CHAPTER I

SEPARATION AXIOMS

A general topological space is too general for many purposes. Consequently, in order to achieve more deeper and fruitful results up to a certain extent, some extra conditions are frequently imposed on the space considered. Amongst these restrictions, the separation axioms which are of our present interest, are basically most common.

One may begin by asking the question — 'what is the central idea that gives rise to the introduction of various types of separation axioms which occur in a topological space and links them in a natural way'? In the algebra of sets any two disjoint sets may be taken to be separated, while in topology 'separation' implies more than mere disjointness. A motivating idea of separation axiom is to make the points and sets of a space topologically distinguishable. What we called separation axioms, enable us to state with precision that a topological space has a rich enough supply of open sets to serve the purpose we may have in mind. The supply of open sets possessed by a topological space is intimately linked to the supply of continuous functions on it; in general, more open sets there are, the more continuous functions a space has. One may think of a
discrete space on which all functions are continuous but it goes too far. The separation axioms make it possible for us to ascertain that our spaces have enough continuous functions without being discrete.

A separation axiom may be correspondingly attached to a certain type of requisite distinguishability amongst points and various kinds of sets. One may have an immediate conjecture of infinitely many separation axioms, but it is merely a bit of an ideal. In fact only a few of them are distinct and worthwhile to study.

1.1. Various Degrees of Separation

In the context of general topology, the concept of separation axioms permits us, on the one hand, to distinguish between points, or points and sets, or between sets and sets, and on the other hand to restrict the generality of the space so that it may be able to possess certain 'nice' properties. To achieve these objectives, various separation axioms were introduced in the past few decades and various degrees of separation type properties were recognised.

In this section we list some of the degrees of separation studied by various authors.
(1) Any set $A$ is said to be weakly separated from a set $B$ if there exists an open neighbourhood of $A$ disjoint from $B$ i.e., $A \cap \text{cl-}B = \emptyset$.

(2) Any two sets $A$ and $B$ are said to be topologically distinguishable if at least one of the sets is weakly separated from the other.

(3) Any two sets $A$ and $B$ are said to be weakly separated if each of the sets is weakly separated from the other.

(4) Any two sets $A$ and $B$ are said to be strongly separated if the sets $A$ and $B$ have disjoint open neighbourhoods.

(5) Any two sets $A$ and $B$ are said to be strongly c-separated if $A$ and $B$ have disjoint closed neighbourhoods.

(6) Any two points $x,y$ (respectively any $x$ and a set $A$ with $x \notin A$) are said to be functionally distinguishable if there exists a real valued continuous function $f$ such that $f(x) \neq f(y)$ (respectively $f(x) \notin f(A)$).
(7) Any two sets $A$ and $B$ are said to be completely separated if there exists a real valued continuous function $f$ such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$.

1.2. The Heirarchy of $T_1$-axioms

If we think of topological distinguishability, weak separation, strong separation, strong c-separation and complete separation of every pair of distinct points $x, y$ in any space $X$ the resultant axioms thus obtained are $T_0', T_1', T_2', T_{2a}', CT_2'$ respectively. These are purely points separation axioms.

If we shift our interest to an arbitrary pair of a point $x$ and a closed set $F$ such that $x \notin F$, then seeking weak separation of $x$ from $F$, and their topological distinguishability as well, is unnecessary, for they are already so separated. Of course, the consideration of weak separation and strong separation of each pair of a closed set $F$ and a point $x$ not in $F$ is meaningful and this gives rise to different axioms what are known as, the axioms of $R_0$ and regularity, respectively. Again, strong c-separability of a pair of a point $x$ and a closed set $A$, with $x \notin A$ leads to no new separation axiom because this type of separation is a consequence of strong separability of $x$ and $A$ in a regular space.
Now, consider the separation of a pair of disjoint closed sets. We know that disjoint closed sets are weakly separated, and strongly c-separated closed sets are strongly separated. Strong separation of each pair of disjoint closed sets leads us to the normality axiom. Further, in a space, normality and strong c-separation of disjoint closed sets are equivalent properties. In a space if each pair of weakly separated sets is strongly separated also then the space is said to possess the axiom of complete normality. A normal space in which each closed set is $G_δ$-set is known as perfectly normal.

It remains to discuss the distinguishabilities in which the 'separation' is sought through the continuous functions. By considering the functional distinguishability of any pair of distinct points and that of any given closed set $A$ and $x$, not in $A$, we have the axioms which are called the completely Hausdorff and the $x ∉ A$ axioms, respectively. In a space any pair of distinct points $x, y$ which are functionally distinguishable, are completely separated, and in a normal space, any pair of disjoint closed sets are completely separated. Therefore a consideration of complete separation between any closed set $A$ and $x$ not in $A$, is natural. This becomes especially interesting for it introduces another separation axiom, the axiom of complete regularity. It may be mentioned that the axioms of
regularity and complete regularity together with $T_0$-axiom give rise to what are known as respectively $T_3$ and $T_{3a}$ (or $T_{3^*}$) spaces. A space satisfying the $T_{3a}$ axiom is usually called a Tychonoff space.

Now we give below the implications that exist amongst the various $T_1$-axioms discussed above.

\[(1) \quad T_6 \Rightarrow T_5 \Rightarrow T_4 \Rightarrow T_{3a} \Rightarrow T_3 \Rightarrow T_{2a},\]

\[(2) \quad T_{3a} \Rightarrow CT_2 \Rightarrow T_{2a} \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0;\]

where $T_6 = T_1 +$ perfectly normal, $T_5 = T_1 +$ completely normal, $T_4 = T_1 +$ normal.

1.3. Axioms Between $T_0$ and $T_1$

A number of separation axioms between $T_0$ and $T_1$ were investigated by C.E. Aull and W.J. Thron in [12]. It was observed there, that all of them can be stated in terms of the concept of derived sets.

By $\text{cl-} \{x\}$, $\{x\}'$ we shall mean the closure and the derived set of $\{x\}$, respectively and $\text{ker-} \{x\}$ is defined to be the set $\{y : x \in \text{cl-} \{y\}\}$. 
It is easy to see that in a $T_1$-space any two disjoint finite sets are weakly separated and that in a $T_0$-space any two disjoint finite sets may fail, in general, to be topologically distinguishable. This consideration led Aull and Thron [12] to the introduction of two separation axioms which are known to be $T_F$ and $T_{FP}$. Also it was noted that every homogeneous $T_F$ space is a $T_1$-space.

A space $X$ is called $T_F$ iff any one of the following equivalent conditions is satisfied.

