CHAPTER - VIII

ON SIMULTANEOUS FIXED POINT THEOREMS
IN NORMED LINEAR SPACES

8.1 Opial [1] gave a fixed point theorem for
non-expansive self-mappings of a weakly compact convex
subset K of a Banach space X satisfying a condition
called Opial's condition which we presently define:
X is said to satisfy Opial's condition if for \{x_n\} any
sequence converging weakly to x, then

\[ \lim_{n \to \infty} \inf \| x_n - y \| > \lim_{n \to \infty} \inf \| x_n - x \| \]

for \( x \neq y \). Panja and Bainab [1] recently proved
simultaneous fixed point theorems for a family of mappings
in a weak compact subset of a normed linear space satisfying
Opial's condition. Their main result is as follows:

THEOREM A. Let K be a weak compact subset of a normed
linear space X satisfying Opial's condition and \{T_n\}
be a family of self-mappings on K such that for all values
of n and n'

\[(8.1.1) \ | | T_n(x) - T_{n'}(y) | | \leq a | | x - y | | + b ( | | x - T_n(x) | | + | | y - T_{n'}(y) | | )
+ c ( | | x - T_n(y) | | + | | y - T_n(x) | | ) \]
where \( x, y \in K \) and \( a, b, c, \) are non-negative numbers with \( a + 2b + 2c \leq 1 \) and

\[
\text{Inf}_{n,x \in K} \| x - T_n(x) \| = 0,
\]

then \( \{ T_n \} \) have a common fixed point in \( K \).

Let the family of self-mappings \( \{ T_n \} \) on \( K \) be such that for all values of \( n \) and \( n' \),

\[
\| T_n(x) - T_n'(y) \| \leq \alpha_1 \max \left\{ \| x - y \|, \frac{1}{2} \left( \| x - T_n(x) \| + \| y - T_n'(y) \| \right) \right\}
\]

\[+ \alpha_2 \left( \| x - T_n(x) \| + \| y - T_n'(y) \| \right) \]

where \( x, y \in K \) and \( \alpha_1, \alpha_2 \geq 0 \) with \( \alpha_1 + \alpha_2 \leq 1 \), or

\[
\| T_n(x) - T_n'(y) \| \leq \alpha_1 \max \left\{ \| x - y \|, \frac{1}{2} \left( \| x - T_n(y) \| + \| y - T_n'(x) \| \right) \right\}
\]

\[+ \alpha_2 \left( \| x - T_n'(x) \| + \| y - T_n(y) \| \right) \]

where \( x, y \in K \) and \( \alpha_1, \alpha_2 \geq 0 \) with \( \alpha_1 + 2\alpha_2 \leq 1 \).

It is easy to see that (8.1.1) \( \Rightarrow \) (8.1.3) and (8.1.1) \( \Rightarrow \) (8.1.4).
Our object in this section is to generalize Theorem A and some more results of Panja and Baisnab [1] by replacing condition (8.1.1) by condition (8.1.3) or (8.1.4).

**Theorem 1.** Let $K$ be a weak compact subset of normed linear space $X$ satisfying Opial's condition and $\{T_n\}$ be a family of self-mappings on $K$ such that conditions (8.1.2) and (8.1.3) hold. Then $\{T_n\}$ have a common fixed point in $K$.

**Proof.** Since $\inf_{n,x \in K} ||x - T_n(x)|| = 0$, there exists a sequence $\{x_{n_i}\} \in K$ such that $||x_{n_i} - T_{n_i}(x_{n_i})|| \to 0$ as $i \to \infty$ and there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ converging weakly to an element $z$ (say) in $K$. Now for any $\{T_{r_i}\} \in \{T_n\}$, we have

$$||x_{n_{i_k}} - T_{r_i}(z)||$$

$$\leq ||x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})|| + ||T_{n_{i_k}}(x_{n_{i_k}}) - T_{r_i}(z)||$$

$$\leq ||x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})|| + \alpha_1 \max \left\{||x_{n_{i_k}} - z||, \right\}$$

$$\frac{1}{2} (||x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})|| + ||z - T_{r_i}(z)||)$$
\[ + \alpha_2(||x_{n_{1k}} - T(z)|| + ||z - T_{n_{1k}}(x_{n_{1k}})||) \]

\[ \leq ||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| \]

\[ + \alpha_1 \max \left\{ ||x_{n_{1k}} - z||, \frac{1}{2}(||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| + ||z - T(z)||) \right\} \]

\[ + \alpha_2(||x_{n_{1k}} - T(z)|| + ||z - x_{n_{1k}}|| + ||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})||) \]

