THE THIRD CHAPTER

The Operation Subtraction :
THE OPERATION SUBTRACTION

(1) \((+a) - (+b) = S_p^2 (+a \cap b \cap a \cap b)\)

where

\[
S_p (L_g \cap R_g)
\]

is the section of the first kind for the operator number "b".

(2) \((+a) - (+b) = S_p^1 (+a \cap b \cap a \cap b)\)

where

\[
S_p (L_g \cap R_g)
\]

is the section of the second kind for the operator number "b".

Note: The kind of the section of the result of subtraction changes from the first kind to the second kind and from the second to the first kind according as the section for the number "b" is taken to be of the first kind or that of the second kind.
IMPORTANT OF SIGNED NUMBERS:

+ a , -a and + b , -b

In forming the section of the number which is the resulting number of the operation "subtraction" it is a matter of basic importance to take into account the sign of the operand number as well as the sign of the operation-number or the operator number responsible for the operation "subtraction".

Further, since operation subtraction is a binary operation hence in every there are two numbers involved in this operation. In

( a) - ( b)

the number "a" is the operand number and the number "b" is the operator number. In this approach it is necessary to take into account the signs (plus or minus) of both the operand number and the operator number.

THEREFORE THERE WILL BE FOUR BASIC CASES OF THE ABOVE EXPRESSION. THEY ARE

(1) 
( +a) - ( +b)

(2) 
( +a) ( -b)

(3) 
( -a) ( +b)

(4) 
( -a) - ( -b)
we now begin to discuss the formation of the result of the operation subtraction between two numbers by means of Dedekind's sections.

First of all we consider the cases in which both the operand number "a" and the operator number are rational.

Case no. (1): The number "a" is rational and positive and the number "b" too is rational and positive.

The formula is (showing procedure of getting it)

\[(a + b) - (a + b) = S_p \left( \frac{L_p}{R_p}, \frac{R_p}{L_p} \right) \]

\[S_p \left( \frac{a + b}{R_p}, \frac{a - b}{L_p} \right)\]

section of the second kind for the result of subtraction.

where

\[S_p \left( \frac{a + b}{R_p}, \frac{a - b}{L_p} \right)\]

is the section of the first kind selected for the operator number "b".

Note: We notice that the result

\[S_p \left( \frac{a + b}{R_p}, \frac{a - b}{L_p} \right)\]

is a section of the second kind.

**Thus the type of the section of the result has changed from the first kind to the second kind.**
AND WHERE

(1) \[ a - R_S \] means that from the operand number \( a \) is subtracted, turn by turn, every number of \( R_S \) which is the sliding upper class of the section of the first kind taken for the operand number \( b \).

(2) \[ a - L_P \] means that from the operand number \( a \) is subtracted, turn by turn, every number of \( L_P \) which is the fixed lower class of the section of the first kind selected for the operator number \( b \).

Therefore,

(1) \[ a - R_S \] is the lower class of the section for the resulting number obtained due to the operation "subtraction" between the numbers \( a \) and \( b \).

(2) \[ a - L_P \] is the upper class of the section for the resulting number obtained due to the operation "subtraction" between the numbers \( a \) and \( b \).

AND THUS,

(1) \[ a - R_S \] is a sliding lower class of the section as indicated by the symbol \( R_S \).

(2) \[ a - L_P \] is a fixed upper class of the section as indicated by the symbol \( L_P \).

HENCE FINALLY,

The section of the result of the operation "subtraction"

\[
S_2( \Downarrow a - R_S, \Downarrow a - L_P )
\]
is a section of the second kind. Therefore we notice that when for the operator number Sb" the section of the first kind is taken then for the resulting number the section of the second kind comes out. This change of section from first kind to the second kind is a wonderful phenomenon in the study of operations by sections. More interesting things will be brought to light regarding the interchanging of the kinds of sections in the discussions coming later in this part of the study on the operation "subtraction".

EXAMPLE: (1) Operand number is +3 and the operator number is +2.

