CHAPTER V

ON THE STRONG LOGARITHMIC SUMMABILITY OF A SERIES ASSOCIATED WITH FOURIER SERIES AND OF A SERIES ASSOCIATED WITH THE DERIVED FOURIER SERIES.

5.1. Let \( f \in L(-\pi, \pi) \) and be periodic outside with a period \( 2\pi \) and let its Fourier series be

\[
(5.1.1) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x) .
\]

Then the differentiated Fourier series of (5.1.1) is

\[
(5.1.2) \quad \sum_{n=1}^{\infty} n \left\{ b_n \cos nx - a_n \sin nx \right\} = \sum_{n=1}^{\infty} n B_n(x) .
\]

Here we study the series

\[
(5.1.3) \quad \sum_{n=1}^{\infty} \frac{S_n^* - f}{n}
\]

where

\[
(5.1.4) \quad S_n^* = \left\{ \frac{1}{2} a_0 + \sum_{u=1}^{n} A_u(x) \right\} - \left\{ \frac{a_n \cos nx + b_n \sin nx}{2} \right\}
\]

and the series

\[
(5.1.5) \quad \sum_{n=1}^{\infty} \frac{S_n(x)}{n}
\]

where

\[
(5.1.6) \quad S_n(x) = \sum_{\gamma=1}^{n} \gamma B_\gamma(x) .
\]
We set

\[(5.1.7) \quad \Phi(t) = \Phi_X(t) = \left\{ \varphi(x+t) + \varphi(x-t) - 2\varphi(x) \right\}, \]

\[(5.1.8) \quad \Phi_1(t) = \frac{1}{t} \int_0^t \varphi(u) \, du, \]

\[(5.1.9) \quad \Psi(t) = \left\{ \varphi(x+t) - \varphi(x-t) \right\}, \]

\[(5.1.10) \quad q(t) = \frac{\Psi(t)}{4 \sin t/2} \]

and designate by \( S_{\gamma}^*(x) \) the partial sum of order \( \gamma \) of the series \( (5.1.3) \) and by \( \gamma_{\gamma}^*(x) \) the partial sum of order \( \gamma \) of the series \( (5.1.5) \).

The present chapter is devoted to the discussion of the strong logarithmic summability of the series \( (5.1.3) \) and \( (5.1.5) \) and here the following theorems have been proved.

**Theorem 1.** Let \( \{ \gamma_j \} \) be an increasing sequence of positive integers satisfying the following conditions:

a) There exists a constant \( c > 2 \) such that

\[ (\gamma_{j+1} - \gamma_j) / \gamma_j \geq \frac{c}{j} \]

b) There exists a lacunary sequence \( j_0 \) \((j_0 + 1, \gamma_j, \gamma_{j+1})\) such that \( \gamma_{j+1} / \gamma_j < B \), \( B \) being a constant.

If \( p \) and \( q \) are the conjugate indices (i.e., \( p + \frac{1}{q} = 1 \)) and \( q > p > 1 \); and if

\[(5.1.11) \quad \int_0^t |\varphi(\omega)|^b \, d\omega = o(t \log t)^{-1}, \]
**Theorem 2.** If the sequence \( \{ n_j \} \) satisfies the conditions a) and b); if for the conjugate indices \( p \) and \( q \), such that \( q \leq p \leq 1 \), we have

\[
\int_0^t |g(u)|^p \, du = o(t \log^{p-1} t),
\]

\[
\int_0^t |g(u)| \, du = o(t)
\]

and if

\[
\frac{1}{K} \int_0^K g(t) \cos \sec \frac{t}{2} \, dt
\]

exists as a Cauchy integral, then

\[
\sum_{j=1}^K \int \left| s_{n_j}(x) - s \right|^{q_p} = o(\log K)
\]

for all positive \( q \leq q_p \), \( s \) being a constant.