(a) Every $x$ and every finite set $F$ such that $x \not\in F$ are topologically distinguishable.

(b) For any $x \in X$, $y \in \{x\}'$ implies $\{y\}' = \emptyset$.

(c) For all $x, y \in X$, $x \neq y$, either $\text{cl-}\{x\} \cap \text{cl-}\{y\}$ or $\text{ker-}\{x\} \cap \text{ker-}\{y\}$ is degenerate.

Again a space $X$ is known to be $T_{FP}$ iff any one of the following equivalent conditions is satisfied.

(a) Any two disjoint finite sets are topologically distinguishable.

(b) Either of the following cases holds
(i) \( \ker\{x\} \subseteq \{x\} = \emptyset \) for all but at most one \( x \in X \).

(ii) \( \{x\}' = \emptyset \) for all but at most one \( x \in X \).

It was noted by C.T. Yang (see J.L. Kelley page 56) that in a space the derived set of each set is closed iff the derived set of each point is closed. A space in which the derived set of every point is closed is defined to be a \( T_D \)-space [12]. Two axioms namely \( T_{UD} \) and \( T_{DD} \) each of which is a slight variation of \( T_D \)-axiom were introduced in [12]. A \( T_{UD} \)-space is a space in which the derived set of every point is the union of disjoint closed sets. A \( T_{DD} \)-space is a \( T_D \) space such that the derived sets of any two distinct points are disjoint.

A \( T_D \) space is characterized as a space in which every point can be expressed as the intersection of an open set and a closed set. It is interesting to note that every finite \( T_0 \)-space is a \( T_D \)-space and that every discrete space of Alexandroff is a \( T_D \)-space. Further it was shown by W.J. Thron [67] that if any two \( T_D \)-spaces are lattice equivalent, then they are homeomorphic. \( T_D \)-spaces were also studied by R.E. Larson [44] and K.H. Pahk [54] in the context of minimal topologies. Larson has shown that a \( T_D \)-space is minimal \( T_D \) iff finite unions of point-closures are point-closures and that any subspace of a minimal \( T_D \)-space is minimal \( T_D \). S.M.
Robinson and Y.C. Wu [57] have shown that if \( \{X_i : i \in I\} \) is an infinite collection of \( T_D \) spaces which are not \( T_1 \) then their product space is not a \( T_D \)-space. Further, using the cardinality restriction they have investigated a hierarchy of separation axioms weaker than \( T_D \). For an infinite cardinal \( m \) they call a space \( X \) to be a \( T^{(m)} \) space provided that for each \( x \in X \), \( \{x\} = F \cap (\cap \{O_i : i \in I\}) \), where \( F \) is closed, each \( O_i \) is open and \( \text{card}(I) = m \).

Another axiom between \( T_0 \) and \( T_1 \) was introduced by J.W.T. Youngs [73] in connection with a study of locally connected spaces. He defined a space \( X \) to be a \( T_y \) space iff for all \( x, y \in X \), \( x \neq y \), \( \text{cl-}\{x\} \cap \text{cl-}\{y\} \) is degenerate. Aull and Thron [12] have shown that a space is \( T_y \) iff for all \( x, y \in X \), \( x \neq y \), \( \text{ker-}\{x\} \cap \text{ker-}\{y\} \) is degenerate; and have also introduced the axiom \( T_{ys} \), a slightly stronger version of \( T_y \)-axiom.

A space is said to be \( T_{ys} \) iff any one of the following equivalent conditions is satisfied.

(a) For all \( x, y \in Y \), \( x \neq y \), \( \text{cl-}\{x\} \cap \text{cl-}\{y\} \) is either \( \emptyset \) or \( \{x\} \) or \( \{y\} \).

(b) The derived sets of any two distinct points are weakly separated.
(c) The closures of the derived sets of any two distinct points are disjoint.

Among the various axioms between \( T_0 \) and \( T_1 \), discussed above, the following implications are known:

\[
\begin{align*}
(1) \quad & T_1 \implies T_{DD} \implies T_D \implies T_{UD} \implies T_0, \\
(2) \quad & T_1 \implies T_{FF} \implies T_Y \implies T_F \implies T_{UD}, \\
(3) \quad & T_1 \implies T_{DD} \implies T_{YS} \implies T_Y, \\
(4) \quad & \text{Normality} + T_{UD} \implies T_D, \\
(5) \quad & \text{Normality} + T_F \implies T_Y, \\
(6) \quad & \text{Complete normality} + T_F \implies T_{DD}.
\end{align*}
\]

In [12] Aull and Thron raised a question whether there exists a separation axiom which is weaker than \( T_1 \) and at the same time it implies \( T_1 \)-axiom in the presence of normality. This problem has been solved affirmatively in S. M. Kim [43], S. M. Robinson and Wu [57], and Y.K. Choudhary and B.C. Singhai [16]. S.M. Kim has given four axioms, namely \( T_\alpha \), \( T_\beta \), \( T_\alpha \), and \( T_\beta \), each of which serves the purpose. Robinson and Wu [57] have noted two conditions which they call strong \( T_0 \) and strong \( T_D \), both of which are weaker than \( T_1 \)-axiom and give the \( T_1 \)-axiom in the presence of normality. For the sake of brevity, we give below only the axioms of Robinson and Wu [57].
**Strong $T_0$** A space $X$ is a strong $T_0$ provided that, for each $x \in X$, $\{x\}'$ is either empty or is the union of non-empty closed sets such that the intersection of this family is empty and at least one of the non-empty members is compact.

**Strong $T_D$** A space $X$ is strong $T_D$ if for each $x \in X$, $\{x\}'$ is empty or is a union of a finite family of nonempty closed sets such that the intersection of this family is empty.

Keeping the same objective in mind, another axiom, $T_{C_0}$, which is given below, was introduced by Choudhary and Singhai [16]. It is shown there that $T_{C_0}$-axiom is weaker than both the axioms strong $T_0$ and strong $T_D$; and is independent of the $T_0$-axiom and implies the $T_1$-axiom in the presence of normality.

**$T_{C_0}$-axiom** A space $X$ is said to satisfy the $T_{C_0}$-axiom iff for every $x \in X$, either $\{x\}'$ is empty or $\{x\}'$ contains two nonempty disjoint closed sets.