Then,

\[(+3) \rightarrow (+2) = 1 + S_p ( L_p \cdot R_p )\]

\[= +3 \text{ section of first kind for } 2\]

\[= 2 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad R_p \left( +3 \cdot R_p, +3 \cdot L_p \right)\]

\[= 2 \quad \frac{1}{3} \quad (+2) \quad \frac{1}{3} \quad \frac{1}{2}\]

\[= 2 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad R_p \left( R_p, L_p \right)\]

\[= 2 \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \quad L_p \cdot R_p\]

\[= \text{ section of the second kind for the number } +1 \text{ which is the result.}\]

Note:

\[\frac{1}{R_p} \quad \frac{1}{L_p}\]

\[\text{means the lower sliding class which we write}\]

\[\text{in accordance with the discussions contained}\]

\[\text{in the part of forming sections.}\]
(i) The section here for the operator number 2 is taken of the first kind, it is

\[ \mathcal{S}_{P} \left( \frac{\ell - 2}{L}, \frac{\ell}{R} \right) \]

and the section of the result is

\[ \mathcal{S}_{P} \left( \frac{\ell - 1}{L}, \frac{\ell}{R} \right) \]

which is a section of the second kind; hence we conclude that change of the type of the section takes place.

(ii) SIMILARLY, if for the operator number the section of the second kind were taken, then the result would come to be a section of the first kind.

Details of working:

The operand number is

\[ \frac{\ell + 3}{L} \]

and taking the first kind of section for the operator number we have

\[ \frac{\ell + 2}{L}, \frac{\ell + 2}{R} \]

To form the lower class of the result-number \( \ell + 1 \):

\[ \frac{a - \ell + b}{R} \]

We subtract from the operand number \( \ell + 3 \) all the numbers, one after the other, which are contained in the
Upper class \( ^{\frac{1}{2}} R_S \) of the operator number \( \sqrt{\frac{1}{2}} \). By doing so we get the following series of numbers:

\[
\frac{1}{3} - \frac{\sqrt{2}}{R_S} = (\frac{1}{3}) - (\frac{\sqrt{2}}{2} \ e) = 1 \ e, \ e \text{ tending to zero},
\]

\[
(\frac{1}{3}) \ 3 = 0,
(\frac{1}{3}) \ 4 = -1,
(\frac{1}{3}) \ 5 = -2,
(\frac{1}{3}) \ 6 = -3.
\]

Hence the lower class \( ^{\frac{1}{2}} L_S \) for the number \( \sqrt{\frac{1}{2}} \) which is the result of the operation "subtraction" contains the set of numbers:

\[
^{\frac{1}{2}} L_S = 1 \ e, 0, -1, -2, -3, \ldots
\]

and all the numbers contained in the portions of the continuum marked by these numbers.

Noted...

To get the lower class of the result, from the operand number all the numbers of the upper class of the section of the operand number have been subtracted and not the numbers of the lower class of the same. This point of the study on the operation subtraction is of great importance.

Symbolically we now get:

\[
\frac{1}{3} \ + \ 2 \ - \ 1 = \frac{1}{3} \ R_S - L_S = \ldots 1
\]
(2) To calculate the upper class of the section of the difference $(\div 3) - (\div 2)$:

The operand number is

$$\div 3$$

and section for operator number is

\[ L_F \quad R_S \]

\[ + \quad + \]

\[ -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \]

which is a section of the first kind.

In accordance with the formula given above the method is as follows:

$$ + a - \frac{+b}{L_F} = (\div 3) - (\div 2) $$

We subtract from the operand number $\div 3$ all the numbers turn by turn, which are contained in the lower class $\div 2$ of the operator number $\div 2$. By doing so we get the following series of numbers:

\[
\begin{align*}
(\div 3) - 2 & = 1 \\
(\div 3) - 1 & = 2 \\
(\div 3) - 0 & = 3 \\
(\div 3) - (-1) & = 4 \\
(\div 3) - (-2) & = 5 \\
(\div 3) - (-3) & = 6 \\
\end{align*}
\]

Hence the upper class of the section of the result is
is given by

\[
(\frac{2}{3}) - L_p = 1, 2, 3, 4, 5, 6, \ldots
\]

which is symbolically written as

\[
(\frac{2}{3}) - L_F = L_F
\]

\[
= \frac{1}{1}
\]

\[
= L_F
\]

= upper class of the difference

\[
:= \frac{1}{1}_F
\]

( the subscript "F" in \( L \) indicates that the upper class is a fixed one. That is the upper class is replaced by the letter \( \frac{1}{1}_F \).

