CHAPTER IV

$R_1$-TOPOLOGICAL SPACES

The hierarchy of 'Regularity Axioms' studied by A.S. Davis [4] in 1961 included besides the $R_0$-axiom studied in chapter II of the present thesis, the $R_1$-axiom. The $R_1$-axiom is independent of $T_0$ and of $T_1$, but strictly weaker than $T_2$. However, $T_2 = R_1 + T_0$. Further, $R_1$-spaces were examined in some detail by M.G. Murdeshwar and S.A. Naimpally in [6] where they weakened the hypothesis from $T_2$ to $R_1$ in some well known results. They [7] also studied the $R_1$-axiom in the context of quasi-uniformity. The $R_1$-axiom is found to be well behaved as it is hereditary, productive, projective and topologically invariant. Recently, it has been noted in [3] that the $R_1$-property is preserved under almost homeomorphism. It was shown in [6] that in a paracompact space the $R_1$-axiom implies normality. This type of result in the context of various generalised paracompact properties has been recently discussed in [5]. Further, G.D. Richardson [9] has shown that in a locally compact $T_1$-space, the closed compact neighbourhoods of each point form a neighbourhood base.

In the present chapter we give a number of characterizations of the $R_1$-axiom and show that the
axioms $R_1$ and weakly Hausdorff introduced by B. Banaschewski and J. M. Maranda [2] are equivalent. Further, it is shown in an $R_1$-space that the closure of a compact set is the union of the closures of its points, and that locally compactness property is open hereditary. Finally, a condition is obtained under which a dense locally compact subset of an $R_1$-space is open.

1. Definitions

**Definition 1.1.** : In a topological space $(X, \mathcal{T})$, a set $A$ is said to be a point-closure set if it can be expressed as the closure of a singleton set, and the complement of a point-closure set will be called a co-point-closure set.

**Definition 1.2.** [4] : A topological space $(X, \mathcal{T})$ is said to be an $R_1$ if for each pair of points $x, y \in X$, such that $\text{cl-}\{x\} \neq \text{cl-}\{y\}$ there are disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$.

2. Characterizations of $R_1$-axiom.

Every $R_1$-space is $R_0$ and in an $R_0$-space for any open set $U$, $x \in U$ implies $\text{cl-}\{x\} \subseteq U$. In fact, it was noted in [6] that a topological space is $R_1$ iff whenever
\( cl\{x\} \neq cl\{y\} \), \( cl\{x\}\) and \( cl\{y\}\) have disjoint open neighbourhoods.

It is worthwhile to note that if in the axiom of regularity the closed set is replaced by a point-closure set then the axiom so obtained is nothing but a restatement of the \( R_1 \)-axiom. This is so, for, in a topological space \( cl\{x\} \neq cl\{y\} \) implies either \( x \notin cl\{y\} \) or \( y \notin cl\{x\} \) and conversely. Thus, we have the following result.

A topological space \((X, \mathcal{T})\) is \( R_1 \) iff for any point-closure set \( A \) and \( x \) such that \( x \in X \setminus A \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \), \( A \subset V \).

Persuing the above approach corresponding to the other characterizations of the axiom of regularity, we have the following interesting reformulations of the \( R_1 \)-axiom.

**Theorem 2.1.** In a topological space \((X, \mathcal{T})\) the following statements are equivalent.

(a) \((X, \mathcal{T})\) is \( R_1 \).

(b) For every co-point-closure set \( G \) and \( x \in G \) there exists an open set \( U \) such that \( x \in U \), \( cl\{U\} \subseteq G \).
(c) Any point-closure set is the intersection of all closed neighbourhoods containing it.

Proof: (a) implies (b): If \( G \) be a co-point-closure set containing \( x \) then \( X - G \) is a point-closure set such that \( x \not\in X - G \). Therefore, there are disjoint open sets \( U \) and \( V \) such that \( x \in U, (X - G) \subseteq V \) and hence, the sets \( \text{cl-}U \) and \( (X - G) \) are disjoint. Therefore, the open set \( U \) is such that \( x \in U, \text{cl-}U \subseteq G \).