Now if \( \max \left\{ ||x_{n_{1k}} - z||, \frac{1}{2}(||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| + ||z - T(z)||) \right\} \)

\[ = \frac{1}{2}(||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| + ||z - T(z)||) \]

then

\[ ||x_{n_{1k}} - T(z)|| \leq ||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| \]

\[ + \frac{\alpha_1}{2}(||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| + ||z - T(z)||) \]

\[ + \alpha_2(||x_{n_{1k}} - T(z)|| + ||z - x_{n_{1k}}|| + ||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})||) \]

\[ \leq ||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| \]

\[ + \frac{\alpha_1}{2}(||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})|| + ||z - x_{n_{1k}}|| + ||x_{n_{1k}} - T(z)||) \]

\[ + \alpha_2(||x_{n_{1k}} - T(z)|| + ||z - x_{n_{1k}}|| + ||x_{n_{1k}} - T_{n_{1k}}(x_{n_{1k}})||) \]

which yields
\[
\|x_{n_{i_k}} - T_{R}(z)\| \leq \frac{1 + \frac{\alpha_1}{2} + \alpha_2}{1 - \frac{\alpha_1}{2} - \alpha_2} \|x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})\| + \frac{\alpha_1 + \alpha_2}{1 - \frac{\alpha_1}{2} - \alpha_2} \|z-x_{n_{i_k}}\|
\]

and

\[
\lim_{k \to \infty} \inf \|x_{n_{i_k}} - T_{R}(z)\| \leq \frac{\frac{\alpha_1}{2} + \alpha_2}{1 - \frac{\alpha_1}{2} - \alpha_2} \lim_{k \to \infty} \inf \|x_{n_{i_k}} - z\|.
\]

Therefore
\[
\lim_{k \to \infty} \inf \|x_{n_{i_k}} - T_{R}(z)\| \leq \lim_{k \to \infty} \inf \|x_{n_{i_k}} - z\|
\]

and by Opial's condition \( z = T_{R}(z) \).

Similarly if \( \|x_{n_{i_k}} - z\| \) is maximum of the two members in the bracket then

\[
\|x_{n_{i_k}} - T_{R}(z)\| \leq \|x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})\| + \alpha_1 \|x_{n_{i_k}} - z\|
\]

\[
+ \alpha_2 (\|x_{n_{i_k}} - T_{R}(z)\| + \|z-x_{n_{i_k}}\|) + \|x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})\|
\]

which implies

\[
\|x_{n_{i_k}} - T_{R}(z)\| \leq \frac{1+\alpha_2}{1-\alpha_2} \|x_{n_{i_k}} - T_{n_{i_k}}(x_{n_{i_k}})\| + \frac{\alpha_1+\alpha_2}{1-\alpha_2} \|x_{n_{i_k}} - z\|
\]
and

\[ \lim_{k \to \infty} \inf \| x_{n_{1k}} - T_r(z) \| \leq \frac{\alpha_1 + \alpha_2}{1 - \alpha_2} \lim_{k \to \infty} \inf \| x_{n_{1k}} - z \|, \]

Hence again we have \( \lim_{k \to \infty} \inf \| x_{n_{1k}} - T_r(z) \| \leq \lim_{k \to \infty} \inf \| x_{n_{1k}} - z \|, \)

and by Opial's condition \( z = T_r(z). \)

This completes the proof.

**Theorem 2.** Let \( K \) be a weak compact subset of a normed linear space \( X \) satisfying Opial's condition and \( \{ T_n \} \) be a family of self-mappings on \( K \) such that the conditions (8.1.2) and (8.1.4) hold. Then \( \{ T_n \} \) have a common fixed point in \( K. \)

**Proof.** As in Theorem 1.

**Remark.** Theorem 1 and 2 assert existence of fixed point common to \( T_n \). If condition (8.1.2) of these theorems is strengthened then common fixed point so obtained can also be determined by a scheme of iteration as shown in the next theorems.

