CHAPTER IV

FIXED POINT THEOREMS FOR MULTIFUNCTIONS

4.1 Let \( CB(X) \) be the set of all non-empty closed bounded subsets of a metric space \((X,d)\) and \( D \) be the Hausdorff Pompeiu metric on \( CB(X) \)

\[
D(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\}
\]

where \( A, B \in CB(X) \) and \( d(x,A) = \inf_{y \in A} d(x,y) \).

Let \( A, B \in CB(X) \) and \( \lambda > 1 \). In what follows, the following well-known fact will be used: For each \( a \in A \), there is a \( b \in B \) such that \( d(a,b) \leq \lambda D(A,B) \).

During the past few years several authors have written a lot about the conditions in which multifunctions admit fixed points. Some recent results are due to Rus [2], Ray [1], Iseki [4], Toru [1], Achari [3] and others. Recently, the following result was given by Popa [2].

THEOREM A. Let \((X,d)\) be a non-empty complete metric space and \( F : (X,d) \rightarrow CB(X) \) a multifunction. If
(4.1.1) \( F : (X,d) \longrightarrow (X,d) \) is u.w.c.,

(4.1.2) \( D(Fx,Fy) \leq k \max \left\{ \frac{d(x,Fx)d(y,Fy)}{d(x,y)}, \frac{d(y,Fx)d(x,Fy)}{d(x,y)} \right\} \)

for all \( x,y \in X, \ x \neq y \) and \( 0 < k < 1 \), then \( F \) has fixed points.

Our objective here is to establish a fixed point theorem for extension of mapping considered by Popa. Before the statement of our theorem we mention some lemmas which will be required in the sequel.

**Lemma 1.** ([2], Theorem 3) Let \( X \) and \( Y \) be two topological spaces. For the multifunction \( F : X \longrightarrow Y \), with \( Y \) is \( T_3 \) space and for which \( F(x_0) \) is a strictly paracompact set, the concept of multifunction u.w.c. in \( x_0 \) coincides with the concept of multifunction u.s.c. in \( x_0 \).

**Lemma 2.** ([2], Lemma 1). Let \( X \) and \( Y \) be two topological spaces and let \( G \subseteq Y \) be some open set. The multifunction \( F : X \longrightarrow Y \) has a closed graph if and only if for each \( x \in X \) and \( y \in Y \) so that \( (x,y) \in G(F) \), there are two open sets \( U \subseteq X \) and \( V \subseteq Y \), containing \( x \) and \( y \),
respectively, so that \( F(U) \cap V = \emptyset \).

**Theorem 1.** Let \((X,d)\) be a non-empty complete metric space and \(F : (X,d) \longrightarrow CB(X)\) a multifunction. If (4.1.1) holds and

(4.1.3) \[
D(Fx, Fy) \leq k \max \left\{ \frac{d(x, y)}{d(x, Fx) d(y, Fy)}, \frac{d(y, Fx) d(x, Fy)}{d(x, y)} \right\},
\]

then \( F \) is a multifunction with fixed points.

**Proof.** Choose a real number \( q \) with \( 1 < q < \frac{1}{k} \).

Let \( x_0 \in X \) and \( x_1 \in F(x_0) \). Then there is an \( x_2 \in F(x_1) \) so that \( d(x_1, x_2) \leq q D(F(x_0), F(x_1)) \). Applying (4.1.3), we have

\[
d(x_1, x_2) \leq q D(F(x_0), F(x_1))
\]

\[
\leq q k \max \left\{ \frac{d(x_0, x_1)}{d(x_0, F(x_0)) d(x_1, F(x_1))}, \frac{d(x_1, F(x_0)) d(x_0, F(x_1))}{d(x_0, x_1)} \right\},
\]

\[
\frac{1}{2} \frac{d(x_0, F(x_0)) d(x_0, F(x_1))}{d(x_0, x_1)}
\]

for all \( x, y \in X, x \neq y \) and \( 0 < k < 1 \), then \( F \) has fixed points.
\[ \leq q k \max \left\{ d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \right. \\
\left. \frac{d(x_0, x_1)d(x_1, x_2)}{d(x_0, x_1)}, \frac{d(x_1, x_2)}{d(x_0, x_1)} \right\}, \]

\[ \frac{1}{2} \frac{d(x_0, x_1)d(x_0, x_2)}{d(x_0, x_1)} \}

\[ \leq q k \max \left\{ d(x_0, x_1), d(x_1, x_2), \frac{1}{2} [d(x_0, x_1) + d(x_1, x_2)] \right\} \]

which implies

\[ d(x_1, x_2) \leq q k d(x_0, x_1). \]

