CHAPTER V

ON CERTAIN FIXED POINT MAPPINGS IN HAUSDORFF SPACES

5.1 During the past few years a number of authors such as Singh and Zorzitte [1], J.S.W. Wong [1], Fisher and Khan [1], Fisher and Ray [1], Popa [4] etc. have obtained a number of interesting results on fixed points for different type of contractive mappings in Hausdorff spaces. Our objective here is to establish some fixed point theorems for mappings satisfying a more generalized condition.

We first present the following theorem:

THEOREM 1. Let S and T be continuous self-mappings of a Hausdorff space X. Let F be a continuous symmetric mapping of X x X into \( \mathbb{R}_+ \) satisfying the following conditions:

(5.1.1) \( F(x,x) = 0 \) for all \( x \in X \) and \( F(x,y) \neq 0 \) for all \( x, y \in X, \ x \neq y \).

(5.1.2) there exist \( p, q \in \mathbb{N} \) such that

\[
F(S^p x, T^q y) < \max \left\{ \frac{F(x,x)}{F(x,y)} \left[ F(x,y), F(x,S^p x), F(y, T^q y), \frac{F(x,S^p x)}{F(x,y)} \right] \right\}
\]

\[
\leq \min \left\{ \frac{F(x,T^q y), F(y,S^p x)}{F(x,y)} \right\}
\]

for all \( x, y \in X, \ x \neq y \). If for some \( x_0 \in X \), the sequence
\{x_n\}_{n=1}^{\infty}$ consisting of distinct points $x_n$ has a convergent subsequence, where $S^p x_{2n} = x_{2n+1}$ and $T^q x_{2n+1} = x_{2n+2}$ for all $n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, then $S$ and $T$ have a unique common fixed point.

**Proof.** Let $F_n = F(x_n, x_{n+1})$ for $n \in \mathbb{N}_0$. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (5.1.2), we have

$$F_{2n+1} = F(S^p x_{2n}, T^q x_{2n+1})$$

$$< \max \left\{ F(x_{2n}, x_{2n+1}), F(x_{2n}, x_{2n+2}), \frac{F(x_{2n}, x_{2n+1})F(x_{2n+1}, x_{2n+2})}{F(x_{2n}, x_{2n+1})} \right\}$$

$$U \min \left\{ F(x_{2n}, x_{2n+2}), F(x_{2n+1}, x_{2n+1}), \frac{F(x_{2n}, x_{2n+2})F(x_{2n+1}, x_{2n+1})}{F(x_{2n}, x_{2n+1})} \right\}$$

which implies $F_{2n+1} < F_{2n}$ for all $n \in \mathbb{N}_0$. Similarly we can prove that $F_{2n+2} < F_{2n+1}$ for all $n \in \mathbb{N}_0$. Thus $\{F_n\}_{n=0}^{\infty}$ is decreasing and hence converges to some real number $w \in \mathbb{R}_+$. Since $\{x_n\}_{n=0}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=0}^{\infty}$
in $X$ which converges to some $z \in X$, we may take

$$\lim_{k} x_{2n_{k}}.$$

Since $F$ and $S$ are continuous,

$$\text{(5.1.3)} \quad F(z, S^{p}z) = F(\lim_{k} x_{2n_{k}}, S^{p} \lim_{k} x_{2n_{k}})$$

$$= F(\lim_{k} x_{2n_{k}}, \lim_{k} x_{2n_{k}+1})$$

$$= \lim_{k} F(x_{2n_{k}}, x_{2n_{k}+1})$$

$$= w = \lim_{k} F(x_{2n_{k}+1}, x_{2n_{k}+2}) = F(S^{p}z, T^{q}S^{p}z).$$

If $z \neq S^{p}z$, then it follows from (5.1.2) and (5.1.3) that

$$F(S^{p}z, T^{q}S^{p}z) < \max \left\{ F(z, S^{p}z), F(z, S^{p}z), F(S^{p}z, T^{q}S^{p}z), \frac{F(z, S^{p}z)F(S^{p}z, T^{q}S^{p}z)}{F(z, S^{p}z)} \right\}$$

$$\cup \min \left\{ F(z, T^{q}S^{p}z), F(S^{p}z, S^{p}z), \frac{F(z, T^{q}S^{p}z)F(S^{p}z, S^{p}z)}{F(z, S^{p}z)} \right\}$$

i.e.