**Corollary to Theorem 2.** If the sequence \( \{ n_j \} \) satisfies the conditions a) and b); if for some \( q \leq 2 \) we have

\[
\int_0^t |g(u)|^2 \, du = o(t \log^{2/q} t),
\]

\[
\int_0^t |g(u)| \, du = o(t)
\]

and if the integral (5.1.20) exists in the sense of Cauchy, then

\[
\sum_{j=1}^K \int \left| s_{n_j}(x) - s \right|^{q_p} = o(\log K),
\]
converges is the existence of the integral (5.1.13).

Since (5.3.2) holds under the hypotheses of Theorem 1, this lemma holds true, it being proved in section 8 of chapter I.

**Lemma 2.** If

\[
(5.3.4) \quad \int_0^t |g(u)| \, du = o(t),
\]

then a necessary and sufficient condition that

\[
(5.3.5) \quad \frac{1}{\alpha} \int_0^\kappa g(t) \cos \frac{t}{\alpha} \, P(\kappa, t) \, dt
\]

converges is the existence of the integral (5.1.20)

This lemma has also been proved in section 8 of chapter I.

**5.4. Proof of Theorem 1.** $S_n^*(x)$, the partial sum of order $n$ of the series (5.1.3) is given by (1.8.3), when Lemma 1 holds true, as

\[
S_n^*(x) - S^* = -\frac{1}{\alpha} \int_0^\kappa \frac{\phi_1(t)}{t} \sin nt \, dt + o(1)
\]

\[
= -\frac{1}{\alpha} \int_0^{\gamma/\alpha} \frac{\phi_1(t)}{t} \sin nt \, dt - \int_0^{\gamma/\alpha} \frac{\phi_1(t)}{t} \sin nt \, dt + o(1)
\]

\[
= J_1 + J_2 + o(1),
\]
say, $s^*$ being a constant.

If $\Xi_1(t) = \int_0^1 f_1(u) \, du$, then since the existence of the integral of Theorem 5.13 implies $\Xi_1(t) = o(t)$, we have for $\varphi_0 / n = c$,

$$J_1 = -\frac{1}{\lambda} \int_0^c \Xi_1(t) \frac{\sin \pi t}{t} \, dt$$

$$= -\frac{1}{\lambda} \left[ \sin \pi c \Xi_1(c) \frac{c}{\pi} \right] + \frac{1}{\lambda} \int_0^c (\cos \pi t - \frac{\pi}{\lambda} \sin \pi t) \Xi_1(t) \, dt$$

$$= o(1) + \int_0^c o\left( \frac{n}{\ell} + \frac{n}{\ell^2} \right) \, dt$$

$$= o(1) ,$$

when $\varphi_0$ is fixed. Hence as $n_j \to \infty$ we have

$$S_{n_j}(x) - S^* = -\frac{1}{\lambda} \int_0^c \Xi_1(t) \sin \pi n_j t \frac{1}{t} \, dt + o(1) ,$$

provided that $c = O\left( \frac{1}{n_j} \right)$.

Consider now some $\lambda = 2^j$. In order to prove the theorem we shall just prove that

$$\sum_{J=\lambda}^{3\lambda-1} \frac{1}{J} \left| S_{n_j}(x) - S^* \right| = o\left( \frac{\log \lambda}{\lambda} \right)$$

for $\lambda \to \infty$. Or in view of (5.4.1) we shall first prove that

$$\sum_{J=\lambda}^{3\lambda-1} \frac{1}{J} \left| \int_{c}^{\pi} \frac{f_1(t)}{t} \sin \pi n_j t \, dt \right| = o\left( \frac{\log \lambda}{\lambda} \right)$$
for $\lambda \to \infty$. Since
\[
\frac{n_{2\lambda}}{n_\lambda} = \frac{n_{2\lambda+1}}{n_{2\lambda}} = \frac{n_{j\omega+1}}{n_{j\omega}} \leq B,
\]
we have $\frac{1}{n_\lambda} = O\left(\frac{1}{n_{2\lambda}}\right)$.