Suppose \( x_3, \ldots x_n, \ldots \) could be chosen so that \( x_n \in F(x_{n-1}) \)
and \( d(x_n, x_{n+1}) \leq q D(F(x_{n-1}), F(x_n)). \)

Repeating the above argument, we obtain

\[ d(x_n, x_{n+1}) \leq (qk)^n d(x_0, x_1). \]

Then by routine calculation, it can be shown that the
sequence \( \{x_n\} \) is Cauchy and since \((X,d)\) is complete, we
have \( x_n \longrightarrow x. \)

The space \((X,d)\) being metric, it is paracompact
and as \( F(x) \) is a closed set, for all \( x \in X \), \( F(x) \) is a
paracompact set in \( X \) and according to theorem 2.6 of
Kessler [1], F(x) is strictly paracompact. Then according to lemma 1, F is u.s.c., according to lemma 2, F has a closed graph and then from \( x_n \in F(x_{n-1}) \) results \( x \in F(x) \), so F has fixed points.

**REMARK.** During the proof, we have supposed that \( x_n \neq x_{n-1} \), because if for a certain \( n \), \( x_n = x_{n-1} \) results that \( x_{n-1} \) is a fixed point.

4.2 We denote

\[
\delta(A,B) = \sup \{ d(a,b) \mid a \in A, b \in B \},
\]

where \( A, B \in CB(x) \). If A consists of a single point 'a', we write

\[
\delta(A,B) = \delta(a,B). \quad \text{If} \quad \delta(A,B) = 0, \quad \text{then}
\]

\[
A = B = \{ a \} \quad \text{(Rus[2], Lemma 1)}.
\]

Let \( S : X \rightarrow X \) be a multifunction. Denote

\[
T(S) = \{ x \in X \mid x \in Sx \}.
\]

We now prove some common fixed point theorems for two multifunctions defined on a non-empty complete metric space which satisfy a new contractive type condition. Popa [3] has also obtained some results along this line.
In order to prove our results, we need the following lemmas:

**LEMMA 3.** Let \((X,d)\) be a non-empty complete metric space and \(F, G : (X,d) \rightarrow CB(X)\) be two multifunctions. If

\[
(4.2.1) \quad D(Fx,Gy) \leq k \max \left\{ \frac{d(x,y) + d(x, Gy)[1 + d(x, Fx) + d(y, Fx)]}{2[1 + \delta(x,y)]}, \frac{d(y, Fx)[1 + d(y, Gy) + d(x, Gy)]}{2[1 + \delta(x,y)]} \right\}
\]

holds for all \(x, y \in X\), \(0 < k < 1\) and \(T(F) \neq \emptyset\), then \(T(G) \neq \emptyset\) and \(T(F) = T(G)\).

**PROOF.** Let \(u \in T(F)\), then \(u \in F(u)\). By (4.2.1) we have

\[
d(u,Gu) \leq D(Fu,Gu)
\]

\[
\leq k \max \left\{ d(u,u), \frac{d(u,Gu)[1 + d(u,Fu) + d(u,Fu)]}{2[1 + \delta(u,u)]}, \frac{d(u,Fu)[1 + d(u,Gu) + d(u,Gu)]}{2[1 + \delta(u,u)]} \right\}
\]

\[
\leq \frac{k}{2} d(u,Gu)
\]
which implies $d(u, Gu) = 0$. Since $Gu$ is closed, this shows that $u \in Gu$, which implies $T(F) \subseteq T(G)$.

Analogously, $T(G) \subseteq T(F)$.

**Lemma 4.** Let $(X, d)$ be a non-empty complete metric space. Let $F, G : (X, d) \rightarrow CB(X)$ be two multifunctions. If

$$D^2(Fx, Gy) \leq k \max \left\{ d(x, Fx)d(y, Gy), d(x, Gy)d(y, Fx), \frac{1}{2} d(x, Gy)d(y, Gy), \frac{1}{2} d(x, Fx)d(y, Fx) \right\}$$

holds for all $x, y \in X$, $0 < k < 1$ and $T(F) \neq \emptyset$, then $T(G) \neq \emptyset$ and $T(F) = T(G)$.