$$F(S^{p}z, T^{q}S^{p}z) < F(S^{p}z, T^{q}S^{p}z)$$

which is a contradiction. Thus $z = S^{p}z$. Similarly $z = T^{q}z$. 
Let $z_1 \neq z$ be another common fixed point of $S^p$ and $T^q$. Then using (5.1.2), we have

$F(z_1, z) = F(S^p z_1, T^q z)$

$< \max \left\{ \frac{F(z_1, z), F(z_1, S^p z_1), F(z, T^q z), F(z_1, S^p z_1) F(z, T^q z)}{F(z_1, z)} \right\}$

$U \min \left\{ F(z_1, T^q z), F(z, S^p z_1), \frac{F(z_1, T^q z) F(z, S^p z_1)}{F(z_1, z)} \right\}$

i.e. $F(z_1, z) < F(z_1, z)$

which is a contradiction. Hence $z$ is the unique common fixed point of $S^p$ and $T^q$.

We now show that $z$ is a common fixed point of $S$ and $T$.

Since $S^p S z = S S^p z = Sz$, $Sz$ is also a fixed point of $S$.

By the uniqueness of $z$, it follows that $Sz = z$. Similarly $Tz = z$. Suppose $z_2 \neq z$ is another common fixed point of $S$ and $T$. Then

$F(z_2, z) = F(S^p z_2, T^q z)$

$< \max \left\{ \frac{F(z_2, z), F(z_2, S^p z_2), F(z, T^q z), F(z_2, S^p z_2) F(z, T^q z)}{F(z_2, z)} \right\}$

$U \min \left\{ F(z_2, T^q z), F(z, S^p z_2), \frac{F(z_2, T^q z) F(z, S^p z_2)}{F(z_2, z)} \right\}$

i.e. $F(z_2, z) < F(z_2, z)$
a contradiction. Hence $z_2 = z$. This completes the proof.

Since every sequence in a compact Hausdorff space has a convergent subsequence, we have

**Theorem 2.** Let $S$ and $T$ be continuous self-mappings of a compact Hausdorff space $X$. Let $F$ be a continuous symmetric mapping of $X \times X$ into $R_+$ satisfying the condition (5.1.1) and (5.1.2). Then $S$ and $T$ have a unique common fixed point.

5.2 In this section we extend a result of Ray and Chatterjee ([1], Theorem 1) and some other known results to a more general case.

Let $R_+$ be the set of non-negative real numbers and $N$ the set of positive integers.

**Theorem 3.** Let $S$ and $T$ be continuous self-mappings of a Hausdorff space $X$. Let $F$ be a continuous symmetric mapping of $X \times X$ into $R_+$ satisfying the condition (5.1.1) and the following:

(5.2.1) $F(x,z) \leq F(x,y) + F(y,z)$ for all $x,y,z \in X$,

(5.2.2) there exist $p,q \in N$ such that

$$F(S^px,T^qy) < h(F(x,y),F(x,S^px),F(y,T^qy),F(x,T^qy),$$
\( F(y, s^p x), (F(x, y))^{-1} F(x, s^p x) F(y, T^q y), \\
(2F(x, y))^{-1} F(x, T^q y) F(y, s^p x), (2F(x, y))^{-1} F(y, T^q y) F(y, s^p x) \)

for all \( x, y \in X \), \( x \neq y \), where \( h : (R^*)^9 \rightarrow R^+ \) is a non-decreasing in each coordinate variable and satisfies the condition:

\[(5.2.3) \quad g(t) = h(t, t, t, 2t, 2t, t, t, t, t) \leq t \quad \text{for all} \quad t > 0.\]

If for some \( x_0 \in X \), the sequence \( \{ x_n \}_{n=1}^{\infty} \) consisting of distinct points \( x_n \) has a convergent subsequence, where \( x_{2n+1} = s^p x_{2n} \) and \( x_{2n+2} = T^q x_{2n+1} \) for all \( n \in N_0 = N \setminus \{ 0 \} \), then \( S \) and \( T \) have a unique common fixed point.