Again, since $\lambda \leq j \leq 2\lambda - 1 < 2\lambda$, we have $\frac{1}{n_\lambda} = O\left(\frac{1}{n_j}\right)$.

Therefore we can replace $\tau = O\left(\frac{1}{n_j}\right)$ in (5.4.2) by $\pi/n_\lambda$.

Thus we have to show that
\[
\tau = \sum_{j=\lambda}^{2\lambda-1} \frac{1}{j} \left| \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \sin n_j t \, dt \right|^q = o\left(\frac{\log \lambda}{\lambda}\right).
\]

To prove this, consider the sum,
\[
\Omega_\lambda = \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \sin n_j t \, dt
\]
\[
= \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \sin n_j t \, dt
\]
where the coefficients $\xi_j$ are arbitrary.

One can write $n_\lambda = \lambda^{I+i\tau}$, where $I = I(\lambda)$ is an integer and $\tau = \tau(\lambda)$ is such that $0 < \tau \leq 1$.

Obviously $I > 0$; also $I \leq A$. But $I = 0$ only if $\tau = 1$. If $I > 0$, we have

\begin{align*}
(5.4.3) \quad \Omega_\lambda &= \left\{ \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \, dt \right\} + \left\{ \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \, dt \right\} + \left\{ \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \, dt \right\} + \left\{ \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \, dt \right\} \\
&= \int_{\pi/n_\lambda}^{\pi} \frac{\Phi_j(t)}{t} \, dt.
\end{align*}
\[(90)\]
\[\quad + \int_{\pi \lambda/\pi}^{\pi \lambda/\pi} + \frac{\phi(t)}{t} \left( \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{\lambda} \sin (\xi_j t) \right) dt\]
\[= \Theta_1 + \Theta_2 + \ldots + \Theta_i + \ldots + \Theta_{\lambda} + \nu,\]

say. In the case when \(I = 0\), we have only \(\Theta = \nu\) and in this case as we have already remarked, \(f = 1\).

Consider the integral

\[\Theta_i = \int_{\pi \lambda/\pi}^{\pi \lambda/\pi} \frac{\phi(t)}{t} \left( \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{\lambda} \sin (\xi_j t) \right) dt.\]

Substituting \(t = u \lambda/\pi\) in \(\Theta_i\), we have

\[\Theta_i = \int \frac{\phi(u \lambda/\pi)}{u \lambda/\pi} \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{\lambda} \sin \left( \frac{\xi_j \lambda u}{\pi} \right) \right\} du.
\]

Now, since \(i > 1\), \(n_j > n\lambda\) and \(j < 2\lambda\), we have from hypothesis a) of Theorem 1

\[(5.4.4) \quad \frac{(n_{j+1} - n_j) \lambda^i}{\lambda \lambda} > \frac{(n_{j+1} - n_j) \lambda}{n_j} > \frac{(n_{j+1} - n_j) j}{\lambda n_j} \quad > \frac{\xi}{\lambda} > 1.\]

Hence the difference of \(n_{j+1} \lambda^i/\lambda\lambda\) and \(n_j \lambda^i/\lambda\lambda\) is greater than 1. Therefore, all the integers \([n_j \lambda^i/\lambda\lambda]\) are different. (Henceforth we shall denote by \([\gamma]\), the integral part of \(\gamma\)).

Instead of studying the integral \(\Theta_i\), we shall study the integral
\[
\Theta^* = \int_0^{\kappa} \frac{\Phi_i(\mu u)}{u} \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \sin(n_j \mu u) \right\} du,
\]

where

\[(5.4.5) \quad 0 < \mu < 1, \quad (n_{j+1} - n_j) \mu > 1 \quad \text{and} \quad n_j \mu > \lambda.\]

It can be easily seen that these properties are verified by \( \mu = \lambda^i / n_\lambda\); the first, because \( \lambda^i / n_\lambda = \lambda^i / \lambda = I + i\) and \(I - i > 0\) and \(0 < \mu < 1\), the second simply by replacing \(\lambda^i / n_\lambda\) by \(\mu\) in (5.4.4) and the third, because \(n_j \mu > n_\lambda\); \(\lambda^i > \lambda\), therefore \(n_j \lambda^i / n_\lambda > \lambda\).

