6.1 Iseki [3] proved the following fixed point theorem for 2-metric space which was obtained by Ćirić [5] for 1-metric space.

**Theorem A.** Let $T$ be an orbitally continuous mapping of a bounded complete 2-metric space $X$ into itself. If $T$ satisfies the condition:

\[(6.1.1) \quad \min \{d(Tx,Ty,a), d(x,Tx,a), d(y,Ty,a)\} \leq q \min \{d(x,Ty,a), d(y,Tx,a)\} \]

for all $x,y,a \in X$, and for some $q$ with $0 < q < 1$, then for each $x \in X$, the sequence \( \{T^n x\} \) (\( n = 1, 2, 3, \ldots \)) converges to a fixed point of $T$.

Recently, Mishra [1] generated this result in the following manner.

**Theorem B.** Let $T$ be an orbitally continuous mapping of a bounded complete 2-metric space $X$ into itself. If $T$ satisfies the condition:
(6.1.2) \[ \min \{d(Tx,Ty,a),d(x,Tx,a),d(y,Ty,a)d(Tx,T^2x,a),d(y,T^2x,a)\} \]
\[ - \min \{d(x,Ty,a),d(y,Tx,a),d(x,T^2x,a),d(Ty,T^2x,a)\} \]
\[ \leq q d(x,y,a) \]

for all \(x,y,a \in X\), and for some \(q\) with \(0 < q < 1\), then for each \(x \in X\), the sequence \(\{T^n x\}\) \((n = 1,2,3, \ldots)\) converges to a fixed point of \(T\).

Motivated by the technique of Taskovic [1], Khan [3] established the following generalization of Theorem A.

**THEOREM C.** Let \(T\) be an orbitally continuous mapping of a complete bounded 2-metric space \(X\) into itself. If \(T\) satisfies the condition:

(6.1.3) there exist real numbers \(\alpha_1, \alpha_2, \alpha_3, \beta\) for every \(x,y,a \in X\) such that

\[ \alpha_1 + \alpha_2 + \alpha_3 > \beta, \quad \beta - \alpha_2 \geq 0, \quad \beta - \alpha_3 \geq 0 \]

and

\[ \alpha_1 d(Tx,Ty,a) + \alpha_2 d(x,Tx,a) + \alpha_3 d(y,Ty,a) \]
\[ - \min \{d(x,Ty,a),d(y,Tx,a)\} \leq \beta d(x,y,a) \]

for all \(x,y,a \in X\), then for each \(x \in X\), the sequence \(\{T^n x\}\) converges to fixed point of \(T\).

In this section, we first obtain the generalization of Theorem B. Before the statement of our theorem we mention
the following lemma of S.L. Singh [2] which is required in
the sequel.

**Lemma.** Let \( \{y_n\} \) be a sequence in a complete 2-metric space
\( X \). If there exists \( h \in (0,1) \) such that

\[
d(y_n, y_{n+1}, a) \leq h d(y_{n-1}, y_n, a)
\]

for all \( n \) and for all \( n \in X \), then the sequence \( \{y_n\} \)
converges to a point in \( X \).

**Theorem 1.** Let \( T \) be an orbitally continuous mapping of
a complete 2-metric space \( X \) into itself satisfying

\[
(6.1.4) \quad \alpha_1 d(Tx, Ty, a) + \alpha_2 d(x, Tx, a) + \alpha_3 d(y, Ty, a)
+ \alpha_4 d(Tx, T^2x, a) + \alpha_5 d(y, T^2x, a)
- \min \{d(x, Ty, a), d(y, Tx, a), d(x, T^2x, a), d(Ty, T^2x, a)\}
\leq \beta d(x, y, a)
\]

for all \( x, y, a \in X \), where \( \alpha_1 (i = 1, 2, \ldots, 5) \) and \( \beta \) are
real numbers with \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 > \beta \) and \( \beta - \alpha_2 \geq 0 \).
Then \( T \) has a fixed point in \( X \).