**Theorem 3.** Let \( K \) be a weak compact subset of a normed linear space \( X \) satisfying Opial's condition and \( \{ T_n \} \) be
a family of self-mappings on $K$ satisfying condition (8.1.3). If there is $x \in K$ such that $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$, where $\{x_n = T_n(x_{n-1}), x_0 = x\}$, then the family $\{T_n\}$ have a common fixed point $z$ in $K$ and $\{x_n\}$ converges weakly to $z$.

**Proof.** Here $\{T_n\}$ satisfies all conditions of Theorem 1, an application of which yields a common fixed point $z$ for $\{T_n\}$ where $z$ is weak limit of a subsequence $\{x_{n_1}\}$ of $\{x_n\}$. Next to show that $\{x_n\}$ converges weakly to $z$, at first we prove that for any common fixed point $w$ (say) of $\{T_n\}$, the sequence $\{||x_n - w||\}$ is convergent.

Now $||x_{n+1} - w|| = ||T_{n+1}(x_n) - T_{n+1}(w)||$

\[
\leq \alpha_1 \max \{||x_n - w||,\}
\]

\[
\frac{1}{2}(||x_n - T_{n+1}(x_n)|| + ||w - T_{n+1}(w)||)
\]

\[
+ \alpha_2 (||x_n - T_{n+1}(w)|| + ||w - T_{n+1}(x_n)||)
\]

\[
= \alpha_1 \max \{||x_n - w||, \frac{1}{2}(||x_n - x_{n+1}||)\}
\]

\[
+ \alpha_2 (||x_n - w|| + ||w - x_{n+1}||)
\]

If $\max\{||x_n - w||, \frac{1}{2}(||x_n - x_{n+1}||)\} = \frac{1}{2} ||x_n - x_{n+1}||$, then
\[ \begin{align*}
\|x_{n+1} - w\| & \leq \frac{\alpha_1}{2} (\|x_n - x_{n+1}\| + \alpha_2 (\|x_n - w\| + \|w - x_{n+1}\|)) \\
& \leq \frac{\alpha_1}{2} (\|x_n - w\| + \|w - x_{n+1}\|) + \alpha_2 (\|x_n - w\| + \|w - x_{n+1}\|)
\end{align*} \]

which implies
\[ \|x_{n+1} - w\| \leq \frac{\frac{\alpha_1}{2} + \alpha_2}{1 - \frac{\alpha_1}{2} - \alpha_2} \|x_n - w\| \leq \|x_n - w\|. \]

Similarly, if \( \|x_n - w\| \) is maximum then
\[ \|x_{n+1} - w\| \leq \alpha_1 \|x_n - w\| + \alpha_2 (\|x_n - w\| + \|w - x_{n+1}\|) \]

which yields
\[ \|x_{n+1} - w\| \leq \frac{\frac{\alpha_1}{2} + \alpha_2}{1 - \frac{\alpha_1}{2} - \alpha_2} \|x_n - w\| \leq \|x_n - w\|. \]

Thus in both cases \( \{\|x_n - w\|\} \) is a decreasing sequence and therefore convergent.

Next, if possible, let \( z' \neq z \) be another weak limiting point of \( \{x_n\} \). Then there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) converging weakly to \( z' \) and by the above argument \( z' \) is another common fixed point of \( \{T_n\} \). Since \( X \) satisfies Opial's condition

\[ \lim_{i \to \infty} \inf_{j \neq i} \|x_{n_i} - z\| < \lim_{i \to \infty} \inf_{j \neq i} \|x_{n_i} - z'\|. \]
Hence

$$\lim_{n \to \infty} ||x_n - z|| < \lim_{n \to \infty} ||x_n - z'||,$$

since \( \{ ||x_n - z|| \} \) and \( \{ ||x_n - z'|| \} \) are convergent. Now changing the role of \( \{ x_{n_1} \} \) by \( \{ x_{p_1} \} \), we have

$$\lim_{n \to \infty} ||x_n - z'|| < \lim_{n \to \infty} ||x_n - z||$$

which is a contradiction. Therefore \( z = z' \). Thus \( \{ x_n \} \) converges weakly to \( z \). This completes the proof.