(b) implies (c): Let \( F \) be a point-closure-set and let \( F^* = \bigcap \{ \text{cl-}G : G \text{ is open and } F \subseteq G \} \). In fact, we will show that \( F = F^* \). Clearly, \( F \subseteq F^* \) and for the reverse inclusion we will show that each \( x \not\in F \) implies \( x \not\in F^* \). Now, \( X - F \) is a co-point-closure set and so for any \( x \not\in F \), there exists an open set \( U \) such that \( x \in U \subseteq \text{cl-}U \subseteq X - F \). Therefore, \( x \not\in \text{cl-}(X - \text{cl-}U) \) and hence, \( \text{cl-}(X - \text{cl-}U) \) being a closed neighbourhood of \( F \), \( x \not\in F^* \). It follows that any point-closure set is the intersection of all closed neighbourhoods containing it.

(c) implies (a): Obvious. This completes the proof of the theorem.

It is shown in chapter II, theorem 2.1, that a topological space is \( R_0 \) iff for every pair of points \( x, y \)
either \( \ker\{x\} = \ker\{y\} \) or \( \ker\{x\} \cap \ker\{y\} = \emptyset \). What can be said in this direction for an \( R_1 \)-space is given by the next result.

**Theorem 2.2.** A topological space \( (X, \mathcal{T}) \) is \( R_1 \) iff for each pair of points \( x, y \in X \), such that \( \ker\{x\} \neq \ker\{y\} \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U, y \in V \). In fact, the sets \( U \) and \( V \) are such that \( \ker\{x\} \subseteq U, \ker\{y\} \subseteq V \).

**Proof:** It immediately follows from the lemma 2.1 of chapter II, which says that in a topological space, for any pair of points \( x, y, \text{cl-} x \neq \text{cl-} y \) iff \( \ker\{x\} \neq \ker\{y\} \).

In 1961, B. Banaschewski and J.M. Maranda [2] discussed the notion of a weakly Hausdorff property in connection with the proximity functions, a certain class of mappings from a set of all subsets of \( X \) into the set of all filters on \( X \). In their terminology it was shown that a strict extension \( X^* \) of a space \( X \) is weakly Hausdorff iff its filter trace on \( X \) is weakly Hausdorff. It has been noted by them there that \( T_0 \)-axiom amounts to saying that distinct points have distinct neighbourhood filters.

**Definition 2.1.** [2]: A topological space \( (X, \mathcal{T}) \) is said to be weakly Hausdorff if any two distinct neighbourhood
filters of points are incompatible.

Any two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be incompatible if the filter generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ is improper, i.e. it coincides with the set of all subsets of the space (for details we may refer to [2]).

The fact that both the concepts of weakly Hausdorff and $R_1$-axiom are not distinct is given by the next result.

**Theorem 2.3.** A topological space is $R_1$ iff it is weakly Hausdorff.

**Proof:** It follows from the fact that in a topological space $(X, \mathcal{T})$, for any pair of points $x, y \in X, \text{cl} \{x\} \neq \text{cl} \{y\}$ iff their neighbourhood filters are distinct, and that $x$ and $y$ have disjoint open neighbourhoods iff their neighbourhood filters are incompatible.

3. Some Further Results on $R_1$-Spaces.

It was noted in [6] that in an $R_1$-space any compact set $F$ and $x$ such that $\text{cl} \{x\} \cap F = \emptyset$, have disjoint open neighbourhoods. The proof of this result depends
on the fact that for each $y \in F$, $\text{cl-} \{x\} \neq \text{cl-} \{y\}$.
Therefore, it is easy to see in an $R_1$-space that in each of the following cases any compact set $F$ and $x$ such that $x \not\in F$, have disjoint open neighbourhoods.

i) $\{x\} \cap \text{cl-} F = \emptyset$.

ii) Each $y \in F$ implies $\text{cl-} \{y\} \subset F$.

iii) Each $y \in F$ implies $\text{cl-} \{x\} \neq \text{cl-} \{y\}$.

iv) Each $y \in F$ implies $\ker- \{x\} \neq \ker- \{y\}$.