\[
\frac{1}{1}_F
\]

Therefore, the section of the transform is given by

\[
(\frac{2}{3}) - (\frac{2}{2}) = (\frac{2}{3}) - S_P(\frac{2}{L_F}, \frac{1}{R_S})
\]

\[
= S_P(\frac{2}{3} - L_F, \frac{2}{3})
\]

\[
= S_P(\frac{2}{3} - (\frac{2}{3}) - (\frac{2}{2}) = (\frac{2}{3}) - (\frac{2}{2})
\]

\[
= S_P(\frac{1}{1}, \frac{1}{L_F})
\]

\[
= S_P(\frac{2}{L_S}, \frac{2}{R_F})
\]

section of the second kind for the operation "subtraction".

Note: For the operator number the first kind \( S^1_P \) of section was selected, and the section for the result -number comes out to \( S^2_P \) a section of the second kind, that is \( S^2_P \).
Thus finally if for an operator number \( b \) the first-kind section is selected, that is if

\[
\overline{\mathrm{f}} \cdot b = L_p \xrightarrow{R_s} S_p \]

Then,

\[
(\overline{\mathrm{f}} \cdot a) - (\overline{\mathrm{f}} \cdot b) = L_s \xrightarrow{R_p} S_p
\]

\[
= S_p
\]

the result is a section of the second kind.

SIMILARLY, if for the operator number \( b \) the section of the second kind is selected, that is if

\[
\overline{\mathrm{f}} \cdot b^* = L_s \rightarrow R_p = S_p
\]

then,

\[
(\overline{\mathrm{f}} \cdot a) - (\overline{\mathrm{f}} \cdot b) = L_p \rightarrow S = S_p
\]

\[
= S_p
\]

the result is a section of the first kind.

Note: The changing of the kind of the section from the first to the second and vice versa in the operation 2subtraction provides a very interesting study of Dedekind's sections, and thus aspect of the THEORY has been brought to light for the first time.
Similarly the method applies in all the rest three cases which are

(1) \((+a) \cdot (-b)\)

(2) \((-a) (+ b)\)

(3) \((-a) (-b)\)

Hence, when "a" and "b" are both rational numbers,

Thus,

(1) \((+a) (+ b) = \frac{2}{3} \left( \begin{array}{cc} L_3 & R_p \\ L_3 & R_p \end{array} \right)\)

(2) \((-a) (-b) = \frac{2}{3} \left( \begin{array}{cc} (+a) (-b) & (+a) (-b) \\ (+a) (-b) & (+a) (-b) \end{array} \right)\)

(3) \((-a) (+b) = \frac{2}{3} \left( \begin{array}{cc} (-a) (+b) & (-a) (+b) \\ (-a) (+b) & (-a) (+b) \end{array} \right)\)

(4) \((+a) (-b) = \frac{2}{3} \left( \begin{array}{cc} (-a) (-b) & (-a) (-b) \\ (-a) (-b) & (-a) (-b) \end{array} \right)\)

where

\[ \left( \begin{array}{cc} 1 & +b \\ +b & +b \end{array} \right) = \frac{2}{3} \left( \begin{array}{cc} L_3 & R_p \\ L_3 & R_p \end{array} \right) \]

is the section of the first kind representing the operator number "b".

And,

\[ \left( \begin{array}{cc} a & b \\ L_3 & R_p \end{array} \right) = \left( \begin{array}{cc} a & b \\ L_3 & R_p \end{array} \right) \]

\[ \left( \begin{array}{cc} a & b \\ R_p & R_p \end{array} \right) = \left( \begin{array}{cc} a & b \\ L_3 & R_p \end{array} \right) \]

for the purpose of carrying out the operation "SUBTRACTION".
The kind of the section changes from first to the second in all the four formulae (1), (2), (3), (4).