Writing \(\delta_j = (n_j \mu - [n_j \mu])/2\)

and \(\Delta_j = (n_j \mu + [n_j \mu])/2\),

so that \(0 \leq \delta_j < \frac{1}{2}\) and \(\Delta_j > [n_j \mu]\) . To find a convenient bound of the integral \(\Theta^*\), consider the integral

\[
\Theta^{**} = \int_0^{\kappa} \frac{\Phi_i(\mu u)}{u} \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \sin([n_j \mu] u) \right\} du
\]

and the difference \(\Theta^* - \Theta^{**}\).

From Hölder's inequality, we have

\[
|\Theta^{**}| \leq \left\{ \int_0^{\kappa} \left| \Phi_i(\mu u) \right|^p \frac{1}{u} \right\}^{1/p} \left\{ \int_0^{\kappa} \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \sin([n_j \mu] u) \right\}^{1/q} du
\]

\[(5.4.6) \quad = kL,\]
say. From (5.4.5) we have \( \lambda \mu \leq n_j < n_2 \lambda = 2 \lambda^{1+1}. \)

Hence \( \log(\lambda/\mu) = O(\log \lambda) \) for \( \lambda \) large. Now,
\[
K^b = \int_{\lambda/\mu}^{\mu/\lambda} \frac{\phi_1(b)}{\mu b} \, du = \mu^{b-1} \int_{\lambda/\mu}^{\mu/\lambda} \frac{\phi_1(b)}{b} \, dt < \mu^{b-1} \int_{\lambda/\mu}^{\mu/\lambda} \frac{\phi_1(b)}{b} \, dt.
\]

Integrating by parts and using (5.1.11) we have
\[
K^b = \mu^{b-1} \left\{ o \left( \frac{\log (b-1) u}{u} \right) \right\}_{\lambda/\mu}^{\mu/\lambda} + \int_{\lambda/\mu}^{\mu/\lambda} o \left( \frac{\log (b-1) u}{u} \right) \, du
\]
\[
= \mu^{b-1} \left\{ o \left( \frac{1}{\mu^{b-1}} \log (b-1) \lambda \right) + \int_{\log 1/\mu}^{\log \lambda} o \left( e (b-1) u \right) \, du \right\}
\]
\[
= o \left( \lambda^{b-1} \log (b-1) \lambda \right) + \mu^{b-1} \left\{ \log (b-1) \left( \sum_{j=1}^{\lambda} \frac{e (b-1) u}{u} \right) \right\}
\]
\[
= o \left( \lambda^{b-1} \log (b-1) \lambda \right).
\]

Therefore, we have
\[
(5.4.7) \quad K = o \left( \lambda^{1+1-1} \log 1/\mu \right).
\]

Again, since \( q > b \) and since all the \( \lceil n_j \mu \rceil \) are different, \( L \) is smaller, by virtue of the theorem of Hausdorff-Young, than the product of \( \left( \sum_{j=1}^{\lambda} \frac{\eta_j}{j} \right)^b \) and an absolute constant. Hence
\[(5.4.8) \quad L = O \left( \sum_{j=\lambda}^{2\lambda-1} \left| \xi_j \right|^p \right)^{1/p} = O \left\{ \frac{1}{\lambda} \left( \sum_{j=\lambda}^{2\lambda-1} \left| \xi_j \right|^p \right)^{1/p} \right\} \]

From \((5.4.6)\), \((5.4.7)\) and \((5.4.8)\) we have

\[(5.4.9) \quad \left| \Theta^* \right| = O \left( \frac{\log \lambda}{\lambda^{1/p}} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} \left| \xi_j \right|^p \right\}^{1/p} \]

