chi \in \Lambda \).

Evidently, the axiom \((\text{PH}_2)\) is stronger than the axioms \((\text{PH}_1)\) and \((\text{PH}_1)\). Therefore, in the above theorem the axiom \((\text{PH}_2)\) may be used in place of the two axioms \((\text{PH}_1)\) and \((\text{PH}_1)\). The axiom \((\text{PH}_2)\) is of the type of the axiom III of paraproximity introduced by E. Hayashi [3].

**Theorem 3.2.** : In a topological space \((X, \tau)\), if for each \( x \in X, \{x\}' \) is finite then \((X, \tau)\) is an \( R_D \)-space.

**Proof** : Since every topological space has a compatible proximity, the proof of the above theorem is the same.
as that of the theorem 3.1, except that, we have used in place of the axiom $(PH_1)$, the axiom $(P_1)$ which works due to the finiteness of $\{x\}'$ for each $x \in X$.

**Corollary 3.1.** Every finite topological space $(X, \tau)$ is an $R_D$-space.

**Corollary 3.2.** Every finite $T_0$-topological space $(X, \tau)$ is a $T_D$-space.

Some characterizations of the $R_D$-axiom will be presented in Chapter VI at a place where they appear in a natural manner.
REFERENCES