In fact, they have noted that the $T_{C_0}$-axiom implies $T_1$-axiom in the presence of any one of the so-called normality axioms (the normality axioms are discussed in the section 1.5 of this chapter).
1.4. The Heirarchy of Regularity Axioms

As an equivalent reformulation of the $R_0$-axiom we know that any two topologically distinguishable points are weakly separated. This leads us to think of the other types of 'separation' between topologically distinguishable points. The axioms, by which an arbitrary pair of topologically distinguishable points are strongly separated or strongly c-separated, are known, respectively, as $R_1$ and $R_{1a}$. It has already been noted above that the functional distinguishability coincides with complete separation, so far as pairs of distinct points are concerned. These considerations led K.K.Dube and D.N.Misra [24] to introduce the completely $R_1$-axiom under which topologically distinguishable points are functionally distinguishable.

Since in a $T_0$-space each pair of distinct points is topologically distinguishable, therefore, in such a space, the axioms completely $R_1$, $R_{1a}$, $R_1$, $R_0$ are equivalent to the axioms completely $T_2$, $T_{2a}$, $T_2$, $T_1$ respectively.

The following implications, among the various regularity axioms discussed above, are obvious due to their definitions.
where $\text{CR} = \text{complete regularity}$, $\text{CR}_1 = \text{completely } R_1$
$\text{CT}_2 = \text{completely Hausdorff}$, $R = \text{regularity}$,

$R_0$-axiom was initially introduced by N.A. Shanin [58] and rediscovered by A.S. Davis [19]. Further, $R_1$-axiom was introduced by Davis [19] and also independently by B. Banaschewski and J. M. Maranda [14] in a different context. More results on these axioms may be seen in [20], [23], [50]. In [19] A.S. Davis called the axioms $R_0$, $R_1$ and that of regularity — 'The hierarchy of regularity axioms'. The axiom $\times T_\delta A$ discussed earlier was introduced by W. T. Van Est and H. Freudenthal [27]. The work of C.E. Aull [11] also deserves mention in this context.

Further, in [22], K.K. Dube and D.N. Misra have recently introduced some axioms what are called $R_Y$, $R_{YS}$ $R_D$, $R_{UD}$. Each of these axiom is independent of the $T_0$-axiom but strictly weaker than the $R_0$-axiom. This provides also
an extension of the hierarchy of 'regularity axioms'. Some more on $R_D$-spaces may be seen in [23] and [25].

A topological space $X$ is said to be an

1. $R_{YS}$-space iff for all $x,y$ in $X$, $cl\{-x\} \neq cl\{-y\}$ implies $cl\{-x\} \cap cl\{-y\}$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.

2. $R_y$-space iff for all $x,y$ in $X$ $cl\{-x\} \neq cl\{-y\}$ implies $cl\{-x\} \cap cl\{-y\}$ is degenerate.

3. $R_D$-space iff for all $x$ in $X$, $cl\{-x\} \cap ker\{-x\} = \{x\}$ implies $\{x\}'$ is closed.

4. $R_{UD}$ space iff for all $x$ in $X$, $cl\{-x\} \cap ker\{-x\} = \{x\}$ implies $\{x\}'$ is the union of disjoint closed sets.

It has been noticed that in a $T_o$ space, the axioms $R_y$, $R_{YS}$, $R_D$ and $R_{UD}$ respectively become equivalent to the corresponding axioms $T_y$, $T_{YS}$, $T_D$, $T_{UD}$ of Aull and Thron [12]. In a normal space the $R_{UD}$ axiom coincides with the $R_D$ axiom. Further the following implications have been found to be true.

1. $R_o \implies R_{YS} \implies R_y$.

2. $R_o \implies R_D \implies R_{UD}$.

3. $R_y \implies R_{UD}$. 
1.5. The Heirarchy of Normality Axioms.

Some variants of the standard separation axioms are obtained by substituting point-closure sets in place of points or closed sets in the $T_1$-type of separation axioms which have been discussed earlier.

By a point-closure set we mean a set which is the closure of a singleton set.

In order to have a distinct axiom, the consideration of weak separation, strong separation, strong c-separation and functional separation of each pair of a point $x$ and a point-closure set $A$ with $x \notin A$ in the space leads us no where, for, these types of separations amount to a reformulation of the $R_0$, $R_1$, $R_{1a}$, and completely $R_1$-axioms, respectively. In the normality axiom if one of the closed sets in a pair of disjoint closed sets is replaced by a point-closure set then the resulting axiom is known as the $N_2$-axiom, i.e., in a space $N_2$, an arbitrary disjoint pair of closed set and a point-closure set is strongly separated. The 'complete' separation between each disjoint pair of a point-closure set and a closed set is known as the $N_2a$-axiom. The strong separation and the strong c-separation for each pair of disjoint point-closure sets have been recorded as distinct separation axioms and are called $N_1$ and $N_{1a}$, respectively.
Also, a space $X$ is known to satisfy the $N_0$-axiom iff for each $x \in X$, $\{x\}$ does not contain two nonempty disjoint closed sub sets. If $N$ stands for normality then it is easy to see that

$$N \implies N_{2a} \implies N_2 \implies N_{1a} \implies N_1 \implies N_0.$$ 

The $N_1$-axioms discussed above are due to S.C. Choudhary and B.C. Singhai [16]. There they call these axioms 'The heirarchy of normality axioms'. It is interesting to note that an axiom of the heirarchy of normality axioms, in the presence of the $R_0$-axiom, gives rise to the corresponding axiom of the heirarchy of regularity axioms, and that in the presence of $T_1$-axiom, gives rise to the corresponding axiom of the heirarchy of $T_1$-axioms.

It may be added here that the axioms $N_1$ and $N_2$ were introduced as early as 1954 by C.T. Yang [72], where he called them $T_2'$ and $T_3'$ conditions respectively and he had shown there that in a paracompact space each of these axioms is equivalent to normality.

1.6. Separation for Compact Sets.

In 1965, C.E. Aull [7] introduced and studied the separation axioms which have been concerned more with the nature of sets that are being separated. In this
important class of separation axioms the compact sets replace points and/or closed sets in the statement of separation axioms. Following his notation the letter \( J \) is used to indicate that a point has been replaced by a compact set and the letter \( K \) is used to indicate that a closed set has been replaced by a compact set.