**Proof.** Let $u \in T(F)$, then $u \in Fu$. By (4.2.2) we have

$$d^2(u, Gu) \leq D^2(Fu, Gu) \leq k \max \left\{ d(u, Fu)d(u, Gu), d(u, Gu)d(u, Fu), \frac{1}{2} d(u, Gu)d(u, Gu), \frac{1}{2} d(u, Fu)d(u, Fu) \right\}$$

i.e.,

$$d^2(u, Gu) \leq \frac{k}{2} d^2(u, Gu)$$

which implies $d(u, Gu) = 0$. Since $Gu$ is closed, this shows that $u \in Gu$, which implies $T(F) \subseteq T(G)$.

Analogously, $T(G) \subseteq T(F)$.

**Theorem 2.** Let $(X, d)$ be a non-empty complete metric space and $F, G : X \rightarrow CB(X)$ two multifunctions such that
for all \( x, y \in X \) the inequality (4.2.1) holds where \( 0 < k < 1 \). Then \( F \) and \( G \) have common fixed points and \( T(F) = T(G) \).

**Proof.** Choose a real number \( q \) with \( 1 < q < \frac{1}{k} \). Let \( x_0 \in X \) and \( x_1 \in F(x_0) \). Then there is an \( x_2 \in G(x_1) \) so that \( d(x_1, x_2) \leq q D(Fx_0, Gx_1) \). Suppose \( x_3, x_4, \ldots, x_{2n-1}, x_{2n}, \ldots \) could be chosen so that \( x_{2n-1} \in F(x_{2n-2}), x_{2n} \in G(x_{2n-1}) \) and \( d(x_{2n-1}, x_{2n}) \leq q D(F(x_{2n-2}), G(x_{2n-1})) \), \( d(x_{2n-2}, x_{2n-1}) \leq q D(F(x_{2n-2}), G(x_{2n-3})) \).

Now applying (4.2.1) we have

\[
d(x_{2n-1}, x_{2n}) \leq q k \max\left\{ d(x_{2n-2}, x_{2n-1}) \right\}
\]

\[
\frac{d(x_{2n-2}, Gx_{2n-1})[1 + d(x_{2n-2}, Fx_{2n-2}) + d(x_{2n-1}, Fx_{2n-2})]}{2[1 + \delta(x_{2n-2}, x_{2n-1})]}
\]

\[
\frac{d(x_{2n-1}, Fx_{2n-2})[1 + d(x_{2n-1}, Gx_{2n-1}) + d(x_{2n-2}, Gx_{2n-1})]}{2[1 + \delta(x_{2n-2}, x_{2n-1})]}
\]

\[
\leq q k \max\left\{ d(x_{2n-2}, x_{2n-1}) \right\}
\]

\[
\frac{d(x_{2n-2}, x_{2n})[1 + d(x_{2n-2}, x_{2n-1})]}{2[1 + \delta(x_{2n-2}, x_{2n-1})]}
\]
If \( d(x_{2n-2}, x_{2n-1}) \) is maximum of the two terms of the bracket, then

\[
d(x_{2n-1}, x_{2n}) \leq qk d(x_{2n-2}, x_{2n-1}).
\]

And if \( \frac{d(x_{2n-2}, x_{2n})[1 + d(x_{2n-2}, x_{2n-1})]}{2[1 + \delta(x_{2n-2}, x_{2n-1})]} \) is maximum, then

\[
d(x_{2n-1}, x_{2n}) \leq \frac{qk}{2} d(x_{2n-2}, x_{2n})
\]

i.e.

\[
d(x_{2n-1}, x_{2n}) \leq \frac{qk}{2-qk} d(x_{2n-2}, x_{2n-1}) < qk d(x_{2n-2}, x_{2n-1}).
\]

Thus we have

\[
d(x_{2n-1}, x_{2n}) \leq qk d(x_{2n-2}, x_{2n-1}),
\]

where \( qk < 1 \).

Similarly we have

\[
d(x_{2n-2}, x_{2n-1}) \leq qk d(x_{2n-3}, x_{2n-2}).
\]

Repeating the above argument, we obtain

\[
d(x_n, x_{n+1}) \leq (qk)^n d(x_0, x_1).
\]