**Theorem 4.** Let \( K \) be a weak compact subset of a normed linear space \( X \) satisfying Opial's condition and \( \{ T_n \} \) be a family of self-mappings on \( K \) satisfying condition (8.1.4). If there is \( x \in K \) such that \( ||x_{n+1} - x_n|| \to 0 \) as \( n \to \infty \) where \( \{ x_n = T_n(x_{n-1}), x_0 = x \} \), then the family \( \{ T_n \} \) have a common fixed point \( z \) in \( K \) and \( \{ x_n \} \) converges weakly to \( z \).

**Proof.** As in Theorem 3.

Next we prove:
THEOREM 5. Let $K$ be a weak compact subset of a normed linear space $X$ satisfying Opial's condition and let $\{T_n\}$ be a sequence of self-mappings on $K$ satisfying condition (8.1.3) and having the fixed points $u_n$. If $\{T_n\}$ converges uniformly to a mapping $T: K \rightarrow K$, then any weak limiting point of $\{u_n\}$ is a fixed point of $T$.

PROOF. Since $\{T_n\}$ converges uniformly to $T: K \rightarrow K$, $T$ also satisfies condition (8.1.3). Let $u_0$ be any weak limiting point of $\{u_n\}$. Then there is a subsequence $\{u_{n_1}\}$ of $\{u_n\}$ which converges weakly to $u_0$.

Now

$$||u_{n_1} - T(u_0)||$$

$$\leq ||u_{n_1} - T(u_{n_1})|| + ||T(u_{n_1}) - T(u_0)||$$

$$\leq ||T_{n_1}(u_{n_1}) - T(u_{n_1})|| + \alpha_1 \max\{||u_{n_1} - u_0||, \frac{1}{2}(||u_{n_1} - T(u_{n_1})|| + ||u_0 - T(u_{n_1})||)\}$$

$$+ ||u_0 - T(u_0)||$$

If $\max\{||u_{n_1} - u_0||, \frac{1}{2}(||u_{n_1} - T(u_{n_1})|| + ||u_0 - T(u_0)||\})$}

$$= \frac{1}{2}(||u_{n_1} - T(u_{n_1})|| + ||u_0 - T(u_0)||),$$

then
\[ ||u_{n_1} - T(u_o)|| \leq ||T_{n_1}(u_{n_1}) - T(u_{n_1})|| \]

\[ + \frac{\alpha_1}{2} (||u_{n_1} - T(u_{n_1})|| + ||u_o - T(u_o)||) \]

\[ + \alpha_2 (||u_{n_1} - T(u_o)|| + ||u_o - T(u_{n_1})||) \]

\[ \leq ||T_{n_1}(u_{n_1}) - T(u_{n_1})|| \]

\[ + \frac{\alpha_1}{2} (||T_{n_1}(u_{n_1}) - T(u_{n_1})|| + ||u_o - u_{n_1}|| + ||u_{n_1} - T(u_o)||) \]

\[ + \alpha_2 (||u_{n_1} - T(u_o)|| + ||u_o - u_{n_1}|| + ||T_{n_1}(u_{n_1}) - T(u_{n_1})||). \]

Thus

\[ ||u_{n_1} - T(u_o)|| \leq \frac{1 + \frac{\alpha_1}{2} + \alpha_2}{\alpha_1 - \frac{1}{2} - \alpha_2} ||T_{n_1}(u_{n_1}) - T(u_{n_1})|| \]

\[ + \frac{\alpha_1}{2} + \alpha_2 \]

\[ \frac{1 - \frac{\alpha_1}{2} - \alpha_2}{||u_{n_1} - u_o||}, \]

and

\[ \lim_{i \to \infty} \inf ||u_{n_1} - T(u_o)|| \leq \frac{\frac{\alpha_1}{2} + \alpha_2}{1 - \frac{\alpha_1}{2} - \alpha_2} \lim_{i \to \infty} \inf ||u_{n_1} - u_o||. \]

Therefore
\[
\lim_{i \to \infty} \inf \|u_{n_1} - T(u_0)\| \leq \lim_{i \to \infty} \inf \|u_{n_1} - u_0\|,
\]

and by Opial's condition, \( u_0 = T(u_0) \).

