Chapter First

THE INTRODUCTION:

In the study of numbers we find that the aggregate of rational numbers (positive, negative, integral and fractional) form the basis of all "Number Theory". Every rational number is capable of being expressed by \( p/q \) where \( p \) and \( q \) are positive/negative integers and \( q \) is not zero.

Now the point is that the aggregate of rational numbers is by no means sufficient to perform all operations in Mathematics, or to express all situations during the process of calculations of problems which Mathematics undertakes to solve. For example there is every necessity to find the operations like

\[
(2)^{1/2}, (3)^{1/3}, (5)^{2/3}, \ldots
\]

These cannot be defined by means of rational numbers. But there is every necessity to have irrational numbers like the numbers given above. Hence the study of irrational numbers is made from the viewpoint of Dedekind, the great German mathematician.

1. **Dedekind's Theory of real numbers**:

(a) According to Dedekind's Theory of Real Numbers, it is possible to extend the domain of real numbers by bringing in the field the new type of real numbers called the irrational numbers. So far there existed the aggregate of rational numbers only which are viewed, defined or thought of symbolically as \( p/q \). But this way of defining the rational number is not applicable in the case of the new variety of real numbers called the irrational numbers.
(b) He thought that it was possible to invent a definition of real numbers so that the same definition was applicable to both the already existing real numbers called the rational numbers and the irrational numbers to be incorporated newly in the field of real numbers. Therefore if such a new definition of a real number could be found then the domain of real numbers would get extended by the entry of the new type of real numbers called the irrational numbers.

(c) Hence in Dedekind's Theory of Real Numbers the existence of the aggregate of rational numbers is accepted in the very beginning.

(d) To understand the spirit of this theory it is necessary to make a study of the irrational number $\sqrt{2}$.

(i) It is not possible to express this irrational number by means of $p/q$, the definition which is applicable in the case of every rational number.

(ii) By actually going through the process of finding the square-root of $\sqrt{2}$ we see that

$$\sqrt{2} = 1.414212\ldots$$

and we find that the process of extracting the square-root does not end.

(iii) As the process of square-root finding advances, we get a series of the approximate values

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

which on being squared give the values

$$(1)^2, (1.4)^2, (1.41)^2, (1.414)^2, (1.4142)^2, \ldots$$

(iv) Now $\zeta = 1$, which is a rational number less than 1.
(1.4)^2 = 1.96, which is a rational number less than 2

(1.41)^2 = 1.9881, which is a rational number less than 2

(1.414)^2 = 1.999396, which is a rational number less than 2

\[
(1.4142)^2 = 1.99996164, \text{which is a rational number less than 2}
\]

and so on, etc., etc., etc.,

(e) Hence we find that every number is greater than the preceding number and approaches the number 2.

(f) The method of adding the digit "1" to the last digit of the number obtained after chopping off the process of finding the square-root of \(\sqrt{2}\). (Hence of any irrational number)

During the process of square-root finding we get numbers (which are the approximate values of \(\sqrt{2}\))

1, 1.4, 1.41, 1.414, 1.4142, ---------

Now add the digit "1" to the last digit of the number obtained by the process of chopping off the square-root finding after zero, first, second, third, etc., places of decimal. By doing so we get the following new numbers (rational)

\[
\begin{align*}
1 + 1 &= 2 \\
1.4 + 1 &= 1.5 \\
1.41 + 1 &= 1.42 \\
1.414 + 1 &= 1.415 \\
1.4142 + 1 &= 1.4143 \\
\end{align*}
\]

Now the series of numbers
locate the position of the irrational number $\sqrt{2}$ on the continuum. On squaring the above numbers we get that

$$(2)^2 = 4$$

$$(1.5)^2 = 2.25$$

$$(1.42)^2 = 2.0164$$

$$(1.415)^2 = 2.00225$$

$$(1.4143)^2 = 2.00024449$$

And for a comparative study we write them in the form

4.0000000000
2.2500000000
2.0164000000
2.0022500000
2.00024449

from which it follows that each succeeding number is nearer and nearer to the number 2. But this is happening from the right hand side of the continuum towards the number 2.

From the above discussion we now arrive at the stage in which there are two series of numbers on either side of $\sqrt{2}$. The members of the two series are such that

$$1 < \sqrt{2} < 2$$

$$1.4 < \sqrt{2} < 1.5$$

$$1.41 < \sqrt{2} < 1.42$$
\[ 1.414 < \sqrt{2} < 1.415 \]
\[ 1.4142 < \sqrt{2} < 1.4143 \]

---

as \((1.4)^2 = 1.96\), which is less than 2
and \((1.5)^2 = 2.25\), which is greater than 2.

Hence the stage

\[ 1.4 < \sqrt{2} < 1.5 \]

expresses the possibility of locating the position of the approximate value of \(\sqrt{2}\) between the rational numbers 1.4 and 1.5. Similar consideration is applicable in the cases of the other pairs of rational numbers.

Note: The general case: \(\sqrt{n}\)

According to this continue process we shall first express every irrational number in the form

\[ a_{1}b_{1}c_{1}d_{1}e_{1}f_{1}\]

where \(a_{1}, b_{1}, c_{1}, d_{1}, e_{1}, f_{1}\) are all positive integers including zero. The pairs of numbers between which the value of the irrational number lies are

1. \(a_{1}b_{1}\) and \(a_{1}(b_{1} + 1)\)
2. \(a_{1}bc_{1}\) and \(a_{1}b(c_{1} + 1)\)
3. \(a_{1}bcd_{1}\) and \(a_{1}bc(d_{1} + 1)\)
4. \(a_{1}bced_{1}\) and \(a_{1}bced(e_{1} + 1)\)
5. \(a_{1}bce_{1}d_{1}\) and \(a_{1}bce_{1}d(f_{1} + 1)\)

---

of which the first pair \(a_{1}b_{1}\) and \(a_{1}(b_{1} + 1)\) is quite sufficient to locate the position of the irrational number.