A space \( X \) in which topological distinguishability of each pair of a compact set \( M \) and \( x \in X - M \) (resp. each pair of disjoint compact sets) is observed, is said to satisfy the \( J_0 \)-axiom (respectively, \( J J_0 \)-axiom). The axiom \( J_1 \) by which any pair of a compact set \( M \) and \( x \) with \( x \notin M \) is weakly separated is found to be equivalent to the axiom \( JJ_1 \) in which any pair of disjoint compact sets are weakly separated. Again, \( J_1^* \)-axiom by which each compact set is closed is also equivalent to the \( J_1 \)-axiom. The axiom \( J_2 \) \((=K_3)\) that enables the strong separation of each compact set \( M \) and point \( x \in X - M \) in the space \( X \), and the axiom \( JJ_2 (=JK_3 = KK_4) \) that enables the strong separation of any pair of disjoint compact sets were also found to be each equivalent to the Hausdorff axiom. The axiom \( J_3 (=K_4) \) analogous to the \( T_3 \)-axiom in the present context is defined to be \( T_1 \) and such that it provides the strong separation of each disjoint pair of a closed set and a compact set. Again, in \( KK_5 \) axiom, analogous to the \( T_5 \)-axiom, any two sets weakly separated through compact sets are strongly separated.
Here the term 'weakly separated through compact sets' means that each is contained in a compact set disjoint from the other.

Applying the above technique of C.E. Aull [9], in case of the axiom of complete regularity and the completely Hausdorff axiom, no new axiom is obtained. However, it may be added that the axiom analogous to the $T_{2a}$-axiom, i.e., the axiom $J_{2a}$ in which each pair of a compact set $M$ and point $x \in (X \setminus M)$ in the space $X$ is strongly $c$-separated is a new one and needs to be studied. We feel that in order to extend the class of separation axioms the substitution of paracompact sets or that of any other compact type of sets in place of the points and / or closed sets in the statement of standard separation axioms may also be worth studying.

To summarize the relations between above $J_1$ and $K_1$ axioms, the implications are given below.

(1) $J_1 = J'_1 = JJ_1$, $T_2 = J_2 = JJ_2$, $T_3 = J_3$,
(2) $J_1 \Rightarrow JJ_0 \Rightarrow J_0$,
(3) $T_5 \Rightarrow KK_5 \Rightarrow T_2 \Rightarrow J_1 \Rightarrow (JJ_0 + T_1)$,
(4) $(JJ_0 + T_1) \Rightarrow T_1 \Rightarrow J_0 \Rightarrow T_F \Rightarrow T_0$,
(5) In a compact space, $KK_5 \Rightarrow T_5$ and $J'_0 \Rightarrow T_D$,
(6) In a second countable space, $(T_1 + JJ_0) \Rightarrow KK_5$. 
It may be mentioned here that the condition $J_1$ was discussed earlier by E. Hewitt [39] and R. Vidyathanaswamy [68], in connection with minimal Hausdorff and maximal compact spaces. A compact Hausdorff topology is maximal compact but not conversely. It was noted by A. Ramanathan [55][56] (see also [9]) that a $J_1$, compact space is maximal compact and that conversely a maximal compact space is $J_1$. This shows the advantage that $J_1$ spaces have over $T_2$ in this situation.

Again, the $J_1$ spaces were also studied by E. Halfar [35], H.F. Cullen [18], A. J. Insell [40], N. Levin [45] and A. Wilansky [71]. Levin called $J_1$ spaces 'CC' and Wilansky called them 'KC' spaces. So it was natural that a considerable amount of duplication in the study of these spaces occurred. A. Wilansky has presented a systematic account of the study of these spaces in [71]. H.F. Cullen [18] has shown that in a $J_1$-space every convergent sequence has exactly one limit to which it converges.

E. Halfar [35] has shown that a necessary and sufficient condition for a first countable space to be Hausdorff is that it be a $J_1$-space. S.P. Franklin [23] gives an example of a Fréchet, $J_1$, non $T_2$-space. Again, Wilansky has given an example of a compact, $J_1$, non $T_2$-space and shown that if a $J_1$-space $X$ is such that the family of compact neighbourhoods of each of its points is a local base then it is a
Hausdorff space. Again, he has shown that in the one point compactification space of a $J_1$-space, every sequence converges to at most one point; and further that the one point compactification space $X^*$ of a $J_1$-space $X$ is a $J_1$-space if and only if $X$ is a $K$-space. By a $K$-space is meant a space in which each subset $A$ such that $A \cap K$ is closed for all closed and compact sets $K$, is itself closed.

1.7. **Separation for Some Special Type of Closed Sets.**

In the context of general topology various sets viz. regular closed sets, zero sets, $G_5$-sets play an interesting role in their own right and consequently the separation property for these sets have been investigated from time to time. First we will consider the 'separation' type of properties involving regular closed sets.

If, on the one hand in the axioms of regularity and complete regularity the closed set, and on the other, in the normality axiom one of the two closed sets is replaced in their definitions by a regular closed set, then the axioms so obtained were studied by Singal and his associates [60],[61]. They named them, respectively, the axioms of almost regularity, almost complete regularity and almost normality. Let us denote them by $AR$, $ACR$ and $AN$, respectively, for the sake of
convenience. Pursuing the same approach corresponding to
the $R_0$-axiom one may think of and make a study of the
almost $R_0$-axiom by which a regular closed set and a point
not belonging to it are weakly separated. If $AR_0$ stands
for the almost $R_0$-axiom then $ACR \implies AR \implies AR_0$; and in an
almost $R_0$-space, $AN \implies ACR$.

In 1970, some weak normality type of properties
dealing with the separation of the $G_5$-type of sets were
systematically studied by J. Mack [46] with in the context
of an $\mathcal{M}$-paracompact space. In a space, given an infinite
cardinal number $\mathcal{M}$ a set $A$ is defined to be a $G_{\mathcal{M}}$-set
(respectively, a regular $G_{\mathcal{M}}$-set) if it is the intersection
of at most $\mathcal{M}$ open sets (respectively, at most $\mathcal{M}$ closed sets
whose interiors contain $A$). In particular, if $\mathcal{M} = \aleph_0$,
then these sets become what are known as $G_5$-sets (respec-
tively, regular $G_5$-sets). It is well known that a zero
set of any real valued continuous function is a regular
$G_5$-set and that the intersection of not more than $\mathcal{M}$ such
zero sets is a regular $G_{\mathcal{M}}$-set.

For any infinite cardinal $\mathcal{M}$, a space is said
to be $\mathcal{M}$-normal if each pair of disjoint closed sets of
which one is a regular $G_{\mathcal{M}}$-set, is strongly separated;
and in particular an $\aleph_0$-normal space is termed as a $\delta$-normal
space. It is noted that a normal space is $\mathcal{M}$-normal and that a regular space is normal iff it is $\mathcal{M}$-normal for every cardinal $\mathcal{M}$. However, $\mathcal{M}$-normality together with the Hausdorff axiom may fail to be normal. It is interesting to note that each $\mathcal{M}$-paracompact space is $\mathcal{M}$-normal and hence each countably paracompact space is $\delta$-normal. Recently in [25], some suitable variants of regularity and normality have been shown to hold in a paracompact space.