Similarly,

\[
\text{if } \max \left\{ \|u_{n_1} - u_0\|, \frac{1}{2} \left( \|u_{n_1} - T(u_{n_1})\| + \|u_0 - T(u_0)\| \right) \right\}
\]

\[= \|u_{n_1} - u_0\|, \text{ then}
\]

\[
\|u_{n_1} - T(u_0)\| \leq \|T(u_{n_1}) - T(u_{n_1})\| + a_1 \|u_{n_1} - u_0\|
\]

\[+ a_2 \left( \|u_{n_1} - T(u_0)\| + \|u_0 - T(u_{n_1})\| \right)
\]

\[\leq \|T(u_{n_1}) - T(u_{n_1})\| + a_1 \|u_{n_1} - u_0\|
\]

\[a_2 \left( \|u_{n_1} - T(u_0)\| + \|u_0 - u_{n_1}\| + \|T(u_{n_1}) - T(u_{n_1})\| \right)
\]

Thus

\[
\|u_{n_1} - T(u_0)\| \leq \frac{1 + a_2}{1 - a_2} \|T(u_{n_1}) - T(u_{n_1})\|
\]

\[+ \frac{a_1 + a_2}{1 - a_2} \|u_{n_1} - u_0\|
\]

and

\[
\lim_{i \to \infty} \inf \|u_{n_1} - T(u_0)\| \leq \frac{a_1 + a_2}{1 - a_2} \lim_{i \to \infty} \inf \|u_{n_1} - u_0\|.
\]
Therefore we again have

\[ \lim_{i \to \infty} \inf ||u_{n_i} - T(u_0)|| \leq \lim_{i \to \infty} \inf ||u_{n_i} - u_0||, \]

and by Opial's condition \( u_0 = T(u_0) \).

This completes the proof.

**Theorem 6.** Let \( K \) be a weak compact subset of a normed linear space \( X \) satisfying Opial's condition and let \( \{T_n\} \) be a sequence of self-mappings on \( K \) satisfying condition (8.1.4) and having the fixed points \( u_n \). If \( \{T_n\} \) converges uniformly to mapping \( T : K \to K \), then any weak limiting point of \( \{u_n\} \) is a fixed point of \( T \).

**Proof.** As in Theorem 5.

8.2 Panja and Baisnab([1], Theorem 2.1 and 2.2) also obtained simultaneous fixed point theorems in a weak compact subset of a normed linear space for mappings which are more general than those considered by Reich [1] who proved fixed point theorems for a single mapping in a metric space.

Their main result is as follows:

**Theorem B.** Let \( K \) be a weak compact subset of a normed linear space \( X \). Let \( \{T_n\} \) be a family of self-mappings on \( K \) such that for all values of \( n \) and \( n' \)
(8.2.1) \[ ||T_n(x) - T_n(y)|| \leq a||x-y|| + b(||x-T_n(x)|| + ||y-T_n(y)||), \]

where \( x, y \in K \) and \( 0 \leq a, b < 1 \).

If there is \( x \in K \) such that \( ||x_{n+1} - x_n|| \to 0 \) as \( n \to \infty \)
where \( \{ x_n = T_n(x_{n-1}), x_0 = x \} \), then the family \( \{ T_n \} \) have a
common unique fixed point in \( K \) and \( \{ x_n \} \) converges strongly
to the common fixed point of \( \{ T_n \} \).

Extending the above result we prove the following:

**Theorem 7.** Let \( K \) be a weak compact subset of a normed
linear space \( X \). Let \( \{ T_n \} \) be a family of self-mappings on
\( K \) such that for all values of \( n \) and \( n' \)

(8.2.2) \[ ||T_n(x) - T_{n'}(y)|| \leq a||x-y|| + b(||x-T_n(x)|| + ||y-T_{n'}(y)||) \]

\[ + c(||x-T_{n'}(y)|| + ||y-T_n(x)||) \]

where \( x, y \in K \) and \( a, b, c \geq 0 \) with \( a + 2c < 1 \) and \( b + c < 1 \).