We shall make use of the first pair only in framing a
section for an irrational number.

THE NEW WAY TO FRAME SECTIONS for a rational number and for an irrational number:

(1) According to Dedekind there are two types of sections for a rational number and there is only one type of section for an irrational number. Thus in all there are three types of sections for all the real numbers. Each section consists of two classes known as the lower class and the upper class.

(2) In this new approach

(i) The section for a rational number will be called a PARTIALLY SLIDING SECTION

(ii) The section for an irrational number will be called a TOTALLY SLIDING SECTION

(iii) A Partially Sliding section will be of two types:
   (a) The First type of section
   (b) The Second type of section.

and the totally sliding section shall be called

(c) The Third type of section.

(3) Instead of the sections of rational numbers, the section of real numbers will be framed. That is the lower class and the upper class of each section will be of real numbers. For this purpose the following symbols will be used:

(i) The symbol \((-\infty, a)\) means the aggregate of all real numbers ranging from \(-\infty\) to the number \(a\).

(ii) The symbol \((-\infty, a-e)\) means the aggregate of all the real numbers ranging from \(-\infty\) to \(a-e\), where \(e\) tends to zero.
(iii) The symbol \((a, +\infty)\) means the aggregate of all real numbers ranging from "\(a\)" to \(+\infty\).

(iv) The symbol \((a + e, +\infty)\) means the aggregate of all real numbers ranging from "\(a + e\)" to \(+\infty\).

Accordingly the four kinds of classes (two kinds of lower class and two kinds of upper class of real numbers are)

\[L_F, L_S\] and \[R_F, R_S\]

Therefore we have

\[L_F = (-\infty, a)\]
\[L_S = (-\infty, a-e)\]
\[R_F = (a, +\infty)\]
\[R_S = (a+e, +\infty)\]

(4) To frame section for a rational number "\(a\)"

(i) \(L_F\) is called the \(\text{sliding fixed lower class}\);

(ii) \(L_S\) is called the sliding lower class

(iii) \(R_F\) is called the fixed upper class

(iv) \(R_S\) is called the sliding upper class

Hence the section for a rational number is either

\[a = (L_F, R_S)\]

or,

\[a = (L_S, R_F)\]
The section \( (L_p^a, R_3^a) \) will be called the Partially Sliding section of the first kind and will be written in the form
\[
S_1^a (L_p, R_3)
\]
to specify its kind.

Similarly the section \( (L_3^a, R_p^a) \) will be called the Partially Sliding section of the second kind and will be written in the form
\[
S_2^a (L_3, R_p)
\]
to specify its kind.

(5) i. To frame a section for a positive irrational number \( \sqrt{+n} \)

Suppose \( p.q \) and \( p.(q+1) \) is the first localising pair of numbers for the irrational number \( \sqrt{+n} \). Then the section is
\[
\sqrt{+n} = (L_3^{p.q}, R_3^{p.(q+1)})
\]
This section will be called the Totally Sliding section, and will be written in the form
\[
\sqrt{+n} = S_3^3 (L_3^{p.q}, R_3^{p.(q+1)})
\]
to specify its kind.

ii. To frame section for a negative irrational number \( \sqrt{-n} \)

Suppose \( -p.(q+1) \) and \( -p.q \) is the first pair to locate the irrational number \( n \), then its section is
\[
\sqrt{-n} = S_3^3 (L_3^{-p.(q+1)}, R_3^{-p.q})
\]

(6) Hence we shall have the three types of sections as follows

(1) \( S_1^a \)
(2) \( S_2^a \)
(3) \( S_3^3 \)

and, (3) \( S_T \)
These sections will be the central things in performing the four fundamental operations between any two real numbers. The result in every case shall be a section of one of the above types of sections.

(7) The sections will be called

(1) Section of the **First Kind**

(2) Section of the **Second Kind**

(3) Section of the **Third Kind**

And these may be referred to briefly as

\[
\begin{align*}
S_1^P \\
S_2^P \\
S_3^P \\
S_T
\end{align*}
\]

(8) This new way of framing Dedekind's sections is most generalizing as seen from the viewpoint of performing the four fundamental operations.

(9) Furthermore, changing of section from the First Kind to the Second Kind and vice versa will be the new contribution to Dedekind's Theory of numbers. The change of the section will be seen in the resulting section which would come out as the result of a fundamental operation.

(10) The answer to the question, "Why there are two kinds of sections for the same rational number?" will be found in this thesis at the places where a fundamental operation is carried out between two real numbers in the manners regarding the kind of number (rational or irrational) and in its specification as a positive or negative number.
(11) There is complete uniformity in the process of performing any one of the four fundamental operations between any two real numbers.

(12) While carrying out a fundamental operation the following symbols will be used to indicate the uniformity in doing that operation:

\[
\begin{align*}
1 & \rightarrow L_R \leftarrow R_{L} \\
2 & \rightarrow L_R \leftarrow R_{L} \\
3 & \rightarrow L_R \leftarrow R_{L} \\
3 & \rightarrow L_R \leftarrow R_{L}
\end{align*}
\]
Other research work on Number Theory:

1. F. E. Andrews: Revolving Numbers

2. Paul Erdős: On the Normal Number of Prime factors

3. T. Estermann: A new result in the additive Prime Number Theory