**Proof.** Pick an arbitrary point \( x_0 \in X \) and define a
sequence \( \{x_n\} \) by

\[
x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, 3, \ldots
\]
If for some \( n \), \( x_n = x_{n+1} \) then the result follows immediately. So let \( x_n \neq x_{n+1} \) for each \( n = 0, 1, 2, 3, \ldots \). Then by (6.1.4) for \( x = x_{n-1} \) and \( y = x_n \), we have

\[
\alpha_1 d(Tx_{n-1}, Tx_n, a) + \alpha_2 d(x_{n-1}, Tx_{n-1}, a) + \alpha_3 d(x_n, Tx_n, a) + \alpha_4 d(Tx_{n-1}, T^2 x_{n-1}, a) + \alpha_5 d(x_n, T^2 x_{n-1}, a)
\]

\[
= \min \{ d(x_{n-1}, Tx_n, a), d(x_n, Tx_{n-1}, a), d(x_{n-1}, T^2 x_{n-1}, a) \}
\]

\[
\leq \beta d(x_{n-1}, x_n, a)
\]

i.e.

\[
\alpha_1 d(x_n, x_{n+1}, a) + \alpha_2 d(x_{n-1}, x_n, a) + \alpha_3 d(x_n, x_{n+1}, a) + \alpha_4 d(x_n, x_{n+1}, a) + \alpha_5 d(x_n, x_{n+1}, a)
\]

\[
\leq \beta d(x_{n-1}, x_n, a)
\]

This implies

\[
d(x_n, x_{n+1}, a) \leq q d(x_{n-1}, x_n, a),
\]

for all \( a \in X \), where \( q = \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5} < 1 \).

Hence in view of the lemma of S.L. Singh, the sequence \( \{x_n\} \) converges to some point \( u \in X \). Thus for all \( a \in X \),

\[
\lim_{n \to \infty} d(T^n x_0, u, a) = 0.
\]

Since \( T \) is orbitally continuous, we have

\[
\lim_{n \to \infty} d(T^{n+1} x_0, Tu, a) = 0
\]
for all \( a \in X \).

Therefore by TA-inequality

\[
d(u, Tu, a) \leq d(u, Tu, T^{n+1}x_0) + d(u, T^{n+1}x_0, a) + d(T^{n+1}x_0, Tu, a)
\]

\[\longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.
\]

Hence \( d(u, Tu, a) = 0 \) for all \( a \in X \). Consequently \( Tu = u \).

We now prove another result similar to Theorem 1 but for different type of mappings.

**THEOREM 2.** Let \( T \) be an orbitally continuous mapping of a complete 2-metric space \( X \) into itself satisfying

\[
(6.1.5) \quad \alpha_1[d(Tx, Ty, a)]^2 + \alpha_2d(x, y, a)d(Tx, Ty, a)
\]

\[+ \alpha_3[d(y, Ty, a)]^2 - \min \{d(x, Tx, a)d(y, Ty, a),
\]

\[d(x, Ty, a)d(y, Tx, a)\}

\[\leq \beta d(x, Tx, a)d(y, Ty, a)
\]

for all \( x, y, a \in X \), where \( \alpha_i (i = 1, 2, 3) \) and \( \beta \) are real numbers with \( \alpha_1 + \alpha_2 + \alpha_3 > \beta, \beta - \alpha_2 \geq 0 \). Then the sequence \( \{T^n x\} \) converges to a fixed point of \( T \).

**PROOF.** Pick an arbitrary point \( x_0 \in X \) and define a sequence \( \{x_n\} \) by
\( x_n = T^{x_{n-1}} = T^n x_0, \ n = 1, 2, 3, \ldots \). If for some \( n \), 
\( x_n = x_{n+1} \), then the result is immediate. So let \( x_n \neq x_{n+1} \) 
for every \( n = 0, 1, 2, \ldots \). Then by (6.1.5) for \( x = x_{n-1} \) 
and \( y = x_n \) we have 

\[
\alpha_1 [d(T^{x_{n-1}}, T^{x_n}, a)]^2 + \alpha_2 d(x_{n-1}, x_n, a)d(T^{x_{n-1}}, T^{x_n}, a) \\
+ \alpha_3 [d(x_n, T^{x_n}, a)]^2 - \min \{ d(x_{n-1}, T^{x_{n-1}}, a)d(x_n, T^{x_n}, a), \ \\
d(x_{n-1}, T^{x_n}, a)d(x_n, T^{x_{n-1}}, a) \} \leq \beta d(x_{n-1}, T^{x_{n-1}}, a)d(x_n, T^{x_n}, a)
\]

i.e.

\[
\alpha_1 [d(x_n, x_{n+1}, a)]^2 + \alpha_2 d(x_{n-1}, x_n, a)d(x_n, x_{n+1}, a) \\
+ \alpha_3 [d(x_n, x_{n+1}, a)]^2 \leq \beta d(x_{n-1}, x_n, a)d(x_n, x_{n+1}, a)
\]

This gives 

\[ d(x_n, x_{n+1}, a) \leq q d(x_{n-1}, x_n, a) \]

for all \( a \in X \), where \( q = \frac{\beta - \alpha_2}{\alpha_1 + \alpha_3} < 1. \)

The remaining proof of this theorem is exactly the same as that of Theorem 1.