In a Hausdorff space a compact set is closed but in an $R_1$-space a compact set is closed if it contains the closure of each of its points. In fact we have the following result which is essentially due to Ivan L. Reilly [8].

**Theorem 3.1.** : In an $R_1$-space $(X, \tau)$ for any compact set $F$, $\text{cl-} F = \bigcup \{ \text{cl-} \{x\} : x \in F \}$.

**Proof** : Let $F$ be a compact set of an $R_1$-space $(X, \tau)$ and let $F^* = \bigcup \{ \text{cl-} \{x\} : x \in F \}$. Now, we show that $\text{cl-} F = F^*$. Clearly $F \subseteq F^* \subseteq \text{cl-} F$ and for the reverse inclusion we will show that each $x \not\in F^*$ implies $x \not\in \text{cl-} F$.

Let an arbitrary point $y \in X$ be such that $y \not\in F^*$. Then, for all $x \in F^*$, $\text{cl-} \{x\} \neq \text{cl-} \{y\}$, and hence the space being $R_1$ there exist open sets $U_y^{(x)}$ and $U_x$ such that $x \in U_x$, $y \in U_y^{(x)}$ and $U_x \cap U_y^{(x)} = \emptyset$. Then, the family
\( \mathcal{U} = \{ U_x : x \in F^* \} \) forms an open covering of the compact set \( F \) and hence has a finite subcover \( U_{x_1}, U_{x_2}, \ldots, U_{x_n} \).

Now, the set \( U_y = \bigcap_{i=1}^{n} U_y^{(x_i)} \) is an open neighbourhood of \( y \) which is such that \( U_y \cap F = \emptyset \). It follows that \( y \not\in \text{cl-}F \).

Therefore, \( F^* = \text{cl-}F \).

The above fact is sharpened by the next result.

**Theorem 3.2.** In an \( R_1 \)-space \( (X, \mathcal{T}) \) an almost compact set \( F \) is closed iff \( F \) is such that for \( x \in F \), \( \text{cl-}\{x\} \subseteq F \).

**Proof:** The necessary part is clear and for the sufficiency, in an \( R_1 \)-space \( (X, \mathcal{T}) \), let an almost compact set \( F \) be such that for each \( x \in F \), \( \text{cl-}\{x\} \subseteq F \). For an arbitrary point \( p \in X - F \) and all \( x \in F \), since \( \text{cl-}\{p\} \neq \text{cl-}\{x\} \), there exist open sets \( U_p^{(x)} \) and \( U_x \) such that \( p \in U_p^{(x)} \), \( x \in U_x \) and \( U_x \cap U_p^{(x)} = \emptyset \). This family \( \mathcal{U} = \{ U_x : x \in F \} \) is an open covering of \( F \). Since \( F \) is almost compact, there must be \( x_i \in F \), \( i = 1, 2, \ldots, n \) such that \( F \subseteq \bigcup_{i=1}^{n} \text{cl-}U_{x_i} \). Now, the set \( U_p = \bigcap_{i=1}^{n} U_p^{(x_i)} \) is an open set containing \( p \) such that \( U_p \subseteq X - F \). It follows that \( X - F \) is open and so \( F \) is closed.
In [6] it was shown that one point compactification of a topological space \( X \) is \( R_1 \) iff the space \( X \) is \( R_1 \) and locally compact; and hence, in a locally compact space the \( R_1 \)-axiom equals the complete regularity. Further, G. D. Richardson [9] has extended this result by showing that in a locally compact space, the \( R_1 \)-axiom coincides with the axiom by which the closed compact neighbourhoods of each point form a neighbourhood base. It is well known that the locally compact property is closed hereditary but the next result says that in \( R_1 \)-space it is open hereditary also.

**Theorem 3.3.** : Every open subset of a locally compact \( R_1 \)-space is locally compact.

**Proof** : Let \( G \) be an open set of a locally compact \( R_1 \)-space \((X, \mathcal{T})\) and \( x \) be an arbitrary point such that \( x \in G \). Since a locally compact \( R_1 \)-space is regular, there exists an open set \( U \) such that \( x \in U \subseteq \text{cl-}U \subseteq G \). The space \( X \) is locally compact, therefore there exists a compact neighbourhood \( V \) of \( x \). Now, \( V \cap \text{cl-}U \) is a neighbourhood of \( x \) in \( G \). Again, \( V \cap \text{cl-}U \) is compact because it is a closed subset of the compact set \( V \). It follows that \( G \) is locally compact.

