CHAPTER V

COMPLETELY IRRESOLUTE MAPPINGS

Q. INTRODUCTION:

The class of strongly continuous mappings and that of semi-continuous mappings were introduced by N. Levine in [7] and [8], respectively. The concept of irresolute mappings due to Crossley and Hildebrand [3] is weaker than that of strong continuity but is stronger than that of semi-continuity. However, the concept of irresolute mappings and that of continuous mappings are independent to each other. Again, the notion of complete continuity introduced by Arya and Gupta [1] is weaker than that of strong continuity but is stronger than that of semi-continuity. Moreover, complete continuity and irresolute mappings are the concepts independent of each other.

In this Chapter, we initiate to study a class of completely irresolute mappings. This concept grows in such a way that it is weaker than the strong continuity on one hand and, on the other hand, it is stronger than complete continuity and is also stronger than the concept of irresolute mappings.
1. PRELIMINARIES

A set \( A \) in a space \( X \) is known to be semi-open if there exists an open set \( O \) in \( X \) such that \( O \subseteq A \subseteq \text{cl} \ O \), where \( \text{cl} \ O \) denotes as usual the closure of \( O \) in \( X \). Every open set is semi-open but not conversely. Any union of semi-open sets is semi-open. Complement of a semi-open set in a space \( X \) is a semi-closed set in \( X \). Any closed set is semi-closed but not conversely. Also, the intersection of all the semi-closed sets containing \( A \) is the semi-closure of \( A \) and is denoted by \( \text{scl} \ A \). Since any intersection of semi-closed sets is semi-closed, \( \text{scl} \ A \) is semi-closed and hence, \( A \) is semi-closed iff \( A = \text{scl} \ A \). A set \( U \) in a space \( X \) is termed regularly open(respectively, regularly closed)[4,p.92, Exercise 22] if \( U = \text{Int} (\text{cl} \ U) \) (respectively, \( U = \text{cl} (\text{Int} \ U) \)), where \( \text{Int} \) denotes the interior operator in \( X \). The complement of a regularly open (regularly closed) set in \( X \) is a regularly closed (regularly open) set. Also a regularly open (regularly closed) set is open(closed) but converse may fail. A space \( X \) is semi-\( T_2 \) iff, to each pair of distinct points \( x, y \) of \( X \), there exists a semi-open set \( A \) containing \( x \) such that \( y \notin \text{scl} \ A \).
**DEFINITION 1.1 [7]:** A mapping \( f : X \to Y \) is said to be strongly continuous if \( f(\text{cl } A) \subseteq f(A) \) for each subset \( A \) of \( X \).

Equivalently, \( f : X \to Y \) is strongly continuous iff the inverse image of every subset of \( Y \) is open as well as closed in \( X \) [7].

**DEFINITION 1.2 [8]:** A mapping \( f : X \to Y \) is said to be semi-continuous if the inverse image of every open subset of \( Y \) is semi-open in \( X \).

**DEFINITION 1.3 [3]:** A mapping \( f : X \to Y \) is said to be irresolute if the inverse image of every semi-open subset of \( Y \) is semi-open in \( X \).

**DEFINITION 1.4 [1]:** A mapping \( f : X \to Y \) is said to be completely continuous if the inverse image of every open subset of \( Y \) is a regularly open subset in \( X \).

**DEFINITION 1.5 [6]:** A mapping \( f : X \to Y \) is said to be strongly irresolute if \( f(\text{cl } A) \subseteq f(A) \) for every subset \( A \) of \( X \).
Equivalently, \( f: X \rightarrow Y \) is strongly irresolute iff \( f^{-1}(B) \) is semi-open as well as semi-closed for each subset \( B \) of \( Y \) [6].

Note that every irresolute mapping is semi-continuous but not conversely. Further, a completely continuous mapping is semi-continuous but converse may fail. Also, the concept of completely continuous mappings and that of irresolute mappings are independent to each other. However, a strongly continuous mapping is completely continuous but not conversely.

Recall that the strongly irresolute mappings have been explicitly studied in the earlier Chapter IV and in [6]. There in, it has been found that every strongly continuous mapping is strongly irresolute but the converse may not hold. Further, in the Chapter IV, it has been noticed that every strongly irresolute mapping is irresolute but the converse is not necessarily true.

2. COMPLETELY IRRESOLUTE Mappings:

Starting from the concept of strong continuity, one may arrive, of course by means of implication relations,
at irresolute mappings concept via strongly irresolute. This is one of the ways which has been established in the preceding Chapter IV. The other way, retaining the same, is through the completely irresolute mappings — the concept to which the study of the present chapter is devoted.