Now

\[\Theta^* - \Theta^{**} = \int_{\pi/\lambda}^{\pi} \frac{\phi_1(\mu \omega)}{u} \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \left( \sin \eta_j \mu u - \sin \left[ \eta_j \mu \right] u \right) \right\} du \]

\[(5.4.10) \quad = 2 \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \int_{\pi/\lambda}^{\pi} \frac{\phi_1(\mu \omega)}{u} \sin \eta_j \mu \cos \Delta_j \mu u \, du \]

To find an upper bound for \( \left| \Theta^* - \Theta^{**} \right| \), consider the integral

\[(5.4.11) \quad \mathcal{W} = \int_{\eta}^{\frac{5\mu}{\eta}} \frac{\phi_1(\mu \omega)}{u} \sin \eta \mu \cos \theta \, du \]

where \( 0 < \eta < \frac{\pi}{2} \), \( 0 < \mu < 1 \), \( 0 < \delta < \frac{\pi}{2} \) and \( \mu, \eta, \delta \in \mathbb{R} \).

By replacing \( \mu \omega \) by \( t \), we have

\[\mathcal{W} = \int_{\eta}^{\frac{5\mu}{\eta}} \frac{\phi_1(t)}{t} \sin \left( \frac{\eta}{\mu} t \right) \cos \left( \frac{\mu}{t} t \right) dt \]

\[= \int_{\eta'}^{\frac{5\mu}{\eta'}} \frac{\phi_1(t)}{t} \sin \left( \frac{\eta'}{\mu} t \right) \cos \left( \Delta t \right) dt \]

where \( \eta' = \eta \), \( \xi' = \xi \) and therefore \( 0 < \eta' < \xi' < \frac{\pi}{2} \) and
\( \Delta = \frac{m}{\mu \gamma m} \).

Put
\[
\chi(t) = \frac{\tilde{\psi}_1(t)}{E} \sin \left( \frac{\delta}{\mu} t \right) = \tilde{\psi}_1(t) \varTheta(t).
\]

Then \( \chi(t) \in L \) and we have
\[
W = \int_{\eta'}^{\xi'} \chi(t) \cos \Delta t \, dt = -\int_{\eta'-\pi/\Delta}^{\xi'-\pi/\Delta} \chi(t+\pi/\Delta) \cos \Delta t \, dt
\]

\[
= \left\{ -\int_{\eta'}^{\xi'} + \int_{\eta'-\pi/\Delta}^{\xi'-\pi/\Delta} \right\} \chi(t+\pi/\Delta) \cos \Delta t \, dt.
\]

Hence we deduce
\[
|W| = \left| \frac{1}{2} \int_{\eta'}^{\xi'} \left\{ \chi(t) - \chi(t+\pi/\Delta) \right\} \cos \Delta t \, dt \right|
\]

\[
\leq \frac{1}{2} \int_{\eta'}^{\xi'} |\chi(t+\pi/\Delta) - \chi(t)| \, dt + \frac{\delta}{2} \int_{\eta'}^{\xi'} |\chi_1(t)| \, dt + \frac{5}{\delta \mu} \int_{\eta'}^{\xi'} |\chi_1(t)| \, dt
\]

\[
(5.4.12) \quad \leq \frac{1}{2} \int_{\eta'}^{\xi'} |\chi(t+\pi/\Delta) - \chi(t)| \, dt + \frac{5}{\delta \mu} \int_{\eta'}^{\xi'} |\chi_1(t)| \, dt
\]

where \( \xi = (|E|) = \int_E |\tilde{\psi}_1(t)| \, dt \) tends to zero with \( |E| \), measure of \( E \).
Now, since \( \Theta(t) = t \sin(\delta/\mu t) \), we have

\[ |\Theta| \leq \frac{\delta}{\mu} \quad \text{and} \quad \left| \frac{d\Theta}{dt} \right| \leq C \left( \frac{\delta}{\mu} \right)^2, \]

where \( C = \max \left| \frac{d}{dt} \frac{\sin t}{t} \right| \).