And if for the operator number "b" the second kind of the section is preferred then the corresponding formulae will be

1. \((\uparrow a) \cdot (\uparrow b) = \mathcal{S}_P\left( \frac{1}{L_F \cdot R_S} \right) \)

2. \((\uparrow a) \cdot (\downarrow b) = \mathcal{S}_P\left( \frac{1}{L_F \cdot R_S} \right) \)

3. \((\downarrow a) \cdot (\uparrow b) = \mathcal{S}_P\left( \frac{1}{L_F \cdot R_S} \right) \)

4. \((\downarrow a) \cdot (\downarrow b) = \mathcal{S}_P\left( \frac{1}{L_F \cdot R_S} \right) \)

where

\[ \mathcal{S}_P = \mathcal{S}_P\left( \frac{\uparrow b \cdot \uparrow b}{L_S \cdot R_F} \right) \]

is the section of the second kind for the operator number "b".

And,

\( \frac{L_F}{R_F} (\ a) \cdot (\ b) \)

\( \frac{L_S}{R_S} (\ a) \cdot (\ b) \)

for the purpose of performing the details of the operation "Subtraction".
We have seen that there are in all three types of sections. They are

1. The first kind of section: $S_p$
2. The second kind of section: $S_p$
3. The third kind of section: $S_p$

Either of the sections (the first kind and the second sections) represents a Rational number and the third kind section represents an Irrational number.

Now we notice at this stage: Why there are two kinds of sections to represent a rational number whereas there is only one type or kind of section to an irrational number.

This important question was raised on page no. 38, and there it was suggested that for this question the answer would be provided at that part where operations by means of sections for numbers will be performed.

In the operation "subtraction" which is being discussed on these pages, we have carried it by taking both kinds of the sections $S_p$ and $S_p$ for the operator number "b". And we have found that

1. When for the operator number "b" the section of the first kind is selected and then the operation "Subtraction" is performed then the result comes out a section of the second kind. That is

   \[(a) - (b) = (a) \cdot S_p \quad \text{for the second kind} \quad \text{section of the second kind.}\]
and,

(2) When for the operator number "b" the section of the second kind is taken and then the operation of subtraction is performed between two numbers, then we find that the result comes out to be a section of the first kind. That is,

\[
(a) - (b) = (a) S_p^2 \]
\[
= S_p \]

- section of the first kind.

Thus it has now been established that for carrying out the operation of subtraction between two numbers it is necessary that a rational number should be capable of being represented by the two kinds of sections. In this operation both kinds of the sections (which are \( S_p^1 \) and \( S_p^2 \)) are needed. If there were only one kind of section then the operation of subtraction would have not been possible to carry it out between two numbers. Therefore the existence of the both kinds of sections to represent a rational number finds its fullest justification in the context of the operation SUBTRACTION.

NOTEWORTHY:

The phenomenon of the sections changing their kind from the first to the second and from the second to the first takes place even in the operation MULTIPLICATION.

This will be studied when we take up it later on.
If \[ a = (L_F, R_S) \]
and, \[ b = (L_F, R_S) \]
then,
\[ a - b = (L_F - L_F, R_S - R_S) \]

Thus, operation subtraction by means of sections is not performed in the way the ordinary subtraction between two numbers is performed.

For example, consider the operation subtraction between the two numbers 3 and 2:

Accordingly, we let

\[ 3 = (3, 3) \]

\[
\begin{array}{c|c|c|c|c|c|c|c}
-2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

And,

\[ 2 = (2, 2) \]

\[
\begin{array}{c|c|c|c|c|c|c|c}
-2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Here for both the numbers 3 and 2 the sections of the same kind have been taken, that is sections of the first
kind have been selected.

To form the lower class $L$ of the result

$$3 \cdot 2$$

every number of the lower class $L_F^2$ should be subtracted turn by turn from every number of the lower class $L_F$. On doing so we get

$$L_F^2 \cdot L_F = 3 \cdot L_F^2 \quad (a \text{ part of the section of the result})$$

$$= 3 \begin{array}{c} 2 \end{array} 1$$

$$= 3 \begin{array}{c} 1 \end{array} 2$$

$$= 3 \begin{array}{c} 0 \end{array} 3$$

$$= 3 \begin{array}{c} (-1) \end{array} 4$$

$$= 3 \begin{array}{c} (-2) \end{array} 5$$

$$= 3 \begin{array}{c} (-3) \end{array} 6$$

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And we find that all these numbers 1, 2, 3, 4, 5, 6, do not belong to the lower class of the result.

These numbers are included in the upper class of the section which represents the difference $3 \cdot 2$ as already discussed on pages 78, 79, 80 and 81.