Since \( qk < 1 \), then by routine calculation one can show that \( \{x_n\} \) is a Cauchy sequence and since \( X \) is complete, we have \( \lim x_n = u \) for some \( u \in X \).
If we now suppose that \( d(u, Fu) \neq 0 \), then application of (4.2.1) yields

\[
d(x_{2n}, Fu) \leq D(Gx_{2n-1}, Fu)
\]

\[
\leq k \max \left\{ d(u, x_{2n-1}), \frac{d(u, Gx_{2n-1})[1 + d(u, Fu) + d(x_{2n-1}, Fu)]}{2[1 + \delta(u, x_{2n-1})]} \right\}
\]

\[
d(x_{2n-1}, Fu) \left[ 1 + d(x_{2n-1}, Gx_{2n-1}) + d(u, Gx_{2n-1}) \right]
\]

\[
\leq k \max \left\{ d(u, x_{2n-1}), \frac{d(u, x_{2n})[1 + d(u, Fu) + d(x_{2n-1}, Fu)]}{2[1 + \delta(u, x_{2n-1})]} \right\}
\]

\[
d(x_{2n-1}, Fu)[1 + d(x_{2n-1}, x_{2n}) + d(u, x_{2n})]
\]

\[
\leq k \max \left\{ d(u, x_{2n-1}), \frac{d(u, x_{2n})[1 + d(u, Fu) + d(x_{2n-1}, Fu)]}{2[1 + \delta(u, x_{2n-1})]} \right\}
\]

\[
\leq k \max \left\{ d(u, x_{2n-1}), \frac{d(u, x_{2n})[1 + d(u, Fu) + d(x_{2n-1}, Fu)]}{2[1 + \delta(u, x_{2n-1})]} \right\}
\]

and on letting \( n \rightarrow \infty \), we obtain

\[
d(u, Fu) \leq \frac{k}{2} d(u, Fu)
\]

which implies \( d(u, Fu) = 0 \). Since \( Fu \) is closed, this shows that \( u \in Fu \). By lemma 3, \( u \in GU \) and \( T(F) = T(G) \). This completes the proof.

**Remark.** If \( F = G \) we have the following:
THEOREM 3. Let \((X,d)\) be a non-empty complete metric space and let \(F : (X,d) \rightarrow CB(X)\) be a multifunction such that

\[
D(Fx,Fy) \leq \text{max} \left\{ \frac{d(x,y)[1+d(x,Fx)+d(y,Fy)]}{2[1+\delta(x,y)]}, \right. \\
\left. \frac{d(y,Fx)[1+d(y,Fy)+d(x,Fy)]}{2[1+\delta(x,y)]} \right\}
\]

holds for all \(x,y \in X\) and \(0 < k < 1\). Then \(F\) has fixed points.

Our next result is as follows:

THEOREM 4. Let \((X,d)\) be a non-empty complete metric space and \(F, G : X \rightarrow CB(X)\) two multifunctions such that for all \(x,y \in X\) the inequality (4.2.2) holds where \(0 < k < 1\). Then \(F\) and \(G\) have common fixed points and \(T(F) = T(G)\).

PROOF. Choose a real number \(q\) with \(1 < q < \frac{1}{k}\).

Let \(x_0 \in X\) and \(x_1 \in F(x_0)\). Then there is an \(x_2 \in G(x_1)\) so that \(d(x_1,x_2) \leq q D(Fx_0, Gx_1)\).

Suppose \(x_3,x_4, \ldots x_{2n-1}, x_{2n}, \ldots\) could be chosen so that
\[ x_{2n-1} \in F(x_{2n-1}), \ x_{2n} \in G(x_{2n-1}) \quad \text{and} \]
\[ d(x_{2n-1}, x_{2n}) \leq q \ D(Fx_{2n-2}, Gx_{2n-1}), \]
\[ d(x_{2n-2}, x_{2n-1}) \leq q \ D(Fx_{2n-2}, Gx_{2n-3}). \]

By (4.2.2) for \( x = x_{2n-2} \) and \( y = x_{2n-1} \) we have
\[ d^2(x_{2n-1}, x_{2n}) \leq q^2 D^2(Fx_{2n-2}, Gx_{2n-1}) \]
\[ \leq q^2 \cdot \max \left\{ d(x_{2n-2}, Fx_{2n-2}) \cdot d(x_{2n-1}, Gx_{2n-1}), \right. \]
\[ \left. \frac{1}{2} \ d(x_{2n-2}, Gx_{2n-1}) \cdot d(x_{2n-1}, Fx_{2n-2}), \right. \]
\[ \left. \frac{1}{2} \ d(x_{2n-2}, Gx_{2n-1}) \cdot d(x_{2n-1}, Gx_{2n-1}), \right. \]
\[ \left. \frac{1}{2} \ d(x_{2n-2}, Fx_{2n-2}) \cdot d(x_{2n-1}, Fx_{2n-2}) \right\} \]
\[ \leq q^2 k \left\{ d(x_{2n-2}, x_{2n-1}) \cdot d(x_{2n-1}, x_{2n}), \right. \]
\[ \left. \frac{1}{2} \ d(x_{2n-2}, x_{2n}) \cdot d(x_{2n-1}, x_{2n}) \right\} \]

which simply implies
\[ d(x_{2n-1}, x_{2n}) \leq q^2 k \ d(x_{2n-2}, x_{2n-1}). \]