Hence

\[ \frac{1}{\eta} \int_{\eta}^{\eta'} \left| \chi(t + \frac{\xi}{\Delta}) - \chi(t) \right| dt \]

\[ \leq \frac{1}{\eta} \int_{\eta}^{\eta'} \left( \phi(t + \frac{\xi}{\Delta}) - \phi(t) \right) \phi(t + \frac{\xi}{\Delta}) + \phi(t) \left( \phi(t + \frac{\xi}{\Delta}) - \phi(t) \right) dt \]

(5.4.13) \[ \leq \frac{\delta}{2\mu} \int_{\eta}^{\eta'} \left| \phi(t + \frac{\xi}{\Delta}) - \phi(t) \right| dt + \frac{\xi}{\Delta} C \left( \frac{\delta}{\mu} \right)^2 \int_{\eta}^{\eta'} \left| \phi(t) \right| dt \cdot \]

If we designate by \( \omega(h) \) the integral modulus of continuity of the function \( \phi \), we have from (5.4.12) and (5.4.13)

(5.4.14) \[ |W| \leq \frac{\delta}{\mu} \left( \frac{\xi}{\Delta} \right) + \frac{\delta}{\mu} \omega \left( \frac{\xi}{\Delta} \right) + \frac{C}{\Delta} \frac{\delta^2}{\mu^2}, \]

where \( C \) is a constant depending on the value of \( f(x) \) only.

Since \( \delta < \frac{1}{\Delta}, \mu \Delta = m \quad \text{and} \quad \Delta > m \geq [\Delta] \), we have from (5.4.14)

(5.4.15) \[ |W| \leq \frac{C}{\mu} \left\{ \frac{\xi}{\Delta} + \omega \left( \frac{\xi}{\Delta} \right) + \frac{1}{\Delta} \right\} = \frac{\beta(\lambda)}{\mu}, \]
where $\beta(\lambda) \to 0$ for $\lambda \to \infty$ and depends only on $\lambda$, $C$ being a constant independent of $\gamma$, $\xi$, $\mu$, $\delta$, $m$.

Now in (5.4.15) if $\mu = \sqrt{\beta(\lambda)}$, $|W| = o(1)$. If $\mu < \sqrt{\beta(\lambda)}$ we have from (5.4.11)

$$|W| < \delta \int_0^{2\pi} |\phi_i(\mu u)| du < \frac{1}{2\pi} \int_0^{2\pi} |\phi_i(t)| dt = o(1) \quad \text{for} \quad \lambda \to \infty.$$ 

Finally, if $\mu > \sqrt{\beta(\lambda)}$, we have $|W| \leq C\sqrt{\beta(\lambda)} \to 0$. Thus

(5.4.16) $|W| = o(1)$, uniformly for $\lambda \to \infty$.

From (5.4.10), (5.4.11) and (5.4.16) we obtain

(5.4.17) $|\Omega^* - \Omega^{**}| = \left\{ \sum_{j=\lambda}^{2\lambda-1} |\xi_j|^2 \right\}^{1/2} = o(1) = o\left( \frac{1}{\lambda} \right) \sum_{j=\lambda}^{2\lambda-1} |\xi_j|$

$$= o\left( \frac{1}{\lambda} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} |\xi_j|^2 \right\}^{1/2}$$

(5.4.18) $= o\left( \frac{\log^{1/4+\lambda}}{\lambda^{1/4}} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} |\xi_j|^b \right\}^{1/2}$

From Hölder's inequality. Hence from (5.4.9) and (5.4.18) we have

$$|\Omega^*| = |\Omega^* - \Omega^{**}| + |\Omega^{**}| = o\left( \frac{\log^{1/4+\lambda}}{\lambda^{1/4}} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} |\xi_j|^b \right\}^{1/2}$$