The next result will give a condition under which a dense locally compact subset of an \( R_1 \)-space is open.
Theorem 3.4. : So that in an $R_1$-space a dense locally compact subset $A$ is open it is necessary and sufficient that $x \in A$ implies $\text{cl}-\{x\} \subseteq A$.

Proof: The necessary part is clear and for the sufficiency let $A$ be a dense locally compact subset of an $R_1$-space $(X, \mathcal{T})$ such that for $x \in A$, $\text{cl}-\{x\} \subseteq A$. For each $x \in A$, there must exist a neighbourhood $G$ of $x$ such that $G \cap A$ is compact. This is so because, $A$ is locally compact. Let $U_x$ be an open set such that $x \in U_x \subseteq G$. Let $p \in U_x$ and $\mathcal{U}(p)$ be the family of all neighbourhoods of $p$. Then the family $\mathcal{F} = \{ U_p \cap (G \cap A) : U_p \in \mathcal{U}(p) \}$ is the trace neighbourhood filter on $G \cap A$. Since $A$ is dense in $X$, and since $G \in \mathcal{U}(p)$, the filter $\mathcal{F}$ is proper on $G \cap A$. Now, $G \cap A$ is compact, therefore the filter $\mathcal{F}$ must have a cluster point $q$ in $G \cap A$. Evidently, $q$ is again a cluster point of the neighbourhood filter $\mathcal{U}(p)$. From the assumption that the space $X$ is $R_1$, it is clear that the cluster set of the filter $\mathcal{U}(p)$ is $\text{cl}-\{p\}$. Therefore, $q \in \text{cl}-\{p\}$ and hence, $\text{cl}-\{p\} = \text{cl}-\{q\}$. Thus, we see that for each $p \in U_x$ there exists $q \in A$ such that $\text{cl}-\{p\} = \text{cl}-\{q\}$. But $q \in A$ implies $\text{cl}-\{q\} \subseteq A$ and hence, $p \in A$. Therefore, $U_x \subseteq A$ and it follows that $A$ is open.

It is well known that if $f$ and $g$ are any two continuous function from a topological space $X$ to a $T_2$-space $Y$
then the set  \( A = \{ x : f(x) = g(x) \} \) is closed in \( X \). For an \( R_1 \)-space, we have the following result:

**Theorem 3.5.** If \( f \) and \( g \) are any two continuous functions from any topological space \((X, \mathcal{T})\) to an \( R_1 \)-space \((X^*, \mathcal{T}^*)\) then the set \( A = \{ x : \text{cl}\{f(x)\} = \text{cl}\{g(x)\} \} \) is closed in \( X \).

**Proof:** For any \( y \in X \) such that \( y \notin A \) we have \( \text{cl}\{f(y)\} \neq \text{cl}\{g(y)\} \). Then from the assumption that \((X^*, \mathcal{T}^*)\) is \( R_1 \), there exist open sets \( U \) and \( V \) in \( X^* \) such that \( f(y) \in U \), \( g(y) \in V \) and \( U \cap V = \emptyset \). Clearly, \( f^{-1}(U) \cap g^{-1}(V) \) is an open neighbourhood of \( y \) in \( X \). Again, \( A \cap f^{-1}(V) \cap g^{-1}(U) \) is empty. If it is not so then, let \( z \in X \) be such that \( z \in A \cap \left[ f^{-1}(U) \cap g^{-1}(V) \right] \) which would imply that \( f(z) \in U \), \( g(z) \in V \) and that \( \text{cl}\{f(z)\} = \text{cl}\{g(z)\} \), that is, \( U \cap V \neq \emptyset \). It follows that \( y \) is an interior point of \( X - A \), that is, \( X - A \) is open and so \( A \) is closed.
REFERENCES