**Remark.** We note that the condition of boundedness of 
the space in Theorem A, B and C is not essential.

6.2 Recently, Maiti and Pal [1] obtained an interesting 
extension of Ćirić's fixed point theorem [6] for mappings of an
orbitally complete metric space into itself. Pachpatte [7] also established some extensions of Cirić's fixed point theorem in this direction for mappings of an orbitally complete metric space into itself involving rational expressions.

In this section, we present an analogue of Pachpatte's result in an orbitally complete 2-metric space under slightly different conditions.

**THEOREM 3.** Let $T$ be a self-mapping on a 2-metric space $(X,d)$ satisfying at least one of the following conditions:

(6.2.1) \[ d(x,Tx,a) + d(y,Ty,a) \]

\[ \leq \alpha [d(x,y,a) + \frac{d(y,Tx,a)[1 + d(x,Tx,a) + d(y,Tx,a)]}{[1 + d(x,y,a)]}] \]

\[ 1 \leq \alpha < 2, \]

(6.2.2) \[ d(x,Tx,a) + d(y,Ty,a) \]

\[ \leq \beta [d(x,y,a) + \frac{d(x,Ty,a)[1 + d(x,Tx,a) + d(y,Tx,a)]}{[1 + d(x,y,a)]}] \]

\[ \frac{1}{2} \leq \beta < \frac{2}{3}, \]
\[ (6.2.3) \quad d(x, Tx, a) + d(y, Ty, a) + d(Tx, Ty, a) \]

\[ \leq \gamma \left\{ d(x, y, a) + \frac{d(y, Ty, a)[1 + d(x, Tx, a) + d(y, Tx, a)]}{1 + d(x, y, a)} \right\} \]

\[ 0 \leq \gamma < \frac{3}{2} \]

for all \( x, y, a \in X \). If \((X, d)\) is \((x_0, T)\)-orbitally complete for some \( x_0 \in X \), then the sequence of iterates \( \{ T^n x_0 \} \) converges to a fixed point \( u \in X \) for each \( n = 0, 1, 2, \ldots \).

**Proof.** Define a sequence \( \{ C_n \} \) such that
\[ C_n = d(x_n, x_{n+1}, a), \]
where \( x_n = T^n x (x_0 = x), n = 0, 1, 2, \ldots \). If \( x_n = x_{n+1} \) for some \( n \), then the result is immediate. So let \( x_n \neq x_{n+1} \) for all \( n = 0, 1, 2, \ldots \). Now taking \( x = x_n \) and \( y = x_{n+1} \) in (6.2.1) we have

\[ d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a) \]

\[ \leq \alpha \left\{ d(x_n, x_{n+1}, a) + \frac{d(x_{n+1}, x_{n+1}, a)[1 + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+1}, a)]}{1 + d(x_n, x_{n+1}, a)} \right\} \]

\[ = \alpha d(x_n, x_{n+1}, a) \]

which implies

\[ (6.2.4) \quad C_{n+1} \leq (\alpha - 1) C_n. \]
Similarly, if (6.2.2) and (6.2.3) are true, then correspondingly we obtain

\[(6.2.5) \quad C_{n+1} \leq \left( \frac{2\beta-1}{1-\beta} \right) C_n,\]

and

\[(6.2.6) \quad C_{n+1} \leq \left( \frac{\gamma-1}{2-\gamma} \right) C_n.\]

From (6.2.4) - (6.2.6) we observe that

\[C_{n+1} \leq qC_n\]

for all \(n\), where \(q = \max \left\{ \alpha-1, \frac{2\beta-1}{1-\beta}, \frac{\gamma-1}{2-\gamma} \right\} < 1.\)

In view of the proof of Lemma of S.L. Singh, \(\{x_n\}\) is a Cauchy sequence in \((x_0; T)\)-orbitally complete 2-metric space \(X\), hence converges to some \(u \in X\). Thus for all \(a \in X\),

\[\lim_{n \to \infty} d(T^nx_0, u, a) = 0.\]