If there is \( x \in K \) such that \( ||x_{n+1} - x_n|| \to 0 \) as \( n \to \infty \)
where \( \{ x_n = T_n(x_{n-1}), x_0 = x \} \), then the family \( \{ T_n \} \) has
a common unique fixed point in \( K \) and \( \{ x_n \} \) converges
strongly to the common fixed point of \( \{ T_n \} \).

**Proof.** For any \( m, n \geq 1 \), we have
\[ \|x_m - x_n\| = \|T_m(x_{m-1}) - T_n(x_{n-1})\| \]

\[ \leq a \|x_{m-1} - x_{n-1}\| + b(\|x_{m-1} - T_m(x_{m-1})\| + \|x_{n-1} - T_n(x_{n-1})\|) \]

\[ + c(\|x_{m-1} - T_n(x_{n-1})\| + \|x_{n-1} - T_m(x_{m-1})\|) \]

\[ \leq a(\|x_{m-1} - x_m\| + \|x_m - x_n\| + \|x_n - x_{n-1}\|) \]

\[ + b(\|x_{m-1} - x_m\| + \|x_{n-1} - x_n\|) \]

\[ + c(\|x_{m-1} - x_m\| + 2\|x_m - x_n\| + \|x_{n-1} - x_n\|). \]

Therefore

\[ \|x_m - x_n\| \leq \frac{a + b + c}{1 - a - 2c} (\|x_{m-1} - x_m\| + \|x_{n-1} - x_n\|) \]

\[ \rightarrow 0 \text{ as } m, n \rightarrow \infty. \]

Thus \( \{x_n\} \) is Cauchy.

If \( \mathcal{X} \) is the completion of \( X \), \( \{x_n\} \) being Cauchy in \( X \), it converges strongly to \( z \) in \( X \) and hence weakly to \( z \).

By weak compactness of \( K \), \( \{x_n\} \) has a weak limit \( w \) in \( K \).

Since weak limit of a sequence is unique, it follows \( z = w \) and hence \( \{x_n\} \) converges strongly to \( z \) in \( K \). Now for any \( T \in \mathcal{T} \), we have
\begin{align*}
|z - T_r(z)| & \leq |z - x_n| + |T_n(x_{n-1}) - T_r(z)| \\
& \leq |z - x_n| + a |x_{n-1} - z| \\
& \quad + b (|x_{n-1} - T_n(x_{n-1})| + |z - T_r(z)|) \\
& \quad + c (|x_{n-1} - T_r(z)| + |z - T_n(x_{n-1})|) \\
& \leq |z - x_n| + a (|x_{n-1} - x_n| + |x_n - z|) \\
& \quad + b (|x_{n-1} - x_n| + |z - T_r(z)|) \\
& \quad + c (|x_{n-1} - x_n| + 2|x_n - z| + |z - T_r(z)|) \\
\end{align*}

Thus

\begin{align*}
|z - T_r(z)| & \leq \frac{1 + a + 2c}{1 - b - c} |x_n - z| + \frac{a + b + c}{1 - b - c} |x_n - x_{n+1}| \\
\end{align*}

\[ \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

Hence \( z = T_r(z) \).

For uniqueness, let \( w \neq z \) be another common fixed point of \( \{T_n\} \). Then by (8.2.2), we have
\[ \|z-w\| = \|T_n(z) - T_n(w)\| \]
\[ \leq a \|z-w\| + b(\|z-T_n(z)\| + \|w-T_n(w)\|) + c(\|z-T_n(w)\| + \|w-T_n(z)\|) \]

Therefore

\[ \|z-w\| \leq (a+2c) \|z-w\| < \|z-w\| \]

a contradiction. Thus \( z = w \) is the unique common fixed point of \( \{T_n\} \) and \( \{x_n\} \) converges strongly to \( z \). This completes the proof.

**REMARK.** On taking \( c = 0 \) in Theorem 7, we get Theorem B.