Analogously, we have
\[ d(x_{2n-2}, x_{2n-1}) \leq q^2 d(x_{2n-3}, x_{2n-2}). \]

Repeating the above argument, we obtain

\[ d(x_n, x_{n-1}) \leq (q^2)^n d(x_0, x_1), \]

where \( q^2 < 1 \). Then by routine calculation one can show that \( \{ x_n \} \) is a Cauchy sequence and since \( X \) is complete, we have \( \lim x_n = u \) for some \( u \in X \).

If we now assume that \( d(u, Fu) \neq 0 \), then condition (4.2.2) gives

\[
\begin{align*}
\frac{d^2(x_{2n}, Fu)}{d^2(Gx_{2n-1}, Fu)} & \leq k_{\text{max}} \left\{ d(u, Fu) d(x_{2n-1}, Gx_{2n-1}), \\
\frac{1}{2} d(u, Gx_{2n-1}) d(x_{2n-1}, Fu), \\
\frac{1}{2} d(u, Fu) d(x_{2n-1}, Fu) \right\} \\
& \leq k_{\text{max}} \left\{ d(u, Fu) d(x_{2n-1}, x_{2n}), d(u, x_{2n}) d(x_{2n-1}, Fu), \\
\frac{1}{2} d(u, x_{2n}) d(x_{2n-1}, x_{2n}), \frac{1}{2} d(u, Fu) d(x_{2n-1}, Fu) \right\}
\end{align*}
\]
and on letting $n \to \infty$, we have

$$d^2(u, Fu) \leq \frac{k}{2} d^2(u, Fu)$$

which implies $d(u, Fu) = 0$. Since $Fu$ is closed, this shows that $u \in Fu$. By Lemma 4, $u \in Gu$ and $T(F) = T(G)$. This completes the proof.

**REMARK.** If $F = G$ we have the following:

**THEOREM 5.** Let $(X, d)$ be a non-empty complete metric space and let $F : (X, d) \to CB(X)$ be a multifunction such that

$$D^2(Fx, Fy) \leq k \max \left\{ d(x, Fx)d(y, Fy), d(x, Fy)d(y, Fx), \frac{1}{2} d(x, Fx)d(y, Fy), \frac{1}{2} d(x, Fy)d(y, Fx) \right\}$$

holds for all $x, y \in X$ and $0 < k < 1$. Then $F$ has fixed points.

4.3 Finally we obtain common fixed point theorems for a sequence of multifunctions on a non-empty complete metric space.

**THEOREM 6.** Let $(X, d)$ be a non-empty complete metric space and $\{ F_n \}_{n \in \mathbb{N}}$ be a sequence of multifunctions of $X$ into $CB(X)$ such that
\[ D(F_1 x, F_n y) \leq k \max \left\{ d(x, y), \frac{d(x, F_n y)[1 + d(x, F_1 x) + d(y, F_1 x)]}{2[1 + \delta(x, y)]}, \frac{d(y, F_1 x)[1 + d(y, F_n y) + d(x, F_n y)]}{2[1 + \delta(x, y)]} \right\} \]

holds for all \( x, y \in X, \ 0 < k < 1 \) and \( n \geq 2 \). Then \( \{ F_n \}_{n \in \mathbb{N}} \) has common fixed points and \( T(F_1) = T(F_n) \).

The proof follows by Theorem 2 and Lemma 3.

**Theorem 7.** Let \((X, d)\) be a non-empty complete metric space and \( \{ F_n \}_{n \in \mathbb{N}} \) be a sequence of multifunctions of \( X \) into \( \text{CB}(X) \) such that

\[ D^2(F_1 x, F_n y) \leq k \max \left\{ d(x, F_1 x)d(y, F_n y), d(x, F_n y)d(y, F_1 x), \frac{1}{2} d(x, F_n y)d(y, F_n y), \frac{1}{2} d(x, F_1 x)d(y, F_1 x) \right\} \]

holds for all \( x, y \in X, \ 0 < k < 1 \) and \( n \geq 2 \). Then \( \{ F_n \}_{n \in \mathbb{N}} \) has common fixed points and \( T(F_1) = T(F_n) \).

The proof follows from Theorem 4 and Lemma 4.

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