Further, this introduced concept enables us to make it a point starting from which, with the implication relations view of point, we arrive at the concept of semi-continuity in two ways — one through complete continuity and the other through irresolute mappings.

**DEFINITION 2.1.** A mapping \( f: X \to Y \) is said to be completely irresolute if the inverse image of every semi-open subset in \( Y \) is a regularly open subset of \( X \).

Evidently, a mapping \( f: X \to Y \) is completely irresolute iff the inverse image of every semi-closed subset in \( Y \) is a regularly closed subset of \( X \).

Obviously, every strongly continuous mapping is completely irresolute and every completely irresolute mapping is completely continuous. The converse implications do not hold, in general, as is shown by the following examples.
**EXAMPLE 2.1:** Let \( X = \{a, b, c\} \) with an indiscrete topology. Then, obviously, the identity mapping \( i : X \rightarrow X \) is completely irresolute but neither strongly continuous nor strongly irresolute.

**EXAMPLE 2.2:** Let \( X = \{a, b, c, d\} \) with topology
\[
\mathcal{T} = \{\emptyset, X, \{a, b, c\}, \{c\}, \{a, b\}\}
\]
and \( Y = \{p, q, r\} \) with topology
\[
\mathcal{U} = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}.
\]
If \( f : X \rightarrow Y \) be the mapping defined by \( f(a) = f(b) = p, f(c) = f(d) = r \), then \( f \) is a completely continuous and strongly irresolute mapping which is not completely irresolute.

Also, every completely irresolute mapping is irresolute. But an irresolute mapping may fail to be completely irresolute, as, in Example 2.2, \( f \) is irresolute but not completely irresolute.

In view of Example 2.1 and Example 2.2, it follows that completely irresolute mappings and strongly irresolute mappings are independent notions.

In conclusion, we have the following implications diagram:
3. SOME BASIC PROPERTIES:

In this section, some of the basic properties related to the restriction and the composition of mappings are being discussed.

**Theorem 3.1.** If a mapping \( f : X \rightarrow Y \) and a mapping \( g : Y \rightarrow Z \) are completely irresolute, then \( g \circ f : X \rightarrow Z \) is completely irresolute.

**Proof:** It is straightforward by the definitions themselves.

It may be mentioned that the above result can be sharpened by weakening the condition on the mapping \( g \) as follows.

**Theorem 3.2.** If \( f : X \rightarrow Y \) is completely irresolute and \( g : Y \rightarrow Z \) is irresolute, then \( g \circ f : X \rightarrow Z \) is completely irresolute.
PROOF: Obvious.

Again, the result given in the Theorem 3.1 can also be sharpened by weakening the condition on the mapping $f$.

**THEOREM 3.3.** If $f: X \rightarrow Y$ is completely continuous and $g: Y \rightarrow Z$ is completely irresolute, then $g \circ f: X \rightarrow Z$ is completely irresolute.

**PROOF:** Evidently follows from definitions themselves.

As an immediate consequence of any one of the Theorems 3.1, 3.2 and 3.3, we have the following result.

**COROLLARY 3.1.** If $f: X \rightarrow Y$ is strongly continuous and $g: Y \rightarrow Z$ is completely irresolute, then $g \circ f: X \rightarrow Z$ is completely irresolute.

**DEFINITION 3.1.**[12]: A mapping $f: X \rightarrow Y$ is said to be almost open if the image of every regularly open set is open.

**THEOREM 3.4.** If $f: X \rightarrow Y$ is almost open surjection and $g: Y \rightarrow Z$ is such that $g \circ f: X \rightarrow Z$ is completely irresolute, then $g$ is irresolute.
PROOF: Let \( A \) be any semi-open subset of \( Z \). Then, since 
\( g \circ f \) is completely irresolute, \((g \circ f)^{-1}(A)\) is a regularly 
open subset of \( X \). Since \( f \) is almost open surjection, 
\( f((g \circ f)^{-1}(A)) = g^{-1}(A) \) is a semi-open subset of \( Y \). Hence 
g is irresolute.

**Lemma 3.1** [3]: If \( f: X \rightarrow Y \) is continuous and open, then 
f is irresolute.

**Theorem 3.5**: Let \( f: X \rightarrow \prod_{\alpha \in \Delta} X_\alpha \) be completely irresolute.
For each \( \alpha \in \Delta \), let \( f_\alpha: X \rightarrow X_\alpha \) be defined as 
\( f_\alpha(x) = x_\alpha \) where \( f(x) = (x_\alpha) \). Then each \( f_\alpha \) is completely irresolute.