Hence we prove that

$$3 \cdot 2 \equiv L_F \cdot L_F$$

$$\equiv$$ The lower class of the result.

Similarly, it can be proved that

$$3 \cdot 2 \equiv R_S \cdot R_S$$

$$\equiv$$ The upper class of the result.
Note no. two:

In the previous discussions the sections of the first kind were used. Now here we give an example to perform the operation subtraction in which the sections (for both the operand number and the operator number) have been selected:

We let

\[ 3 = (L_3, R_F^3) \]

\[ \xrightarrow{-2 -1 0 1 2 \circ 4 5 6} \]

\[ = L_3 \rightarrow \circ R_F^3 \]

(section of the second kind)

and,

\[ 2 = (L_3, R_F^2) \]

\[ \xrightarrow{-2 -1 0 1 \circ 2 3 4 5 6} \]

\[ = L_3 \rightarrow \circ R_F^2 \]

(section of the section kind)

We now form the lower class

\[ (3) \quad (2) \]

\[ L_F \]

and the upper class

\[ F(3) \quad (2) \]

\[ F \]
of the section of the result $3 - 2$:

(i) Lower class $3 - R_F^2$:

\[
\begin{align*}
3 - 2 &= 1, \\
3 - 3 &= 0, \\
3 - 4 &= -1, \\
3 - 5 &= -2, \\
3 - 6 &= -3, \\
3 - 7 &= -4,
\end{align*}
\]

Hence the lower class of the section of the desired result consists of the numbers $1, 0, -1, -2, -3, -4, \ldots$ and all the numbers existing in the portions of the continuum marked by the two consecutive integers given in this set of numbers $-a$. The lower class is a "Fixed One." Thus

\[
3 - R_F^2 = 1, 0, -1, -2, -3, -4, \ldots
\]

(ii) Upper class $3 - L_3^2$:

\[
\begin{align*}
3 - (2 - e) &= 1 + e, \\
3 - 1 &= 2, \\
3 - 0 &= 3, \\
3 - (-1) &= 4.
\end{align*}
\]
Hence the upper class of the section of the desired result consists of the numbers 1, 2, 3, 4, 5, 6, 7, and the numbers existing in the portions of the continuum marked by the two consecutive integers given in this set of numbers. The upper class is a "Sliding One". Thus,

\[ 3 \cdot \frac{1}{L_s^2} = 1: (1, 2, 3, 4, 5, 6, 7) \]

Now combining the results in (i) and (ii) we get that

\[ 3 - 2 = (3 - R_F^2, 3 - L_s^2) \]

\[ = (L_F^{(3)}, R_S^{(3)}), (L_F^{(2)}, R_S^{(2)}) \]

\[ = (L_F^{1}, R_S^{1}) \]

(section of the first kind for the resulting number from \( 3 - 2 \))
And we notice that the section has been changed from the second kind to the first kind. Changing of the kind of the type of the second takes place in both the cases when for the operator number either the first kind section or the second kind section is selected.

Therefore for carrying out the operation "subtraction" we need both types of the sections which has been amply shown in the discussions of the operation under study.

This is the reason why there are two types (or kinds) of sections for representing a rational number.

Generalising the results under note no. two we get

1. 
\[(+a) - (+b) = S_p \left( +a, \frac{1}{R_p}, +b, L_s \right)\]

2. 
\[(+a) - (-b) = S_p \left( +a, -\frac{1}{R_p}, +b, L_s \right)\]

3. 
\[(-a) - (+b) = S_p \left( -a, \frac{1}{R_p}, -b, L_s \right)\]

4. 
\[(-a) - (-b) = S_p \left( -a, -\frac{1}{R_p}, -b, L_s \right)\]

where
\[S_p^2 \left( \frac{1}{L_s}, b, \frac{1}{R_p}, b \right)\]

is the section of the second kind for the operator number "b".