**Proof.** Let \( F_n = F(x_n, x_{n+1}) \) for \( n \in N_0 \). Taking \( x = x_{2n} \) and \( y = x_{2n+1} \) in (5.2.2) we get

\[ F_{2n+1} = F(s^p x_{2n}, T^q x_{2n+1}) \]

\[ < h(F(x_{2n}, x_{2n+1}), F(x_{2n}, x_{2n+1}), F(x_{2n+1}, x_{2n+2})F(x_{2n}, x_{2n+2}), \\
F(x_{2n+1}, x_{2n+1}), \\
(F(x_{2n}, x_{2n+1}))^{-1} F(x_{2n}, x_{2n+1}) F(x_{2n+1}, x_{2n+2}), \\
(F(x_{2n}, x_{2n+1}))^{-1} F(x_{2n}, x_{2n+2}) F(x_{2n+1}, x_{2n+1}), \\
(2F(x_{2n}, x_{2n+1}))^{-1} F(x_{2n}, x_{2n+1}) F(x_{2n}, x_{2n+1}), \\
(2F(x_{2n}, x_{2n+1}))^{-1} F(x_{2n+1}, x_{2n+2}) F(x_{2n+1}, x_{2n+1}). \]
If \( F_{2n} < F_{2n+1} \), then

\[
F_{2n+1} < h(F_{2n+1}, F_{2n+1}, F_{2n+1}, 2F_{2n+1}, 0, F_{2n+1}, 0, F_{2n+1}, 0) \leq F_{2n+1}
\]

which is a contradiction. Hence \( F_{2n} \geq F_{2n+1} \) for all \( n \in \mathbb{N}_0 \).

Similarly, we can prove that \( F_{2n+1} \geq F_{2n+2} \) for all \( n \in \mathbb{N}_0 \).

Thus \( \{F_n\}_{n=0}^{\infty} \) is decreasing and hence converges to some real number \( w \in \mathbb{R}_+ \). Since \( \{x_n\}_{n=0}^{\infty} \) has a convergent subsequence \( \{x_{n_k}\}_{k=0}^{\infty} \) in \( X \) which converges to some \( z \in X \), we may take

\[
z = \lim_{k} x_{2n_k}.
\]

Since \( F \) and \( S \) are continuous,

\[
F(z, S^p z) = F(\lim_{k} x_{2n_k}, S^p \lim_{k} x_{2n_k})
\]

\[
= F(\lim_{k} x_{2n_k}, \lim_{k} x_{2n_k+1})
\]

\[
= \lim_{k} F(x_{2n_k}, x_{2n_k+1})
\]

\[
= w = \lim_{k} F(x_{2n_k+1}, x_{2n_k+2})
\]

\[
= F(S^p z, T^q S^p z)
\]

If \( z \neq S^p z \), then it follows from (5.2.2) and (5.2.4) that
\[ F(s^p z, t^q s^p z) < h(F(z, s^p z), F(z, s^p z), F(s^p z, t^q s^p z), F(z, t^q s^p z), F(s^p z, s^p z), (F(z, s^p z))^{-1} F(z, s^p z) F(s^p z, t^q s^p z), (F(z, s^p z))^{-1} F(z, t^q s^p z) F(s^p z, s^p z), (2F(z, s^p z))^{-1} F(z, s^p z) F(z, t^q s^p z), (2F(z, s^p z))^{-1} F(s^p z, t^q s^p z) F(s^p z, s^p z)) \leq g(F(s^p z, t^q s^p z)) \leq F(s^p z, t^q s^p z) \]

which is a contradiction. Thus \( z = s^p z \). Similarly, \( z = t^q z \).

If \( z_1(\neq z) \) were another common fixed point of \( s^p \) and \( t^q \), then

\[ F(z_1, z) = F(s^p z_1, t^q z) \]

\[ < h(F(z_1, z), F(z_1, z_1), F(z, z), F(z_1, z), F(z, z_1), (F(z_1, z))^{-1} F(z_1, z) F(z, z), (F(z_1, z))^{-1} F(z_1, z) F(z, z_1), (2F(z_1, z))^{-1} F(z_1, z_1) F(z_1, z), (2F(z_1, z))^{-1} F(z, z) F(z, z_1)) \leq g(F(z_1, z)) \leq F(z_1, z) \]

which is a contradiction. Hence \( z \) is the unique common fixed
point of $S^q$ and $T^q$.