Again, if each closed set and each zero set disjoint from it are completely separated, then the space is known to be $\delta$-normally separated. A space will be termed weakly $\delta$-normally separated if each regular closed set and each zero set disjoint from it are completely separated. Clearly each normal space is $\delta$-normally separated. Likewise $\delta$-normal separation implies weak $\delta$-normal separation and the converse is true for $\delta$-normal spaces. In [74] P. Zenor introduced the concept of $\delta$-normal separation which he named as the property \( '\&' \). In general, in a space the properties \( ' \) of being $\delta$-normally separated and $'\delta$-normality' are not comparable. If in a space every $G_\delta$-set is a zero-set, then $\delta$-normal separation implies $\delta$-normality but conversely each countably compact space is both $\delta$-normal and $\delta$-normally separated. Also a completely regular Pseudocompact space is weakly $\delta$-normally separated.
1.8. Separation Type of Properties Related to Sequences.

Following C.E. Aull [9] a space $X$ is said to be $S_0$ if every convergent sequence has exactly one limit to which it converges. These spaces were also studied by H.F. Cullen [18], M.G. Murdeshwar and S.A. Naimpally [49], P. Slepian [62], A. Wilansky [71]. Murdeshwar and Naimpally [49] have called such spaces semi-Hausdorff and Wilansky [71] called them by the name US spaces. The $S_0$-axiom (or the semi Hausdorff axiom) is weaker than the $T_2$-axiom but stronger than the $T_1$-axiom. Examples of a $T_1$, non $S_0$-space and of a compact, $S_*$, non $T_2$-space may be seen in [49] and [71] respectively. In [49] the basic properties of semi-Hausdorff spaces were studied and it is shown there that a space $X$ is semi-Hausdorff iff the diagonal set is sequentially closed. A set $B$ in a space $X$ is called sequentially closed if no sequence in $B$ converges to a point in $X \setminus B$. Again, in [9] such a space is characterized as a space in which every countable filter has at most one limit point.

In another paper [10], C.E. Aull has shown that every countably paracompact, first countable, $T_2$-space is $T_3$ and that a space with $\sigma$-locally finite base is metrizable iff it is $T_2$ and countably paracompact. It has been also pointed out there that one may replace the $T_2$-condition, in the latter result, by any condition which together with
first countable axiom implies $T_2$, such as $S_0$-axiom, or $J_1$-axiom. This is true because H. F. Cullen [18] has shown that every $J_1$-space (i.e. a space in which each compact set is closed) is an $S_0$-space and that every first countable, $S_0$-space is Hausdorff. Regarding one point compactification the following two results were noted by A. Wilansky [71].

(1) One point compactification space of a $J_1$-space is an $S_0$-space.

(2) One point compactification of an $S_0$-space $X$ is $S_0$ iff every convergent sequence in $X$ has a relatively compact subsequence, where by a relatively compact sequence we mean that the closure of the set of points of the sequence is compact.

To formulate the various axioms related to sequences the concept of a side point and that of a highly divergent sequence have been found to be useful. We use here again the terminology of C. E. Aull [9]. A highly divergent sequence is a sequence which has no convergent subsequence; and a point $y$ is said to be a side point of a sequence $\{x_n\}$ if $y$ is an accumulation point of the set values of $\{x_n\}$ but no subsequence of $\{x_n\}$ converges to $y$. 
Now, some of the axioms related to sequences are listed below. A topological space $X$ is said to satisfy

$S_0$  If sequences converge to atmost one point.

$S_1$  If $S_0$ is satisfied and every convergent sequence has a subsequence without side points.

$S_2$  If $S_0$ is satisfied and no convergent sequence has a side point.

$S_3$  If every sequentially compact set is closed.

$S_4$  If every sequentially countably compact set is closed.

$S_5$  If $S_0$ is satisfied and every sequentially closed set is closed.

$S_6$  If $S_0$ is satisfied and for $x \in X$, $M \subset X$ such that $x \in M'$, there is a sequence $\{x_n\}$ of points of $M$ converging to $x$.

$H_1$  If $S_0$ is satisfied and every highly divergent sequence has a subsequence without side points.

$H_2$  If $S_0$ is satisfied and no highly divergent sequence has a side point.

All of the above axioms are still not regarded as the separation axioms but the relations of some of them with the separation axioms are among the earliest to be
studied. Here it may be noted that if a given topology satisfies \( S_0, S_1, S_2, S_3, S_4 \) axioms, then any finer topology also satisfies them.

For a discussion of classes of spaces satisfying \( S_5 \) and \( S_6 \), see R. M. Dudley [26] and S. P. Franklin [28]. An example of a non Hausdorff space which satisfied \( S_6 \)-axiom may be seen in M. Fréchet ([30], page 213).

The following implications amongst various \( S_i \) and \( H_i \) axioms discussed above can be summarised as follows.

(1) \( S_6 \Rightarrow S_5 \Rightarrow S_4 \Rightarrow S_3 \Rightarrow S_2 \Rightarrow S_1 \Rightarrow S_0 \).

(2) \( S_5 \Rightarrow H_2 \Rightarrow H_1 \).

(3) \( H_1 + S_3 \Rightarrow S_4 \).

(4) \( T_2 \Rightarrow J_1 \Rightarrow S_2 \Rightarrow S_1 \Rightarrow S_0 \Rightarrow T_1 \).

1.9. Separation Type Properties Involving Enumerability.

We know that a space is a \( T_1 \)-space iff each point of the space is the intersection of all open sets containing it and that a space is \( T_2 \) iff each point of the space is the intersection of all closed neighbourhoods of it. These characterizations of \( T_1 \) and \( T_2 \)-axioms led
C. E. Aull [8] to the introduction of the following two separation type properties which he called \( E_0 \) and \( E_1 \) respectively.

\( E_0 \) A topological space is said to be an \( E_0 \)-space if every point of the space is a \( G_\delta \).

\( E_1 \) A topological space is said to be an \( E_1 \)-space if every point of the space is the intersection of a countable number of closed neighbourhoods.