**Proof**: Let \( p_\alpha \) be the projection of \( \prod_{\alpha \in \Delta} X_\alpha \) onto \( X_\alpha \).
Then, for each \( \alpha \in \Delta \), \( f_\alpha = p_\alpha \circ f \). Since \( f \) is completely 
irresolute and \( p_\alpha \), being continuous and open, is irresolute 
by Lemma 3.1, it follows in view of Theorem 3.2 that \( f_\alpha \) is 
completely irresolute.

The restriction of a completely irresolute mapping 
to a subset of its domain may fail to be completely irresolute 
as is shown by the following example.

**Example 3.1**: Let \( X = \{a, b, c, d\} \) with topology
\[ T = \{ \emptyset, X, \{a, b\}, \{c\}, \{a, b, c\} \} \text{ and } Y = \{p, q, r\} \text{ with topology} \]

\[ \mathcal{U} = \{ \emptyset, Y, \{p, q\} \} \]. Then the mapping \( f: X \rightarrow Y \), defined by \( f(a) = p, f(b) = q, f(c) = f(d) = r \), is completely irresolute. However, the restriction of \( f \) to the set \( A = \{a, d\} \) is not completely irresolute.

**4. Some More Properties:**

**Definition 4.1 [13]:** A space \( X \) is said to be nearly compact if every regularly open cover of \( X \) has a finite subcover.

**Definition 4.2 [14]:** A space \( X \) is said to be \( e \)-compact if every semi-open cover of \( X \) has a finite subcover.

**Theorem 4.1:** Every surjective completely irresolute image of a nearly compact space is \( e \)-compact.

**Proof:** Let \( f : X \rightarrow Y \) be a completely irresolute mapping from a nearly compact space \( X \) onto a space \( Y \). Let \( \{ U_\alpha : \alpha \in \Delta \} \) be any semi-open cover of \( Y \). Thus, \( \{ f^{-1}(U_\alpha) : \alpha \in \Delta \} \) is a regularly open cover of \( X \). Since \( X \) is nearly compact, there exists a finite subfamily
\( \{ f^{-1}(U_{\alpha i}) : i = 1, 2, \ldots, n \} \) of \( \{ f^{-1}(U_{\alpha}) : \alpha \in \Delta \} \) which covers \( X \). It follows then that \( \{ U_{\alpha i} : i = 1, 2, \ldots, n \} \) is a finite subfamily of \( \{ U_{\alpha} : \alpha \in \Delta \} \) which covers \( Y \). Hence, \( Y \) is \( s \)-compact.

**Lemma 4.1:** A mapping \( f : X \to Y \) has a strongly semi-closed graph \( G(f) \) iff, for each \( x \in X \), \( y \in Y \) such that \( y \neq f(x) \), there exist semi-open sets \( U \) in \( X \) and \( V \) in \( Y \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap \text{scl } V = \emptyset \) [Chapter III and [5]].

In view of Example 2.1, a completely irresolute mapping may fail to have a strongly semi-closed graph. However, we have the result given below.

**Theorem 4.2:** If \( f : X \to Y \) is completely irresolute and \( Y \) is semi-T\(_2\), then \( G(f) \) is strongly semi-closed.

**Proof:** Let \( x \in X \), \( y \in Y \) such that \( y \neq f(x) \). Then since \( Y \) is semi-T\(_2\), there exists a semi-open set \( V \) containing \( y \) such that \( f(x) \not\subseteq \text{scl } V \). Therefore \( f^{-1}(\text{scl } V) \) is regularly closed and does not contain \( x \). Thus \( x \in X - f^{-1}(\text{scl } V) = U \), a semi-open set in \( X \). Hence \( f(U) \cap \text{scl } V = \emptyset \). Consequently, \( G(f) \) is strongly semi-closed.
The converse to the above Theorem 4.2 is not necessarily true as is shown by the following example.

**Example 4.1.** Let \( X = \{a, b, c\} \) with topologies

\[
\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}
\]
and \( \mathcal{U} \) = a discrete topology.

Then, obviously, the identity mapping \( i : (X, \mathcal{T}) \rightarrow (X, \mathcal{U}) \) has a strongly semi-closed graph but is not completely irresolute.

**Definition 4.3.** A mapping \( f : X \rightarrow Y \) is said to be pre-semi-closed if the image under \( f \) of every semi-closed set in \( X \) is a semi-closed subset in \( Y \).