This result is equally true for $|\Omega_1|, |\Omega_2|, \ldots, |\Omega_I|$. Hence

(5.4.19) $|\Omega_1| + |\Omega_2| + \ldots + |\Omega_I| = o\left( \frac{\log^{1/4+\lambda}}{\lambda^{1/4}} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} |\xi_j|^b \right\}^{1/2}$. 
Consider now $V$ in (5.4.3). We have

$$
|V| < \int_{\pi/\lambda}^{\pi} \left| \frac{\phi(t)}{t} \right| \sum_{j=\lambda}^{2\lambda-1} \frac{\xi_j}{j} \sin n_j t \, dt
$$

$$
\leq \left\{ \int_{\pi/\lambda}^{\pi} \left| \frac{\phi(t)}{t} \right|^{p/2} \, dt \right\}^{2/p} \left\{ \int_{0}^{2\pi} \left| \frac{\xi_j}{j} \sin n_j t \right|^{q} \, dt \right\}^{1/q}
$$

$$
\leq o\left( \lambda^{V/4} \log^{V/4} \lambda \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{1}{j} \right\}^{1/p}
$$

(5.4.20)

$$
= o\left( \frac{\log^{V/4} \lambda}{\lambda^{V/4}} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{1}{j} \right\}^{1/p}
$$

by the theorem of Hausdorff-Young. From (5.4.3), (5.4.19) and (5.4.20) we have

$$
\Theta = o\left( \frac{\log^{V/4} \lambda}{\lambda^{V/4}} \right) \left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{1}{j} \right\}^{1/p}
$$

But we know that for $\xi_j$ satisfying $\left\{ \sum_{j=\lambda}^{2\lambda-1} \frac{1}{j} \right\}^{1/p} = 1$,

we have $T^{1/q} = \Theta$. Therefore

$$
T^{1/q} = o\left( \frac{\log^{V/4} \lambda}{\lambda^{V/4}} \right) ; \text{ that is to say } T = o\left( \frac{\log \lambda}{\lambda^{V/4}} \right) = o\left( \frac{\log \lambda}{\lambda} \right)
$$

because $p \leq q$. Writing

$$
\lim_{\lambda \to \infty} \frac{\phi(t)}{t} = \sin n_j t \, dt
$$
we have shown that

\[(5.4.21) \quad \sum_{j=1}^{3\lambda-1} \frac{1}{j} \left| U_{n_j} \right| \geq o \left( \frac{\log \lambda}{\lambda} \right) \quad \text{for } \lambda \to \infty.\]

For proving Theorem 1, it suffices to consider the case \( K=3^{2m} - 1 \). Now

\[(5.4.22) \quad \sum_{j=1}^{K} \frac{1}{j} \left| U_{n_j} \right| \geq \left\{ \sum_{l=1}^{[\log K]} + \sum_{l=[\log K]+1}^{K} \right\} \frac{1}{j} \left| U_{n_j} \right| \geq o.\]

Consider the first sum. From (5.1.12) we have

\[\left| U_{n_j} \right| = \mid \int_{n_j/k}^{n_j/k} \frac{\phi(t)}{t} \sin \eta t \mathrm{d}t \leq \int_{n_j/k}^{n_j/k} \frac{1}{t} |\phi(t)| \mathrm{d}t = o(\log n_j),\]

since (5.1.12) implies

\[\int_{t}^{n_j} \frac{1}{t} |\phi(t)| \mathrm{d}t = o(\log t).\]

We obtain

\[\sum_{l=1}^{[\log K]} \frac{1}{j} \left| U_{n_j} \right| \geq \sum_{l=1}^{[\log K]} \frac{1}{j} o(\log \eta n_j) = \sum_{j=1}^{[\log K]} \frac{1}{j} o(\log \eta \log k) = o(\log^2 \log k) = o(\log k).\]

(5.4.23)