The \( E_0 \)-spaces were studied by several mathematicians. For instance, a summary of these results may be seen in [6]. Recently, \( E_1 \)-closed and minimal \( E_1 \)-spaces have been studied by M.K. Singh and Asha Mathur [59]. Aull [8] established various results concerning some of the relations of these \( E_0 \) and \( E_1 \)-axiom to the first countable axiom, locally countably compact, locally countably paracompact properties and some well known separation axioms.

Clearly, every \( E_1 \) space is \( E_0 \) and \( T_2 \). The converse is not true in general. Every \( E_0 \)-space is \( T_1 \). An example of a space which is \( E_0 \) and \( T_2 \) but not \( E_1 \) is provided in [8]. However, it is shown there that every first countable \( T_2 \)-space is \( E_1 \). This result is a sharpening of the fact that first countable \( T_3 \)-space is \( E_1 \). An \( E_0, T_2 \),
Lindelöf space is again an $E_1$-space. In a first countable space the $T_1$-axiom implies the $E_0$-axiom.

Every regular, $E_0$ space is $E_1$. This is so because in a regular space every open neighbourhood of a point contains a closed neighbourhood. It is interesting to note that every countably paracompact subset in an $E_1$ space is closed and that every locally countably paracompact $E_1$-space is $T_3$. Concerning locally countably compact spaces it is known that every locally countably compact $E_1$-space is first countable; and that every regular, $E_0$, locally countably compact space is first countable.

1.10. Separation Type of Properties Involving Base or Subbase Condition.

In many cases it is more convenient to deal with a base or subbase for a topology rather than the topology itself. For example, a space whose topology has a countable base has many pleasant properties. There will generally be many different bases and subbases for a topology and the appropriate choice may depend on the problem under consideration. A base or subbase is useful only if its sets are simple in form or a few in number.
The device of bases and subbases provides also, in a natural way, some interesting ways of characterizing some separation properties. For instance, a space is regular iff closed neighbourhoods of each point form a neighbourhood base, and it is completely regular iff zero-set neighbourhoods of each point form a neighbourhood base. Further, a space is normal iff each closed set has a closed neighbourhood base for its neighbourhood system.

In his paper [47], E. Michael introduced and studied an important class of topological spaces which he called $\mathcal{K}_0$-spaces. There a collection $\mathcal{U}$ of subsets (not necessarily open) of a space $X$ is called a 'pseudo base' for $X$ if whenever $A \subset G$, with $A$ compact and $G$ open in $X$, then $A \subset U \subset G$ for some $U \in \mathcal{U}$. Again, a collection $\mathcal{U}$ is called a 'point-pseudo base' if whenever $x \in G$ with $G$ open in $X$, then $x \in U \subset G$ for some $U \in \mathcal{U}$. A $T_3$-space with a countable pseudo base is called an $\mathcal{K}_0$-space and if a $T_3$-space has a countable point-pseudo base then it is called 'Cosmic'. Both these classes of spaces have proved to be considerably interesting and need to be studied further.

We take up next a brief discussion of the concept of semi regularity and allied spaces. In a semi regular space, regular open sets form a base for the topology. Clearly, every regular space is semi regular but not conversely. A space is
said to be, in an analogous manner, semi-normal if given any closed set \( F \) and an open set \( G \) containing \( F \) there exists a regular open set \( U \) such that \( F \subset U \subset G \); or equivalently, if given any closed set \( F \) and any closed set \( H \) disjoint from \( F \), then exists an open set \( U \) such that \( F \subset \text{cl-}U \) and \( H \cap \text{cl-}U = \emptyset \). It can be easily seen that semi-normality amounts to saying that each closed set \( F \) has a base consisting of regular open sets for all the open sets which contain \( F \). Some interesting results on semi-normality were obtained by several authors among which we mention here the work of G. Viglino [69],[70].

It should be remembered that the term 'semi-normal' which we have discussed above, is also used in the literature for a space which has a normal base. In his paper [31], O. Frink introduced the notion of a normal base to formulate his method of Hausdorff compactification of Tychonoff spaces. For more material on normal bases we may refer to the papers of Alo and Shapiro [2],[3],[4] and [5]. A ring of sets is a family of sets which is closed under finite unions and finite intersections. In a \( T_1 \)-space \( X \), a base \( \mathcal{B} \) for closed sets is called disjunctive if given any closed set \( F \) and any point \( x \) not in \( F \), there exists a closed set \( A \in \mathcal{B} \) such that \( x \in A, A \cap F = \emptyset \). The base is said to be separating if given any two disjoint sets \( A, B \in \mathcal{B} \) there exists members \( C \) and \( D \) of \( \mathcal{B} \) such that
A \subseteq (X - C), B \subseteq (X - D) and (X - C) \cap (X - D) = \emptyset. Now, a base \mathcal{B} for the closed sets of a T_1-space X is called a normal base if it is a disjunctive ring of closed sets that is also separating.

O. Frink showed in [31], that if a T_1-space X has a normal base \mathcal{B}, then the associated Wallman space \omega(\mathcal{B}) consisting of the \mathcal{B} ultrafilters is a Hausdorff compactification of X. More generally speaking, by choosing different normal bases \mathcal{B} for a noncompact T_1-space X, different Hausdorff compactifications may be obtained. It follows that a T_1-space X with a normal base is Tychonoff and conversely that if a space X is Tychonoff then the family of all zero sets forms a normal base. In fact, as an internal characterization of Tychonoff spaces he showed there that a T_1-space is completely regular iff it has a normal base.

These ideas led Alo and Shapiro [4] to introduce the concept of countably productive normal bases (i.e. normal bases closed under countable intersections) to study the corresponding problem of real compactification. The motivation for recent work along this direction has been provided by the problem: 'Is every Hausdorff compactification (respectively, real compactification) a Wallman compactification (respectively, Wallman real compactification)?' Besides the work of Alo and Shapiro important work in this direction has been done by B. Banaschewski [13], A.K. Steiner and E.P. Steiner [64], [65] and by M. Gagrát and S.A. Naimpally [32], [33].
In the paper [34] J. de Groot and J. Aarts remarked that although the axiom of complete regularity is a separation axiom, in none of its usual forms does it look like an intrinsic separation axiom. With the object of making it 'look like one', they obtained such characterizations of complete regularity which fit in naturally between regularity and normality. Following their terminology any two subsets $A$ and $B$ of a set $X$ are said to be screened by the pair $(C,D)$ if $C \cup D = X$, $A \cap D = \emptyset$, and $B \cap C = \emptyset$, (consequently $A \subseteq C$ and $B \subseteq D$). Again, the sets $A$ and $B$ are said to be screened by a finite family $\mathcal{C}$, if $\mathcal{C}$ covers $X$ and each element of $\mathcal{C}$ meets (i.e. has nonempty intersection with) at most one of $A$ and $B$. Assuming the spaces to be $T_1$, they introduced the characterizations (1) and (2) of complete regularity given below. Further, the method extends itself naturally to characterize the Hausdorff axiom, regularity and normality.