Note: The result is a section of the first kind.
Case 1. (5) : When "a" is a positive rational number and "b" is a positive irrational number:

Suppose \( p \cdot q \) and \( p \cdot (q + 1) \) form the first pair of localising numbers for the irrational positive number "b", then the section representing this irrational number is

\[
\begin{align*}
  b &= \frac{L_3}{p \cdot q} \quad \Rightarrow \quad \frac{R_3}{S_3(p \cdot (q + 1))} \\
    &= S_T \left( L_3 \cdot p \cdot q, R_3 \cdot p \cdot (q + 1) \right)
\end{align*}
\]

( a section of the third kind as an irrational number has a section of the third kind only)

And therefore,

\[
\begin{align*}
  (+ a) \cdot (+ b) &= (+ a) \quad S_T \left( L_3 \cdot p \cdot q, R_3 \cdot p \cdot (q + 1) \right) \\
    &= S_T \left( + a, R_3 \cdot p \cdot (q + 1) - a, L_3 \cdot p \cdot q \right) \\
    &= S_T \left( + a, p \cdot (q + 1) , + a, - p \cdot q \right)
\end{align*}
\]

a section of the third kind.

Hence finally we get

\[
\begin{align*}
  (+ a) \cdot (+ b) &= S_T \left( L_3 \cdot p \cdot (q + 1) , + a, - p \cdot q \right)
\end{align*}
\]

Note: Since for an irrational number there is only one type of section (of the third kind \( S_T \)), hence the result too is a section of the third kind.
EXAMPLE: To construct the section for $3 - \sqrt{2}$:

The first pair of numbers for locating the irrational number

$$\sqrt{2}$$

is 1.4 and 1.5. Hence the section (of the third kind) for this number is

\[
\begin{align*}
3 - \sqrt{2} & = 3 \quad S_T \left( \frac{L_S}{R_S} , \frac{R_S}{L_S} \right) \\
& = S_T (3, 1.5, 3 - L_S) \\
& = S_T (3, R_S, 3, 1.5) \\
& = S_T (3, R_S, L_S, 3, 1.4)
\end{align*}
\]

( the noteworthy point here is that $L_S$ and $R_S$ interchange their positions when operation is performed )

Hence,

\[
\begin{align*}
3 - \sqrt{2} & = S_T (3, 1.5, 1.6) \\
& = S_T (L_S, R_S) \\
& = S_T (L_S, 1.5, 1.6) \\
& = S_T (L_S, R_S)
\end{align*}
\]

$L_S$ and $R_S$ have been interchanged to specify the lower class and the upper class of the desired section.

Hence,

\[
3 - \sqrt{2} = S_T (L_S, R_S) \quad \text{Answer}
\]
Case (6): When "a" is a positive irrational number and "b" is also a positive irrational number:

(i) Let \( pq \) and \( q (q - 1) \) be the first localising pair of numbers for the irrational number "a", and,

(ii) Let \( rs \) and \( r (s - 1) \) be the first localising pair of numbers for the irrational number "b"

Therefore the sections for these two numbers are given by

\[
\begin{align*}
(\sqrt[n]{a}) &= (L_S, R_S) \\
(\sqrt[b]{b}) &= (L_S, R_S)
\end{align*}
\]

Therefore,

\[
(\sqrt[n]{a}) (\sqrt[b]{b}) = (L_S, R_S) (L_S, R_S)
\]

\[
= \left( \begin{array}{c}
L_S \\
R_S
\end{array} \right) \left( \begin{array}{c}
pq \\
q (q - 1)
\end{array} \right) \left( \begin{array}{c}
(L_S, R_S) \\
(R_S)
\end{array} \right) \left( \begin{array}{c}
r (s - 1) \\
rs
\end{array} \right) \left( \begin{array}{c}
L_S \\
R_S
\end{array} \right) \\
= \left( \begin{array}{c}
L_S \\
R_S
\end{array} \right) \left( \begin{array}{c}
r (s - 1) \\
r (s - 1)
\end{array} \right) \left( \begin{array}{c}
pq \\
q (q - 1)
\end{array} \right) \left( \begin{array}{c}
(L_S, R_S) \\
(R_S)
\end{array} \right)
\]

(a section of the third kind)

Hence finally we have

\[
(\sqrt[n]{a}) (\sqrt[b]{b}) = (L_S, R_S) (L_S, R_S)
\]

where \( \left( L_S, R_S \right) \) and \( \left( L_S, R_S \right) \) are
the sections for \( a \) and \( b \) respectively.