**Theorem 4.3.** Let \( f : X \rightarrow Y \) be a pre-semi-closed map. Given any subset \( S \subseteq Y \) and any semi-open set \( U \) containing \( f^{-1}(S) \), then there exists a semi-open set \( V \supseteq S \) such that \( f^{-1}(V) \subseteq U \).

**Proof:** Let \( V = Y \setminus f(X \setminus U) \); since \( f^{-1}(S) \subseteq U \), it follows that \( S \subseteq V \), and because \( f \) is pre-semi-closed, \( V \) is semi-open in \( Y \). Also,

\[
\begin{align*}
f^{-1}(V) &= f^{-1}(Y) - f^{-1}[f(X \setminus U)] \\
&\subseteq X - [X - U] = U.
\end{align*}
\]

This completes the proof.
DEFINITION 4.4. [14]: A space $X$ is said to be strongly $e$-normal if, to each pair of disjoint semi-closed subsets $A$ and $B$ of $X$, there exist disjoint semi-open subsets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$.

DEFINITION 4.5.: A space $X$ is said to be mildly $e$-normal if, for each pair of disjoint regularly closed subsets $A$ and $B$ of $X$, there exist disjoint semi-open sets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$.

THEOREM 4.4. : If $f : X \to Y$ be completely irresolute, pre-semi-closed mapping from a mildly $e$-normal space $X$ onto a space $Y$, then $Y$ is strongly $e$-normal.

PROOF: Let $A$ and $B$ be two disjoint semi-closed subsets of $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regularly closed subsets of $X$. Since $X$ is mildly $e$-normal, there exist disjoint semi-open sets $G$ and $H$ such that $f^{-1}(A) \subseteq G$ and $f^{-1}(B) \subseteq H$. There exist then, by Theorem 4.3, semi-open sets $U = Y - f(X - G)$ such that $U \supseteq A$, $f^{-1}(U) \subseteq G$ and $V = Y - f(X - H)$ such that $V \supseteq B$, $f^{-1}(V) \subseteq H$. Evidently, $U$ and $V$ are disjoint. Hence $Y$ is strongly $e$-normal.

DEFINITION 4.6. [10]: A space $X$ is said to be almost $e$-regular if, for every regularly closed subset $F$ and every
point $x \notin F$, there exist disjoint semi-open subsets $U$ and $V$ in $X$ such that $x \in U$ and $F \subset V$.

**Definition 4.7.** [14]: A space $X$ is strongly $s$-regular if to each semi-closed subset $F$ and every point $x \notin F$, there exist disjoint semi-open subsets $U$ and $V$ in $X$ such that $x \in U$ and $F \subset V$.

**Theorem 4.5:** If $f$ is a completely irresolute, pre-semi-closed injection of an almost $s$-regular space $X$ onto a space $Y$, then $Y$ is strongly $s$-regular.

**Proof:** Let $F$ be a semi-closed subset of $Y$ and let $y \notin F$. Then $f^{-1}(F)$ is a regularly closed subset of $X$ such that $f^{-1}(y) = x \notin f^{-1}(F)$. Therefore, $X$ being almost $s$-regular, there exist disjoint semi-open subsets $U$ and $V$ in $X$ such that $f^{-1}(y) \in U$ and $f^{-1}(F) \subset V$. Therefore, by Theorem 4.3, semi-open subsets $G = Y - f(X - U)$ such that $f^{-1}(G) \subset U$, $y \in G$ and $H = Y - f(X - V)$ such that $f^{-1}(H) \subset V$, $F \subset H$. $G$ and $H$ are, obviously, disjoint. Hence $Y$ is strongly $s$-regular.

A space $X$ is not $s$-connected iff it is the union of two non-empty, disjoint, semi-open sets [11]. Also every $s$-connected space is connected [11].
**THEOREM 4.6.1** Let $f : X \to Y$ be a completely irresolute surjection. If $X$ is connected (s-connected), then $Y$ is s-connected.

**PROOF:** Suppose $Y$ is not s-connected. Then $Y$ is the union of two non-empty, disjoint, semi-open subsets $A$ and $B$. Hence $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty, regularly open hence open (semi-open), disjoint and their union is $X$. Hence $X$ is not connected (not s-connected).

**COROLLARY 4.1:** Let $f : X \to Y$ be completely irresolute surjection. If $X$ is connected (s-connected), then $Y$ is connected.
REFERENCES


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