Now we shall show that \(u\) is a fixed point of \(T\). We observe that for a pair of points \(x = x_n, y = u\) and for all \(a \in X\) atleast one of the following holds:

\[(6.2.7) \quad d(x_n, x_{n+1}, a) + d(u, Tu, a) \leq \alpha \left\{ d(x_n, u, a) + \frac{d(u, x_{n+1}, a)[1 + d(x_n, x_{n+1}, a) + d(u, x_{n+1}, a)]}{[1 + d(x_n, u, a)]} \right\} \]
\begin{align}
(6.2.8) \quad & d(x_n, x_{n+1}, a) + d(u, Tu, a) \\
& \leq \beta \left\{ d(x_n, u, a) + \frac{d(x_n, Tu, a)[1 + d(x_n, x_{n+1}, a) + d(u, x_{n+1}, a)]}{1 + d(x_n, u, a)} \right\}
\end{align}

\begin{align}
(6.2.9) \quad & d(x_n, x_{n+1}, a) + d(u, Tu, a) + d(x_{n+1}, Tu, a) \\
& \leq \gamma \left\{ d(x_n, u, a) + \frac{d(u, Tu, a)[1 + d(x_n, x_{n+1}, a) + d(u, x_{n+1}, a)]}{1 + d(x_n, u, a)} \right\}
\end{align}

By proceeding along the sequence \{x_n\} we obtain an infinite set of values of \( n \), say \( \{n_k\} \), such that at least one of the relations (6.2.7) \( \cdots \) (6.2.9) is satisfied by the pair of points \( x = x_{n_k}, y = u \) and all \( a \in X \). Let \( k \rightarrow \infty \), we derive

\begin{align*}
& d(u, Tu, a) \leq 0, \\
& d(u, Tu, a) \leq \beta d(u, Tu, a)
\end{align*}

and

\begin{align*}
& d(u, Tu, a) \leq \frac{\gamma}{2} d(u, Tu, a)
\end{align*}

in cases (6.2.4) \( \cdots \) (6.2.6) respectively. All these possibilities lead to the fact that \( u \) is a fixed point of \( T \). This completes the proof.
6.3 Finally, we study some results on fixed points of mappings on 2-metric spaces satisfying conditions of the contractive type. Sharma [3] and Khan [2] have also obtained some results in this direction.

We recall the following concept introduced by Sharma [3]. A 2-metric space \((X, d)\) is said to have the property \((s)\) if \(d(x, y, z) = 0\) implies that at least two of the points \(x, y, z\) are equal. We now prove the following:

**Theorem 4.** Let \((X, d)\) be a 2-metric space having the property \((s)\) and let \(T\) be a self-mapping on \(X\) with \(d\) continuous such that

\[
(6.3.1) \quad d(Tx, Ty, a) < 4d(x, y, a) - d(x, Tx, a) - d(y, Ty, a)
\]

\[
- \frac{d(x, Tx, a) d(y, Ty, a)}{d(x, y, a)}
\]

for all \(x, y, a \in X\) and all different,

\[
(6.3.2) \quad \text{there exists a point } x_0 \in X \text{ such that the sequence } \{x_n\}, \text{ where } x_n = T_{x_{n-1}}, n \geq 1 \text{ and } x_n \neq x_{n+1}, \text{ has a cluster point } z,
\]

\[
(6.3.3) \quad X - \{\overline{x}_n\} \neq \emptyset, \text{ and}
\]

\[
(6.3.4) \quad T \text{ is } (x_0, T)\text{-orbitally continuous.}
\]
Then $z$ is the fixed point of $T$.

**Proof.** Let $a \in X - \mathcal{C} \overline{\mathcal{F}}$, the existence of which is ensured by (6.3.3). Now for any $n \geq 1$, applying (6.3.1) we have

$$d(x_n, x_{n+1}, a) = d(Tx_{n-1}, Tx_n, a)$$

$$< 4d(x_{n-1}, x_n, a) - d(x_{n-1}, x_n, a) - d(x_n, x_{n+1}, a)$$

$$= \frac{d(x_{n-1}, x_n, a) d(x_n, x_{n+1}, a)}{d(x_{n-1}, x_n, a)}.$$

This gives

$$d(x_n, x_{n+1}, a) < d(x_{n-1}, x_n, a).$$

Thus, $\{d(x_n, x_{n+1}, a)\}$ being a non-increasing sequence of non-negative reals, is convergent. Let

$$d(x_n, x_{n+1}, a) \rightarrow f(a) \geq 0.$$

Also by (6.3.2), there exists a sub-sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightarrow z.$$
Now using (6.3.4), we get

\[ T(x_{n_1}) = x_{n_1+1} \longrightarrow Tz, \]

and

\[ T(x_{n_1+1}) = x_{n_1+2} \longrightarrow T^2z. \]

If \( z \neq Tz \), then

\[
0 < d(z, Tz, a) = \lim_{i \to \infty} d(x_{n_1}^{i}, x_{n_1+1}^{i}, a)
\]

\[ = r(a) \]

\[ = \lim_{i \to \infty} d(x_{n_1+1}^{i}, x_{n_1+2}^{i}, a) \]

\[ = d(Tz, T^2z, a) \]

\[ < 4d(z, Tz, a) - d(z, Tz, a) - d(Tz, T^2z, a) \]

\[
= \frac{d(z, Tz, a) \cdot d(Tz, T^2z, a)}{d(z, Tz, a)}
\]

\[ = 4d(z, Tz, a) - d(z, Tz, a) - 2d(z, Tz, a) \]

\[ = d(z, Tz, a) \]

a contradiction. Thus \( z = Tz \).
This completes the proof.