We now show that $z$ is a common fixed point of $S$ and $T$. Since $S^pS = SS^p = Sz$, $Sz$ is also a fixed point of $S$. By the uniqueness of $z$, it follows that $Sz = z$. Similarly, $Tz = z$.

Suppose $z_2 \neq z$ is another common fixed point of $S$ and $T$. Then

$$F(z_2, z) = F(S^p z_2, T^q z)$$

$$< h(F(z_2, z), F(z_2, z), F(z, z), F(z_2, z), F(z, z_2),$$

$$(F(z_2, z))^{-1} F(z_2, z) F(z, z),$$

$$(F(z_2, z))^{-1} F(z_2, z) F(z, z_2),$$

$$(2F(z_2, z))^{-1} F(z_2, z_2) F(z_2, z),$$

$$(2F(z_2, z))^{-1} F(z, z) F(z, z_2))$$

$$< g(F(z_2, z)) \leq F(z_2, z)$$

which is a contradiction. Hence $z_2 = z$. This completes the proof.

**Corollary 1.** Let $S$ and $T$ be continuous self-mappings of a Hausdorff space $X$. Let $F$ be a continuous symmetric mapping of $XXX$ into $R_+$ satisfying (5.1.1), (5.2.1) and
(5.2.5) there exists \( p, q \in \mathbb{N} \) such that

\[
F(S^p x, T^q y) < \alpha_1(x, y) F(x, y) + \alpha_2(x, y) F(x, S^p x)
+ \alpha_3(x, y) F(y, T^q y) + \alpha_4(x, y) \{ F(x, T^q y) + F(y, S^p x) \}
+ \alpha_5(x, y) (F(x, y))^{-1} F(x, S^p x) F(y, T^q y)
+ \alpha_6(x, y) (F(x, y))^{-1} F(x, T^q y) F(y, S^p x)
+ \alpha_7(x, y) (2F(x, y))^{-1} F(x, S^p x) F(x, T^q y)
+ \alpha_8(x, y) (2F(x, y))^{-1} F(y, T^q y) F(y, S^p x)
\]

for all \( x, y \in X, \ x \neq y \), where the mappings \( \alpha_i : X \times X \rightarrow [0, 1] \)

with the property

(5.26) \( K \equiv \sup_{x, y \in X} \{ \alpha_4(x, y) + \sum_{i=1}^{8} \alpha_i(x, y) \} \leq 1. \)

If for some \( x_0 \in X \) the sequence \( \{ x_n \}_{n=0}^{\infty} \) consisting the
distinct points \( x_n \) has a convergent subsequence, where
\( x_{2n+1} = S^p x_{2n} \) and \( x_{2n+2} = T^q x_{2n+1} \) for all \( n \in \mathbb{N}_0 \), then \( S \)
and \( T \) have a unique common fixed point.

**Proof.** Since

\[
\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 + \frac{\alpha_4}{2} (t_4 + t_5) + \alpha_5 t_6 + \alpha_6 t_7
+ \alpha_7 t_8 + \alpha_8 t_9
\]

\( \leq (\alpha_4 + \sum_{i=1}^{8} \alpha_i) \max \left\{ t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}, t_6, t_7, t_8, t_9 \right\} , \)
we take
\[ h(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9) = K \max \{ t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}, t_6, t_7, t_8, t_9 \}. \]
We can easily see that \( h \) satisfies all the conditions of Theorem 3. Hence \( S \) and \( T \) have a unique common fixed point.

**Remarks.**

(1) In Theorem 3, if ' \( < \)' in (5.2.2) is interchanged with ' \( \leq \)' in (5.2.3), then the result of this theorem is still true. Similarly in Corollary 1, if ' \( < \)' in (5.2.5) is interchanged with ' \( \leq \)' in (5.2.6), then the result of this Corollary remains the same.