(1) A space $X$ is completely regular iff there is a base $\mathcal{B}$ for closed sets of $X$ such that
(a) (Base regularity): If $B \in \mathcal{B}$ and $x \notin B$, then $x$ and $B$ are screened by a pair from $\mathcal{B}$,

(b) (Base normality): Every two disjoint elements of $\mathcal{B}$ are screened by a pair from $\mathcal{B}$.
(2) A space $X$ is completely regular iff there is a subbase $\mathcal{G}$ for closed sets of $X$ such that

(a) (Subbase regularity): If $S \in \mathcal{G}$ and $x \not\in S$ then $x$ and $S$ are screened by a finite subcollection of $\mathcal{G}$.

(b) (Subbase normality): Every two disjoint elements of $\mathcal{G}$ are screened by a finite subcollection of $\mathcal{G}$.

(3) A space $X$ is Hausdorff iff it is (sub)base Hausdorff relative to any (sub)base $\mathcal{B}$ of closed sets i.e., every two points of $X$ are screened by (a finite subcollection of $\mathcal{B}$) a pair of elements of $\mathcal{B}$.

(4) A space $X$ is regular iff it is (sub)base regular to some suitable (sub)base $\mathcal{B}$ of closed sets (as defined in (1) and (2) above).

Complete regularity means (sub)base regularity and (sub)base normality relative to some suitable (sub)base of closed sets; but normality can be characterized as follows:

(5) A space $X$ is normal iff it is (sub) base regular and (sub) base normal relative to the base of all closed sets.
The above characterization (1) of complete regularity was also discussed in [1]. It may be mentioned here that this characterization of complete regularity has also been noted independently by E.F. Steiner in [63]; and is related to the characterization of complete regularity given by O. Frink [31]. This paper of de Groot and Aarts [34] clearly brought out the similarity between the process of complete regularization of $T_1$-spaces on the one hand and certain techniques of Stone–Čech compactification on the other, the similarity which was so strikingly exhibited by H. Herrlich in his lecture notes on 'Topologische Reflexionen und Coreflexionen' [37] through categorical considerations.

1.11. E-Type Separation Properties:

We know that a space is Tychonoff iff it can be embedded in a cube, where by a cube we mean some topological power of a unit closed interval $[0,1]$. By considering the cube in a more general setting, S. Mrówka [48] has initiated the study of certain E-type of separation axioms. The approach of H. Herrlich [36] to this type of problem is still more general. To go through these separation properties we need the embedding lemma and some further terminology.
Let $E_{\alpha}$ be a space for each $\alpha \in A$. The set of functions $p$ defined on the index set $A$ with $p_{\alpha} \in E_{\alpha}$, for each $\alpha \in A$, is called the cartesian product of the sets $E_{\alpha}$ and is denoted by $\prod_{\alpha \in A} E_{\alpha}$ (or simply by $\pi E_{\alpha}$). The product $\pi E_{\alpha}$ endowed with the product topology (Tychonoff topology) is called the product space of the spaces $E_{\alpha}$. In fact, the product topology is the smallest topology on $\pi E_{\alpha}$ with respect to which for every $\beta \in A$, the projection map $\pi_{\beta}: \prod_{\alpha \in A} E_{\alpha} \to E_{\beta}$ defined by $\pi_{\beta}(x) = x_{\beta}$ is continuous.

If all spaces $E_{\alpha}$, $\alpha \in A$, are equal to the space $E$ then the product space is denoted by $E^m$, where $m$ is the cardinality of the index set $A$ and then $E^m$ is called the $m^{th}$ (topological) power of $E$. In particular if $E$ is the closed unit interval $[0,1]$, then $E^m$ is called a cube (with the power $m$).

Let for each $\alpha \in A$, $f_{\alpha}$ be a continuous function defined on a topological space $X$ to a topological space $E_{\alpha}$. Then the evaluation map $e: X \to \prod_{\alpha \in A} E_{\alpha}$ induced by the class of functions $\{f_{\alpha}: \alpha \in A\}$ is defined by

$$\pi_{\alpha} \circ e = f_{\alpha}, \text{ for each } \alpha \in A. \quad \ldots (\ast).$$

That is, for each $x \in X$, $e(x)$ is a point in $\pi E_{\alpha}$ whose $\alpha^{th}$ coordinate is $f_{\alpha}(x)$ for each $\alpha \in A$. 
The above condition given by (*) has a dual role. If the map $e$ is given with the condition (*), then the condition (*) defines the class $\{f_\alpha : \alpha \in A\}$ with $f_\alpha : X \to E_\alpha$ for each $\alpha \in A$, and conversely if such a family $\{f_\alpha : \alpha \in A\}$ is given then the condition (*) defines the map $e$.

Now, we state below the well known embedding lemma (see [41]).

**Embedding Lemma:** Let $F = \{f_\alpha : \alpha \in A\}$ be a collection of functions with $f_\alpha : X \to E_\alpha$, where $X$ and $E_\alpha$, $\alpha \in A$, are topological spaces. Let $e$ be the evaluation map induced by $F$, i.e., $e : X \to \prod_{\alpha \in A} E_\alpha$ such that $\pi_\alpha \circ e = f_\alpha$ for each $\alpha \in A$. We have:

1. **(I)** Evaluation map $e$ is continuous iff each $f_\alpha$ is continuous.

2. **(II)** Evaluation map $e$ is one to one iff the family $F$ satisfies the following condition:
   (a): for every $x, y \in A$, $x \neq y$ there is an $f_\alpha \in F$ with $f_\alpha(x) \neq f_\alpha(y)$.

3. **(III)** Evaluation map $e$ is open iff the family $F$ satisfies the following condition:
(b): for every closed set $B$ in $X$ and $x \in X - B$, there exists an $f_\alpha \in F$ such that $f_\alpha(x) \notin \text{cl-} f_\alpha(B)$.

It is clear now that if $X$ is a Tychonoff space and each space $E_\alpha$ equals the closed unit interval $[0,1]$ then the evaluation map $e$ is one to one, open and continuous and hence, a Tychonoff space can be embedded in a cube. In fact the above embedding lemma reduces the problem of embedding of a space topologically in a cube to the problem of finding a rich set of real valued continuous functions.