As regards the second sum of (5.4.22), we have for \( m \) sufficiently large

\[\left[ \log k \right] + 1 \geq \log k \geq 2 \left[ \log m \right].\]
Hence we obtain

$$
\sum_{j=1}^{K} \frac{1}{j} \left| U_{n_{j}} \right|^{q} \leq \sum_{j=1}^{K} \frac{1}{j} \left| U_{n_{j}} \right|^{q}
$$

(5.4.24)

$$
(5.4.24) = \left\{ \sum_{j=1}^{\lfloor \log m \rfloor - 1} \frac{1}{j} \left| U_{n_{j}} \right|^{q} + \sum_{j=1}^{2m-1} \frac{1}{j} \left| U_{n_{j}} \right|^{q} + \sum_{j=1}^{2m-1} \frac{1}{j} \left| U_{n_{j}} \right|^{q} \right\} = \sum_{j=1}^{K} \frac{1}{j} \left| U_{n_{j}} \right|^{q}.
$$

Since there are $2m - \lfloor \log m \rfloor$ sums in (5.4.24) we have from (5.4.21)

$$
\sum_{j=1}^{K} \frac{1}{j} \left| U_{n_{j}} \right|^{q} = o\left\{ \frac{2m-1}{2^m - 2} + \frac{2m-1}{2^m - 2} + \cdots \right\} \text{ up to } 2m - \lfloor \log m \rfloor \text{ terms}
$$

$$
= o\left\{ \frac{2m+1}{2^m - 2} + \frac{2m+2}{2^m - 2} + \cdots \right\} \text{ up to } 2m \text{ terms}
$$

$$
= o\left\{ 2m \left( \frac{1}{2^m - 2} + \frac{1}{2^m - 2} + \cdots \right) + \left( \frac{1}{2^m - 2} + \frac{1}{2^m - 2} + \cdots \right) \right\} \text{ for } m \geq 2
$$

$$
= o(m)
$$

(5.4.25)

$$
= o(\log k).
$$

Combining (5.4.22), (5.4.23) and (5.4.25) we obtain

(5.4.26)

$$
\sum_{j=1}^{K} \frac{1}{j} \left| U_{n_{j}} \right|^{q} = o(\log k).
$$

Finally, because $S_{\Lambda}^{*}(x) - S^{*} = -\frac{1}{K} \left| U_{n_{j}} \right| + o(1)$, we have from Minkowski's inequality and (5.4.26)

$$
\left\{ \sum_{j=1}^{K} \frac{1}{j} \left| S_{\Lambda}^{*}(x) - S^{*} \right|^{q} \right\}^{\frac{1}{q}} \leq \left\{ \sum_{j=1}^{K} \frac{1}{j} \left| U_{n_{j}} \right|^{q} \right\}^{\frac{1}{q}} + o(\log \frac{1}{q} + k)
$$
\( = o(\log^{1/4} k), \)

\[
\sum_{j=1}^{k} \frac{1}{j} \left| \frac{S_n^*(x) - S^*}{q} \right|^q = o(\log k),
\]

which proves Theorem 1.

5.5. **Proof of Cor. 1.** We note that \( q \geq 2 \), being given, if \( \lambda \) is defined by the equation \( \lambda = q / q - 1 \), the condition

\[
\int_0^\infty |\mathcal{A}^*(u)|^b du = o\left( \frac{1}{t \log \frac{b-1}{b}} \right), \quad 1 < b \leq 2,
\]

is then satisfied, as can be seen easily. The result is then an immediate consequence of the application of Theorem 1.

6. **Proof of Theorem 2.** When Lemma 2 holds good, the partial sum of order \( n \) of the series (5.1.5) is given by (1.8.9) as

\[
S_n(x) - S = -\frac{2}{\lambda} \int_0^\infty q(t) \frac{\sec n \pi t}{t} dt + o(1),
\]

where \( S \) is a constant.

Now the proof runs parallel to that of Theorem 1.

Cor. 2. can be demonstrated as Cor. 1.