(2) If we take \( \alpha_i(x, y) = 1 \) and \( \alpha_i(x, y) = 0 \) for \( i = 2, 3, \ldots, 8 \), and \( p = q = 1 \) in Corollary 1, we get a result of Ray ([3], Theorem 1).

(3) Let \( \alpha_1(x, y) = \lambda_1 \in [0, 1] \) be constants for \( i = 1, 2, 3 \) and \( \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0 \) in Corollary 1. Then it reduces to a result of Ray and Chatterjee ([1], Theorem 1).

(4) Let \( X \) be a metric space with metric \( d = F \). Then Theorem 3 and Corollary 1 hold in metric spaces.

(5) Let the space \( X \) of Corollary 1 be a metric space with metric \( d \). Taking \( p = q = 1 \), \( S = T \), \( F = d \), \( \alpha_1(x, y) = 1 \) and \( \alpha_i = 0 \) for \( i = 2, 3, \ldots, 8 \), we get a result of Edelstein ([2], Theorem 1).
Since every sequence in a compact Hausdorff space has a convergent subsequence, we have

**Theorem 4.** Let $S$ and $T$ be continuous self-mappings of a compact Hausdorff space $X$. Let $F$ be a continuous symmetric mapping of $X \times X$ into $\mathbb{R}_+$ satisfying conditions (5.1.1), (5.2.1) and (5.2.2). Then $S$ and $T$ have a unique common fixed point.

**Theorem 5.** Let $T$ be an orbitally continuous self-mapping of a Hausdorff space $X$. Let $F$ be a continuous symmetric mapping of $X \times X$ into $\mathbb{R}_+$ satisfying (5.1.1), (5.2.1) and (5.2.2) with $p = q = 1$ and $S = T$. If for some $x_0 \in X$ the sequence $\{T^n x_0\}_{n=0}^{\infty}$ has a cluster point $z$ in $X$, then $z$ is a fixed point of $T$ and $\lim_{n} T^n x_0 = z$.

**Proof.** Let $x_n = T^n x_0$ and $F_n = F(x_n, x_{n+1})$ for all $n \in \mathbb{N}_0$. If $x_m = x_{m-1}$ for some $m$, then $z = x_{m-1}$ and the assertion follows. Suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ and let $\lim_{n} T^n x_0 = z$. Then as in the proof of Theorem 3 we observe that $\{F_n\}_{n=0}^{\infty}$ is decreasing. Since $\lim_{i} T^{n_1} x_0 = z$ and $T$ is orbitally continuous, it follows that $Tz = \lim_{i} T^{n_1} x_0$, $T^2 z = \lim_{i} T^{n_1+2} x_0$ and

$$(5.2.7) \quad \lim_{i} F(T^{n_1} x_0, T^{n_1+1} x_0) = F(z, Tz)$$
(5.2.8) \[ \lim_{i} F(T^{n_1+1} x_0, T^{n_1+2} x_0) = F(Tz, T^2z). \]

Since \( \{F_n\}_{n=0}^{\infty} \) is convergent sequence and \( \{F_{n_1}\}_{i=0}^{\infty} \) and \( \{F_{n_1+1}\}_{i=0}^{\infty} \) are its subsequences, it follows from (5.2.7) and (5.2.8) that

\[ \lim_{n} F(T^n x_0, T^{n+1} x_0) = F(z, Tz) = F(Tz, T^2z). \]

If \( z \neq Tz \), then by (5.2.2),

\[ F(z, Tz) = F(Tz, T^2z) \]

\[ \leq h(F(z, Tz), F(z, Tz), F(Tz, T^2z), F(z, T^2z), F(Tz, Tz), \]

\[ (F(z, Tz))^{-1} F(z, Tz) F(Tz, T^2z), \]

\[ (F(z, Tz))^{-1} F(z, T^2z) F(Tz, Tz), \]

\[ (2F(z, Tz))^{-1} F(z, Tz) F(z, T^2z), \]

\[ (2F(z, Tz))^{-1} F(Tz, T^2z) F(Tz, Tz) \]

\[ \leq g(F(z, Tz)) \leq F(z, Tz) \]

which is a contradiction. This proves \( Tz = z \).