Example: To construct the section for \( 3 - \sqrt{2} \):

(i) \( 1.4 \) and \( 1.4 \) \( + \) or \( 1.5 \) is the first localising pair of numbers for the irrational number \( 3 \),

and,

(ii) \( 1.7 \) and \( 1.7 \) \( + \) or \( 1.8 \) is the first pair of numbers for the location of the irrational number \( 3 \).

Therefore the sections for these numbers are

\[
\begin{align*}
3 &= L_S(1.7) \rightarrow R_S(1.8) \\
2 &= L_S(1.4) \rightarrow R_S(1.5)
\end{align*}
\]

Hence now we get

\[
\begin{align*}
\sqrt{3} - \sqrt{2} &= (L_S^3, R_S^3) \rightarrow (L_S^2, R_S^2)
\end{align*}
\]
\[
\begin{align*}
&= (L_S, R_S) - (L_S, R_S) \\
&= (1.7, 1.5) - (1.8, 1.4) \\
&= (L_S, R_S) - (1.7, 1.5) \\
&= (L_S^2, R_S^4) \\
&= L_S (2) - R_S (4) \\
&= \sqrt{3} \sqrt{2} = S_T (L_S, R_S) \\
\end{align*}
\]

Hence finally we get

\[
\sqrt{3} \sqrt{2} = S_T (L_S^2, R_S^4)
\]

**Note:** It should be noted that

(i) The lower class of the result is obtained by

\[
\sqrt{a} \cdot b \\
L_S \quad R_S
\]

and (ii) The upper class is obtained by

\[
\sqrt{a} \cdot b \\
R_S \quad Z_S
\]
But the two classes cannot be obtained from
\[ \sqrt{a} \quad L_S \quad R_S^{b} \]

and,
\[ \sqrt{a} \quad R_S^{b} \quad R_S \]

Case no. (7) : The number "a" is positive rational number and "b" is positive irrational number:

Here the operator number "b" whose section we let to be

\[ b \quad \rightarrow \quad b \]
\[ L_S \quad R_S \]

Since the first pair of locating the irrational number b is has the two numbers p;q and p;(q+1) therefore the sliding section for it is

\[ b \quad \rightarrow \quad b \]
\[ L_{S(p,q)} \quad R_{S(p,(q+1))} \]

Hence we get

\[ a \quad b = a \quad S_T^{b} \]

= a - section of third kind for b

= \( 3^{(b)} \quad 3^{(b)} \)

= \( a \quad 3_T(L_S^{b}, R_S^{b}) \)

= \( a \quad 3_T(L_{S(p,q)}, R_{S(p,(q+1))}) \)

= \( a \quad S_T(L_{S(p,q)}, R_{S(p,(q+1))}) \)

= \( a \quad (p;q), a \quad L_{S(p,(q+1))} \)

= \( a \quad R_{S(p,q)}, a \quad L_{S(p,(q+1))} \)

= \( a \quad R_{S(p,q)}, a \quad L_{S(p,(q+1))} \)

= \( a \quad (p;q), a \quad R_{S(p,(q+1))} \)

required formula.
Example: To obtain the section for \( 3 - \sqrt{2} \).

Here the operator number is 2, we let its section to be

\[
\sqrt{2} = L_3 \rightarrow R_3
\]

And further since the first pair for localising this irrational number is 1.4 and 1.41, that is 1.5, hence the section of the third kind for it is

\[
\sqrt{2} = L_3(1.4) \rightarrow R_3(1.5)
\]

Hence we get

\[
3 - 2 = 3 - S_T(2)
\]

= section of third kind for 2

\[
3 - S_T(2) \left( L_3, R_3 \right)
\]

\[
= 3 - 3 \left( L_3(1.4), R_3(1.5) \right)
\]

\[
= S_T(3) \left( 3 - 3 = L_3(1.4) \right)
\]

\[
= S_T(3) \left( 3 - 1.5 \right)
\]

\[
= S_T(3) \left( 3, 2.5 \right)
\]

\[
= 3 \left( 3, 2.5 \right)
\]

\[
= \text{a section of the third kind for the resulting number}.
\]

Hence finally

\[
3 - 2 = S_T(3, 2.5, 2.6)
\]

\[
= 2 \left( 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3 \right)
\]

\[
\text{(Answer)}
\]