We now obtain a result for a pair of mappings.

**Theorem 5.** Let \((X,d)\) be a 2-metric space having the property \((a)\) and let \(S,T\) be a pair of continuous self-mappings on \(X\) with \(d\) continuous such that

\[
(6.3.5) \quad d(Sx,Ty,a) < 4d(x,y,a) - d(x,Sx,a) - d(y,Ty,a)
\]

\[
- \frac{d(x,Sx,a) d(y,Ty,a)}{d(x,y,a)}
\]

for all \(x,y,a \in X\) and all different,

\[
(6.3.6) \quad \text{there exists a point } x_0 \in X \text{ such that the sequence } \{x_n\} \text{ defined by}
\]

\[
x_n = \begin{cases} 
Sx_{n-1}, & \text{n is odd} \\
Tx_{n-1}, & \text{n is even}
\end{cases}
\]

\(x_n \neq x_{n+1}\) for all \(n\), satisfies condition \((6.3.3)\) and contains a subsequence \(\{x_{n_k}\}\) which converges to a point \(z \in X\). Then either \(S\) or \(T\) has a fixed point \(z\).

**Proof.** Let \(a \in X - \{x_n\}\), the existence of which is ensured by \((6.3.3)\). Now, when \(n\) is odd...
\[ d(x_n, x_{n+1}, a) = d(Sx_{n-1}, Tx_n, a) \]
\[ < 4d(x_{n-1}, x_n, a) - d(x_{n-1}, x_n, a) - d(x_n, x_{n+1}, a) \]
\[ \frac{d(x_{n-1}, x_n, a) d(x_n, x_{n+1}, a)}{d(x_{n-1}, x_n, a)} \]

which gives

\[ d(x_n, x_{n+1}, a) < d(x_{n-1}, x_n, a). \]

Similarly, when \( n \) is even

\[ d(x_n, x_{n+1}, a) = d(Tx_{n-1}, Sx_n, a) = d(Sx_n, Tx_{n-1}, a) \]
\[ < 4d(x_n, x_{n-1}, a) - d(x_n, x_{n+1}, a) - d(x_{n-1}, x_n, a) \]
\[ \frac{d(x_n, x_{n+1}, a) d(x_{n-1}, x_n, a)}{d(x_{n-1}, x_n, a)} \]

which gives

\[ d(x_n, x_{n+1}, a) < d(x_{n-1}, x_n, a). \]

Thus

\[ d(x_n, x_{n+1}, a) < d(x_{n-1}, x_n, a) \]

for all \( n \). Therefore \( \{d(x_n, x_{n+1}, a)\} \) is a non-increasing
sequence of non-negative real numbers and hence converges to a real number $r(a) \geq 0$. By condition (6.3.6) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to $z \in X$. Now choose a subsequence $\{x_{n_{k_1}}\}$ of $\{x_{n_k}\}$ such that all $n_{k_1}$'s are either even or odd. If all $n_{k_1}$'s are even then $x_{n_{k_1}} \to z$ and continuity of $S$ and $T$ imply that

$$Sx_{n_{k_1}} = x_{n_{k_1}} + l \to Sz \text{ and } Tx_{n_{k_1}} + 1 = x_{n_{k_1}} + 2 \to TSz.$$ 

If $Sz \neq z$, then

$$0 < d(Sz,z,a) = \lim d(x_{n_{k_1}}, x_{n_{k_1}} + 1, a)$$

$$= r(a)$$

$$= \lim d(x_{n_{k_1}} + 1, x_{n_{k_1}} + 2, a)$$

$$= d(Sz,TSz,a)$$

$$< 4d(z,Sz,a) - d(z,Sz,a) - d(Sz,TSz,a)$$

$$d(z,Sz,a)$$

$$= d(z,Sz,a)$$

leads to a contradiction and so $Sz = z$. 

Similarly, if all $n_{k_i}$'s are odd, we can prove that $Tz = z$.

Hence a fixed point of either $S$ or $T$ is assured.

This completes the proof of the theorem.