The above embedding lemma has been improved by S. Mrówka in [48], where he considered the condition $(b^*)$ given below, in place of the condition (b) of the lemma.

$(b^*)$ For every closed $B$ in $X$ and $x \in X - B$ there is a finite number of $\alpha_i$ with each $\alpha_i \in A$, $i = 1, 2, \ldots, n$ and a continuous map $f$ from $X$ to $\prod_{i=1}^{n} E_{\alpha_i}$ such that $f(x) \notin \text{cl-} f(B)$.

If a class $F = \{ f_\alpha : \alpha \in A \}$ of continuous functions with $f_\alpha : X \to E_\alpha$ satisfies the condition (a) of the above lemma then it is called a $\{E_\alpha : \alpha \in A\}*$-distinguishing
class for $X$. If the class $F$ satisfies the above condition $(b^*)$ then it is called an $\{E_\alpha : \alpha \in A\}$-separating class for $X$ (if all the spaces $E_\alpha$ are equal to a fixed space $E$, then we call them an $E$-distinguishing and $E$-separating class for $X$, respectively).

Given two spaces $X$ and $E$, $X$ is said to be $E$-completely regular provided $X$ is homeomorphic to a subspace of some topological power $E^n$ of $E$. Now by the embedding lemma it is clear that the space $X$ is $E$-completely regular iff the above two conditions (a) and $(b^*)$ are satisfied. Also, the condition $(b^*)$ alone is necessary and sufficient condition so that a $T_1$-space $X$ is $E$-completely regular.

A subset $B$ of $X$ is said to be $E$-closed (respectively $E$-open) in $X$ provided that for some finite $n$ there exists a closed (respectively, an open) subset $T$ of $E^n$ and a continuous function $f : X \to E^n$ such that $f^{-1}(T) = B$. It is clear that $B \subseteq X$ is $E$-closed iff $X - B$ is $E$-open and that a finite union and finite intersection of $E$-closed ($E$-open) subsets of $X$ is again $E$-closed ($E$-open) in $X$.

We know that a space is completely regular iff the family of all zero-sets forms the base for closed sets.
Similarly the topology of an E-completely regular space can be characterized by means of E-open sets (or E-closed sets). In fact, a T_0-space X is E-completely regular iff the class of all E-open sets of X forms a base for open subsets of X. In particular, if X is E-completely regular, then every open (respectively, closed) subset of X is the union (respectively, intersection) of E-open (respectively, E-closed) sets. Now, we give below some of E-type axioms.

A space X is said to be E-completely Hausdorff provided that the class C(X,E) is E-distinguishing. A space X is said to be E-normal provided that for any two disjoint closed subsets A and B of X, there exist two disjoint E-closed subsets A_l and B_l of X with A ⊆ A_l and B ⊆ B_l. A space X is called strongly E-normal provided that for every two disjoint closed sets A and B of X, there exists a finite number n, a continuous function f : X → \mathbb{R}^n and two disjoint closed sets F_1, F_2 of \mathbb{R}^n such that A ⋂ f^{-1}(F_1) and B ⊆ f^{-1}(F_2).

The following implications are known between the E-type separation properties discussed above.

1. Every E-completely regular space is E-Hausdorff.
2. Every E-normal, T_1-space is E-completely regular.
(3) Every strongly $E$-normal space is $E$-normal.

(4) Every $E$-Hausdorff, compact space is strongly $E$-normal.

In general, an $E$-normal $T_0$-space may fail to be an $E$-completely regular space; and an $E$-normal space may fail to be strongly $E$-normal space.

In [36], H. Herrlich obtained $\mathcal{E}$-type separation properties in way of studying $\mathcal{E}$-compact spaces. He considered there a Hausdorff space $X$ and a class $\mathcal{E}$ of Hausdorff spaces. A subset $U$ of $X$ is called $\mathcal{E}$-open (respectively, $\mathcal{E}$-closed) in $X$ if there exists an open (respectively, closed) subset $V$ of some $E \in \mathcal{E}$ and a $f \in C(X,E)$ such that $f^{-1}(V) = U$. Two subsets $A$ and $B$ of $X$ are called $\mathcal{E}$-separated if there exists an $E \in \mathcal{E}$ and $f \in C(X,E)$ such that $f(A) \cap f(B) = \emptyset$. He defined the space $X$ to be $\mathcal{E}$-compact (respectively, $\mathcal{E}$-regular) if $X$ is homeomorphic to a closed subspace (respectively, a subspace) of product of spaces from $\mathcal{E}$. Again, the space $X$ is called $\mathcal{E}$-normal if every two disjoint closed subsets $A$ and $B$ of $X$ are $K\mathcal{E}$-separated, where $K\mathcal{E}$ stands for the class of all $\mathcal{E}$-compact spaces.

As already mentioned that the approach of studying $\mathcal{E}$-compact spaces due to Herrlich [36] is more
general, in some respect, than that of $E$-compact spaces due to S. Mrówka [48]. The point at which these differ is that Mrówka starts with one single Hausdorff space $E$ and its power spaces $E^m$, while Herrlich starts with a class of Hausdorff spaces.

1.12. Concluding Remarks:

In this brief survey we have attempted only to give an idea of a few separation axioms whose study has been initiated recently, laying emphasis more on the way the hierarchy of separation axioms can be extended and some of the ways by which they can be generalized. The axioms listed above do not form a complete list of all such axioms discovered recently. Even characterizations of only those axioms have been included which lie between the standard axioms.

We have not included in this survey any mention of the important recent work related to minimal $p$-spaces [15] (where $P$ may be any property, e.g., a separation property) nor any reference to the important work connected with some well known problems in which a separation type property plays an important role, as for instance, in the following problem of Michael - Tamano [66].

'What is the space $X$ such that $X \times Y$ is normal for any paracompact space $Y$?'
It may be mentioned, here, that some of the axioms listed above have been studied in the context of some recently discovered structures like Proximities, Syntopogeneous structures, Nearness spaces etc. and also in the context of bitopological spaces. See, for instance S.A. Naimpally and B.D. Warrack [52], A. Császár [17], H. Herrlich [38], J.C. Kelly [42], M.G. Murdeshwar and S.A. Naimpally [51] etc.

The axiom of complete regularity and the Hausdorff axiom continue to be the most widely utilized axioms but as Prof. Naimpally remarked the $R_0$-axiom is becoming increasingly significant, and the situation in a few years might be considerably different from what it is today.
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